97學年 機率導論 期中小考二(共兩面) 授課老師：蔡碧紋

學號：________ 姓名：________

1. (TRUE or FALSE) Let X, Y be any two nonnegative integer-valued random variables. Assume their expectations and varainces are finite.

   (a) $\mathbf{E}(X+Y) = \mathbf{E}(X) + \mathbf{E}(Y)$. $\underline{T}$

   (b) $\mathbf{E}(X \cdot Y) = \mathbf{E}(X) \cdot \mathbf{E}(Y)$. $\underline{F}$

   (c) $\mathbf{E}(X^2) = \sum_{k=0}^{\infty} k^2\, \mathbf{P}(X = k)$. $\underline{T}$

   (d) $\mathbf{E}(X^2) = \sum_{k=0}^{\infty} k\, \mathbf{P}(X^2 = k)$. $\underline{T}$

   (e) $\mathrm{Var}(X) = \sum_{k=0}^{\infty} k^2\, \mathbf{P}(X = k) - [\sum_{k=0}^{\infty} k\, \mathbf{P}(X = k)]^2$. $\underline{T}$

2. (TRUE or FALSE) Let X, Y be two random variables with continuous joint probability density function (joint pdf) $f(x, y)$ and marginal densities $f_X(x)$ and $f_Y(y)$, respectively.

   (a) If the correlation coefficient of X and Y is 0, $\rho_{xy} = 0$, then $f(x, y) = f_X(x) \cdot f_Y(y)$. $\underline{F}$

   (b) $\mathbf{P}(X > 4) = \int_4^{\infty} f_X(x)dx$. $\underline{T}$

   (c) $\mathbf{P}(X \geq 4 \mid Y = 1) = \int_4^{\infty} f_X(x \mid Y = 1)dx$. $\underline{T}$ (the subscription is just a notation, it doesn't matter if one writes f_X, $f_{X|Y}$, or f_1)

   (d) $\mathbf{E}(Y) = \mathbf{E}[\mathbf{E}(X \mid Y)]$. $\underline{F}$

   (e) $\int_{-\infty}^{\infty} f_X(x|Y = 1)dx = 1$. $\underline{T}$

3. $\because X \sim U(0,1) \quad \therefore f(x) = 1 \quad 0<x<1$

$E(X) = \int_0^1 x\, dx = \frac{1}{2}$

(a) $\because Y \mid X=x \sim U(0, x)$

$\therefore f(y|x) = \frac{1}{x} \quad 0<y<x \quad 0<x<1$

(b) $E[Y|X=x] = \int_0^x y\, f(y|x)\, dy = \int_0^y y \frac{1}{x} dy = \frac{x}{2} \quad 0<x<1$

(c) $f(x,y) = f(y|x) f(x) = \frac{1}{x} \cdot 1 = \frac{1}{x} \quad 0<y<x \quad 0<x<1$

(d) $E[Y] = E[E[Y|x]] = E[\frac{X}{2}] = \frac{1}{2}E[x] = \frac{1}{4}$

or $f(y) = \int_y^1 f(x,y)\, dx = \int_y^1 \frac{1}{x} dx = -\ln y \quad 0<y<1$

$E[Y] = \int_0^1 y\, f(y)\, dy = \int_0^1 -y \ln y\, dy = \frac{1}{4}$ (integral by part)

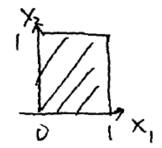
4.

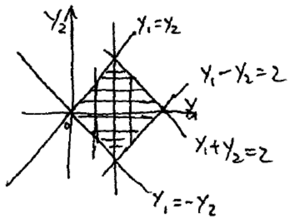
$$\left.\begin{array}{ll} f(x_1) = 1 & 0 < x_1 < 1 \\ f(x_2) = 1 & 0 < x_2 < 1 \\ X_1 \perp X_2 & \end{array}\right| \Rightarrow \text{(a)} \quad f(x_1, x_2) = 1 \qquad \begin{array}{l} 0 < x_1 < 1 \\ 0 < x_2 < 1 \end{array}$$

$$\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = X_1 - X_2 \end{cases} \Rightarrow \quad X_1 = \frac{Y_1 + Y_2}{2} \quad X_2 = \frac{Y_1 - Y_2}{2} \qquad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

(b)

$$f(y_1, y_2) = f(x_1, x_2)\,|J| = \frac{1}{2} \qquad \begin{array}{l} 0 < y_1 + y_2 < 2 \\ 0 < y_1 - y_2 < 2 \end{array}$$





(c)

$$f_{Y_1}(y_1) = \int_{y_2} f(y_1, y_2)\, dy_2 = \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2}\, dy_2 = y_1 & 0 < y_1 \leq 1 \\ \int_{y_1 - 2}^{2 - y_1} \frac{1}{2}\, dy_2 = 2 - y_1 & 1 \leq y < 2 \end{cases}$$

Note that sum of two independent Uniform is a triangular distribution