The convex and monotone functions associated with second-order cone

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Abstract Like the matrix-valued functions used in solutions methods for semidefinite program (SDP) and semidefinite complementarity problem (SDCP), the vector-valued functions associated with second-order cone are defined analogously and also used in solutions methods for second-order cone program (SOCP) and second-order cone complementarity problem (SOCCP). In this paper, we study further about these vector-valued functions associated with second-order cone. In particular, we define so-called SOC-convex and SOC-monotone functions for any given function $f : \mathbb{R} \rightarrow \mathbb{R}$. We discuss the SOC-convexity and SOC-monotonicity for some simple functions, e.g., $f(t) = t^2, t^3, 1/t, t^{1/2}, |t|$, and $|t|_+$. Some characterizations of SOC-convex and SOC-monotone functions are studied and some conjectures about the relationship between SOC-convex and SOC-monotone functions are proposed.

Key words. Second-order cone, convex function, monotone function, complementarity, spectral decomposition

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1 Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called Lorentz cone, is defined by

$$\mathcal{K}^n = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\},$$

where $\| \cdot \|$ denotes the Euclidean norm. If $n = 1$, let $\mathcal{K}^n$ denote the set of nonnegative reals $\mathbb{R}_+$. For any $x, y$ in $\mathbb{R}^n$, we write $x \preceq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$; and write $x \succ_{\mathcal{K}^n} y$ if

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In other words, we have $x \succeq_{K^n} 0$ if and only if $x \in K^n$ and $x \succ_{K^n} 0$ if and only if $x \in \text{int}(K^n)$. The relation $\succeq_{K^n}$ is a partial ordering, but not a linear ordering in $K^n$, i.e., there exist $x, y \in K^n$ such that neither $x \succeq_{K^n} y$ nor $y \succeq_{K^n} x$. To see this, for $n = 2$, let $x = (1, 1), y = (1, 0)$. Then we have $x - y = (0, 1) \notin K^n$, $y - x = (0, -1) \notin K^n$.

Recently, the second-order cone has received much attention in optimization, particularly in the context of applications and solutions methods for second-order cone program (SOCP) [14] and second-order cone complementarity problem (SOCCP), [5, 6, 7, 8]. For those solutions methods, there needs spectral decomposition associated with SOC. The basic concepts are as below. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $x$ can be decomposed as

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)},$$

where $\lambda_1, \lambda_2$ and $u^{(1)}, u^{(2)}$ are the spectral values and the associated spectral vectors of $x$ given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|,$$

$$u^{(i)} = \begin{cases} \frac{1}{\|x_2\|} \left(1, (-1)^i \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ \frac{1}{2} \left(1, (-1)^i w\right), & \text{if } x_2 = 0, \end{cases}$$

for $i = 1, 2$ with $w$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique.

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $K^n (n \geq 1)$ was considered [8, 10]:

$$f^{soc}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \quad (5)$$

If $f$ is defined only on a subset of $\mathbb{R}$, then $f^{soc}$ is defined on the corresponding subset of $\mathbb{R}^n$. The definition (5) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{soc}(x) = x^{1/2}, x^2, \exp(x)$ are discussed in the book of [9]. In fact, the above definition (5) is analogous to one associated with the semidefinite cone $S^n_+$, see [19, 21].

In this paper, we further define so-called SOC-convex and SOC-monotone functions (see Sec. 3) which are parallel to matrix-convex and matrix-monotone functions (see [2, 11]). We study the SOC-convexity and SOC-monotonicity for some simple functions, e.g., $f(t) = t^2, t^3, 1/t, t^{1/2}, \lfloor t \rfloor$, and $\lceil t \rceil$. Then, we explore characterizations of SOC-convex and SOC-monotone functions. In addition, we state some conjectures about the relationship between SOC-convex and SOC-monotone functions. It is our intention to extend the existing properties of matrix-convex and matrix-monotone functions shown as in [2, 11]. As will be seen in Sec. 3, the vector-valued functions associated with SOC are accompanied by Jordan product (will be defined in Sec. 2). However, unlike the matrix multiplication, the Jordan product associated with SOC is not associative which is the main source of difficulty when
we do the extension. Therefore, the ideas for proofs are usually quite different from those for matrix-valued functions. The vector-valued functions associated with SOC are heavily used in the solutions methods for SOCP and SOCCP. Therefore, further study on these functions will be helpful for developing and analyzing more solutions methods. That is one of the main motivations for this paper.

In what follows and throughout the paper, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\| \cdot \|$ is the Euclidean norm. The notation ”:=” means ”define”. For any $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of $f$ at $x$. For any differentiable mapping $F = (F_1, F_2, \ldots, F_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla F(x) = [\nabla F_1(x) \cdots \nabla F_m(x)]$ is a $n \times m$ matrix which denotes the transpose Jacobian of $F$ at $x$. For any symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \succeq B$ (respectively, $A \succ B$) to mean $A - B$ is positive semidefinite (respectively, positive definite).

2 Jordan product and related properties

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their Jordan product as
\[
x \circ y = (x^T y, y_1 x_2 + x_1 y_2).
\]

We write $x^2$ to mean $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. Then $\circ, +$, together with $e = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$ have the following basic properties (see [9, 10]): (1) $e \circ x = x$, for all $x \in \mathbb{R}^n$. (2) $x \circ y = y \circ x$, for all $x, y \in \mathbb{R}^n$. (3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, for all $x, y \in \mathbb{R}^n$. (4) $(x + y) \circ z = x \circ z + y \circ z$, for all $x, y, z \in \mathbb{R}^n$. The Jordan product is not associative. For example, for $n = 3$, let $x = (1, -1, 1)$ and $y = z = (1, 0, 1)$, then we have $(x \circ y) \circ z = (4, -1, 4) \neq x \circ (y \circ z) = (4, -2, 4)$. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$, for all $x \in \mathbb{R}^n$. Thus, we may, without fear of ambiguity, write $x^m$ for the product of $m$ copies of $x$ and $x^{m+n} = x^m \circ x^n$ for all positive integers $m$ and $n$. We define $x^0 = e$. Besides, $\mathcal{K}^n$ is not closed under Jordan product. For example, $x = (\sqrt{2}, 1, 1) \in \mathcal{K}^3$, $y = (\sqrt{2}, 1, -1) \in \mathcal{K}^3$, but $x \circ y = (2, 2\sqrt{2}, 0) \notin \mathcal{K}^3$.

For each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the determinant and the trace of $x$ are defined by
\[
det(x) = x_1^2 - \|x_2\|^2, \quad tr(x) = 2x_1.
\]

In general, $\det(x \circ y) \neq \det(x) \det(y)$ unless $x_2 = y_2$. A vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is said to be invertible if $\det(x) \neq 0$. If $x$ is invertible, then there exists a unique $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ satisfying $x \circ y = y \circ x = e$. We call this $y$ the inverse of $x$ and denote it by $x^{-1}$. In fact, we have
\[
x^{-1} = \frac{1}{x_1^2 - \|x_2\|^2}(x_1 \cdot -x_2) = \frac{1}{\det(x)}(tr(x)e - x).
\]
Therefore, \( x \in \text{int}(\mathcal{K}^n) \) if and only if \( x^{-1} \in \text{int}(\mathcal{K}^n) \). Moreover, if \( x \in \text{int}(\mathcal{K}^n) \), then \( x^{-k} = (x^k)^{-1} \) is also well-defined. For any \( x \in \mathcal{K}^n \), it is known that there exists a unique vector in \( \mathcal{K}^n \) denoted by \( x^{1/2} \) such that \( (x^{1/2})^2 = x^{1/2} \odot x^{1/2} = x \). Indeed,

\[
x^{1/2} = \left( s, \frac{x_2}{2s} \right), \quad \text{where} \quad s = \sqrt{\frac{1}{2} \left( x_1 + \sqrt{x_1^2 - \|x_2\|^2} \right)}.
\]

In the above formula, the term \( x_2/s \) is defined to be the zero vector if \( x_2 = 0 \) and \( s = 0 \), i.e., \( x = 0 \).

For any \( x \in \mathbb{R}^n \), we always have \( x^2 \in \mathcal{K}^n \) (i.e., \( x^2 \succeq_{K^n} 0 \)). Hence, there exists a unique vector \( (x^2)^{1/2} \in \mathcal{K}^n \) denoted by \( \|x\| \). It is easy to verify that \( \|x\| \succeq_{K^n} 0 \) and \( x^2 = \|x\|^2 \) for any \( x \in \mathbb{R}^n \). It is also known that \( \|x\| \succeq_{K^n} x \). For any \( x \in \mathbb{R}^n \), we define \( [x]_+ \) to be the nearest point (in Euclidean norm, since Jordan product does not induce a norm) projection of \( x \) onto \( \mathcal{K}^n \), which is the same definition as in \( \mathbb{R}^n_+ \). In other words, \( [x]_+ \) is the optimal solution of the parametric SOCP: \( [x]_+ = \arg\min \{ \|x - y\| \mid y \in \mathcal{K}^n \} \). It is well-known that \( [x]_+ = \frac{1}{2}(x + \|x\|) \), see Property 2.2(f).

Next, for any \( x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \), we define a linear mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) as

\[
L_x : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad y \rightarrow L_x y := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} y.
\]

It can be easily verified that \( x \odot y = L_x y, \forall y \in \mathbb{R}^n \), and \( L_x \) is positive definite (and hence invertible) if and only if \( x \in \text{int}(\mathcal{K}^n) \). However, \( L_x^{-1} y \neq x^{-1} \odot y, \) for some \( x \in \text{int}(\mathcal{K}^n) \) and \( y \in \mathbb{R}^n \), i.e., \( L_x^{-1} \neq L_x^{-1} \).

The spectral decomposition along with the Jordan algebra associated with SOC entail some basic properties as below. We omit the proofs since they can be found in [9, 10].

**Property 2.1** For any \( x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) with the spectral values \( \lambda_1, \lambda_2 \) and spectral vectors \( u^{(1)}, u^{(2)} \) given as in (3)-(4), we have

(a) \( u^{(1)} \) and \( u^{(2)} \) are orthogonal under Jordan product and have length \( 1/\sqrt{2} \), i.e.,

\[
u^{(1)} \odot u^{(2)} = 0, \quad \|u^{(1)}\| = \|u^{(2)}\| = \frac{1}{\sqrt{2}}.
\]

(b) \( u^{(1)} \) and \( u^{(2)} \) are idempotent under Jordan product, i.e.,

\[
u^{(i)} \odot u^{(i)} = u^{(i)}, \quad i = 1, 2.
\]
(c) \( \lambda_1, \lambda_2 \) are nonnegative (positive) if and only if \( x \in \mathcal{K}^n \) \((x \in \text{int}(\mathcal{K}^n))\), i.e.,
\[
\lambda_i \geq 0, \forall i = 1, 2 \iff x \succeq_{\mathcal{K}_n} 0.
\]
\[
\lambda_i > 0, \forall i = 1, 2 \iff x \succ_{\mathcal{K}_n} 0.
\]

(d) The determinant, the trace and the Euclidean norm of \( x \) can all be represented in terms of \( \lambda_1, \lambda_2 \):
\[
\det(x) = \lambda_1 \lambda_2, \quad tr(x) = \lambda_1 + \lambda_2, \quad \|x\|^2 = \frac{1}{2}(\lambda_1^2 + \lambda_2^2).
\]

Property 2.2 For any \( x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \) with the spectral values \( \lambda_1, \lambda_2 \) and spectral vectors \( u^{(1)}, u^{(2)} \) given as in (3)-(4), we have

(a) \( x^2 = \lambda_1^2 u^{(1)} + \lambda_2^2 u^{(2)} \).

(b) If \( x \in \mathcal{K}_n \), then \( x^{1/2} = \sqrt{\lambda_1} u^{(1)} + \sqrt{\lambda_2} u^{(2)} \).

(c) \( |x| = |\lambda_1| u^{(1)} + |\lambda_2| u^{(2)} \).

(d) \( |x| = [\lambda_1]_+ u^{(1)} + [\lambda_2]_+ u^{(2)} \), \( |x| = [\lambda_1]_- u^{(1)} + [\lambda_2]_- u^{(2)} \).

(e) \( |x| = [x]_+ + [-x]_+ = [x]_+ - [x]_- \).

(f) \( [x]_+ = \frac{1}{2}(x + |x|) \), \( [x]_- = \frac{1}{2}(x - |x|) \).

Property 2.3 (a) Any \( x \in \mathbb{R}^n \) satisfies \( |x| \succeq_{\mathcal{K}_n} x \).

(b) For any \( x, y \succeq_{\mathcal{K}_n} 0 \), if \( x \succeq_{\mathcal{K}_n} y \), then \( x^{1/2} \succeq_{\mathcal{K}_n} y^{1/2} \).

(c) For any \( x, y \in \mathbb{R}^n \), if \( x^2 \succeq_{\mathcal{K}_n} y^2 \), then \( |x| \succeq_{\mathcal{K}_n} |y| \).

(d) For any \( x \in \mathbb{R}^n \), \( x \succeq_{\mathcal{K}_n} 0 \iff (x, y) \succeq_{\mathcal{K}_n} 0 \), \( \forall y \succeq_{\mathcal{K}_n} 0 \).

(e) For any \( x \succeq_{\mathcal{K}_n} 0 \) and \( y \in \mathbb{R}^n \), \( x^2 \succeq_{\mathcal{K}_n} y^2 \implies x \succeq_{\mathcal{K}_n} y \).

In the following propositions, we study and explore more characterizations about spectral values, determinant and trace of \( x \) as well as the partial order \( \succeq_{\mathcal{K}_n} \). In fact, Prop. 2.1-2.4 are parallel results analogous to those associated with positive semidefinite cone, see [11]. Even though both \( \mathcal{K}_n \) and \( \mathcal{S}_n \) belong to self-dual cones and share similar properties, as we will see, the ideas for proving these results are quite different. One reason is that the Jordan product is not associative as mentioned earlier.

Proposition 2.1 For any \( x \succeq_{\mathcal{K}_n} 0 \) and \( y \succeq_{\mathcal{K}_n} 0 \), the following results hold.
(a) If $x \geq_K y$, then $\det(x) \geq \det(y)$, $\text{tr}(x) \geq \text{tr}(y)$.

(b) If $x \geq_K y$, then $\lambda_i(x) \geq \lambda_i(y)$, $\forall i = 1, 2$.

Proof. (a) From definition, we know that

$$\det(x) = x_1^2 - \|x_2\|^2, \quad \text{tr}(x) = 2x_1,$$
$$\det(y) = y_1^2 - \|y_2\|^2, \quad \text{tr}(y) = 2y_1.$$ 

Since $x - y = (x_1 - y_1, x_2 - y_2) \geq_K 0$, we have $\|x_2 - y_2\| \leq x_1 - y_1$. Thus, $x_1 \geq y_1$, and then $\text{tr}(x) \geq \text{tr}(y)$. Besides, the assumption on $x$ and $y$ gives

$$x_1 - y_1 \geq \|x_2 - y_2\| \geq \left| \|x_2\| - \|y_2\| \right|,$$  \hspace{1cm} (7)

which is equivalent to $x_1 - \|x_2\| \geq y_1 - \|y_2\| > 0$ and $x_1 + \|x_2\| \geq y_1 + \|y_2\| > 0$. Hence,

$$\det(x) = x_1^2 - \|x_2\|^2 = (x_1 + \|x_2\|)(x_1 - \|x_2\|) \geq (y_1 + \|y_2\|)(y_1 - \|y_2\|) = \det(y).$$

(b) From definition of spectral values, we know that

$$\lambda_1(x) = x_1 - \|x_2\|, \quad \lambda_2(x) = x_1 + \|x_2\| \quad \text{and} \quad \lambda_1(y) = y_1 - \|y_2\|, \quad \lambda_2(y) = y_1 + \|y_2\|.$$ 

Then, by the inequality (7) in the proof of part (a), the results follow immediately. \qed

**Proposition 2.2** For any $x \geq_K 0$ and $y \geq_K 0$, we have

(a) $\det(x + y) \geq \det(x) + \det(y)$.

(b) $\det(x \circ y) \leq \det(x) \cdot \det(y)$.

(c) $\det(\alpha x + (1 - \alpha)y) \geq \alpha^2 \det(x) + (1 - \alpha)^2 \det(y), \quad \forall 0 < \alpha < 1$.

(d) $\left(\det(e + x)\right)^{1/2} \geq 1 + \det(x)^{1/2}, \quad \forall x \geq_K 0$.

(e) $\det(e + x + y) \leq \det(e + x) \cdot \det(e + y)$.

Proof. (a) For any $x \geq_K 0$ and $y \geq_K 0$, we know $\|x_2\| \leq x_1$ and $\|y_2\| \leq y_1$, which implies

$$\left| \langle x_2, y_2 \rangle \right| \leq \|x_2\| \cdot \|y_2\| \leq x_1y_1.$$
Hence, we obtain
\[
\det(x + y) = (x_1 + y_1)^2 - \|x_2 + y_2\|^2 \\
= (x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) + 2(x_1y_1 - \langle x_2, y_2 \rangle) \\
\geq (x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) \\
= \det(x) + \det(y).
\]

(b) Applying the Cauchy inequality gives
\[
\det(x \circ y) = \langle x, y \rangle^2 - \|x_1y_2 + y_1x_2\|^2 \\
= \left( x_1y_1 + \langle x_2, y_2 \rangle \right)^2 - \left( x_1^2\|y_2\|^2 + 2x_1y_1\langle x_2, y_2 \rangle + y_1^2\|x_2\|^2 \right) \\
= x_1^2y_1^2 + \langle x_2, y_2 \rangle^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2 \\
\leq x_1^2y_1^2 + \|x_2\|^2\cdot\|y_2\|^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2 \\
= \left( x_1^2 - \|x_2\|^2 \right) \left( y_1^2 - \|y_2\|^2 \right) \\
= \det(x) \cdot \det(y).
\]

(c) For any \( x \succeq_{\kappa_0} 0 \) and \( y \succeq_{\kappa_0} 0 \), it is clear that \( \alpha x \succeq_{\kappa_0} 0 \) and \( (1 - \alpha)y \succeq_{\kappa_0} 0 \) for every \( \alpha > 0 \). In addition, we observe that \( \det(\alpha x) = \alpha^2\det(x) \), for all \( \alpha > 0 \). Hence,
\[
\det(\alpha x + (1 - \alpha)y) \geq \det(\alpha x) + \det((1 - \alpha)y) = \alpha^2\det(x) + (1 - \alpha)^2\det(y),
\]
where the inequality is from part (a).

(d) For any \( x \succeq_{\kappa_0} 0 \), we know \( \det(x) = \lambda_1\lambda_2 \geq 0 \), where \( \lambda_i \) are the spectral values of \( x \). Hence, \( \det(e + x) = (1 + \lambda_1)(1 + \lambda_2) \geq (1 + \sqrt{\lambda_1\lambda_2})^2 = (1 + \det(x)^{1/2})^2 \). Then, taking square root both sides yields the desired result.

(e) Again, For any \( x \succeq_{\kappa_0} 0 \) and \( y \succeq_{\kappa_0} 0 \), we have
\[
\left\{ \begin{array}{l}
x_1 - \|x_2\| \geq 0, \\
y_1 - \|y_2\| \geq 0, \\
|\langle x_2, y_2 \rangle| \leq \|x_2\| \cdot \|y_2\| \leq x_1y_1.
\end{array} \right.
\]
Also, we know \( \det(e + x + y) = (1 + x_1 + y_1)^2 - \|x_2 + y_2\|^2 \), \( \det(e + x) = (1 + x_1)^2 - \|x_2\|^2 \) and \( \det(e + y) = (1 + y_1)^2 - \|y_2\|^2 \). Hence,
\[
\det(e + x) \cdot \det(e + y) - \det(e + x + y) \\
= \left( (1 + x_1)^2 - \|x_2\|^2 \right) \left( (1 + y_1)^2 - \|y_2\|^2 \right) - \left( (1 + x_1 + y_1)^2 - \|x_2 + y_2\|^2 \right)
\]
\[2x_1y_1 + 2\langle x_2, y_2 \rangle + 2x_1y_1^2 + 2x_1^2y_1 - 2y_1\|x_2\|^2 - 2x_1\|y_2\|^2 + x_1^2y_1^2 - y_1^2\|x_2\|^2 - x_1^2\|y_2\|^2 + \|x_2\|^2 \cdot \|y_2\|^2 \geq 0,\]

where we multiply out all the expansions to obtain the second equality and the last inequality holds by (8). \[\square\]

**Proposition 2.3** For any \(x, y \in \mathbb{R}^n\), we have

(a) \(\text{tr}(x + y) = \text{tr}(x) + \text{tr}(y)\).

(b) \(\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) \leq \text{tr}(x \circ y) \leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)\).

(c) \(\text{tr}(\alpha x + (1 - \alpha)y) = \alpha \cdot \text{tr}(x) + (1 - \alpha) \cdot \text{tr}(y), \forall \alpha \in \mathbb{R}\).

**Proof.** Part (a) and (c) are trivial. Thus, it remains to verify (b). Using the fact that \(\text{tr}(x \circ y) = 2\langle x, y \rangle\), we obtain

\[
\begin{align*}
\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) &= (x_1 - \|x_2\|)(y_1 + \|y_2\|) + (x_1 + \|x_2\|)(y_1 - \|y_2\|) \\
&= 2(x_1y_1 - \|x_2\||y_2\|) \\
&\leq 2(x_1y_1 + \langle x_2, y_2 \rangle) \\
&= 2\langle x, y \rangle = \text{tr}(x \circ y) \\
&\leq 2(x_1y_1 + \|x_2\||y_2\|) \\
&= (x_1 - \|x_2\|)(y_1 - \|y_2\|) + (x_1 + \|x_2\|)(y_1 + \|y_2\|),
\end{align*}
\]

which completes the proof. \[\square\]

The following two lemmas are well known results in matrix analysis and are key to proving Prop. 2.4 which is an important extension about the function \(\ln \det(\cdot)\) from positive semidefinite cone to SOC.

**Lemma 2.1** For any nonzero vector \(x \in \mathbb{R}^n\), the matrix \(xx^T\) is positive semidefinite (p.s.d.) with only one nonzero eigenvalue \(\|x\|^2\).

**Proof.** The proof is routine, we omit it. \[\square\]

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Lemma 2.2 Suppose that a symmetric matrix is partitioned as \( \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \), where \( A \) and \( C \) are square. Then this matrix is positive definite (p.d.) if and only if \( A \) is positive definite and \( C \succ B^T A^{-1} B \).

\textbf{Proof.} This is Theorem 7.7.6 in [11]. \( \square \)

Proposition 2.4 For any \( x \succ k^n \) and \( y \succ k^n \), we have

(a) the real-valued function \( f(x) = \ln(\text{det}(x)) \) is concave on \( \text{int}(K^n) \).

(b) \( \text{det}(\alpha x + (1 - \alpha)y) \geq (\text{det}(x))^\alpha (\text{det}(y))^{1-\alpha} \), \( \forall 0 < \alpha < 1 \).

(c) the function real-valued \( f(x) = \ln(\text{det}(x^{-1})) \) is convex on \( \text{int}(K^n) \).

(d) the real-valued function \( f(x) = \text{tr}(x^{-1}) \) is convex on \( \text{int}(K^n) \).

\textbf{Proof.} (a) Since \( \text{int}(K^n) \) is a convex set, it is enough to show that \( \nabla^2 f(x) \) is negative semidefinite. From direct computation, we know

\[
\nabla f(x) = \left( \frac{2x_1}{x_1^2 - \|x_2\|^2}, \frac{-2x_2}{x_1^2 - \|x_2\|^2} \right) = 2x^{-1},
\]

and

\[
\nabla^2 f(x) = \begin{bmatrix}
\frac{-2x_1^2 - 2\|x_2\|^2}{(x_1^2 - \|x_2\|^2)^2} & \frac{4x_1x_2}{(x_1^2 - \|x_2\|^2)^2} \\
\frac{4x_1x_2}{(x_1^2 - \|x_2\|^2)^2} & \frac{-2(x_1^2 - \|x_2\|^2)I - 4x_2x_2^T}{(x_1^2 - \|x_2\|^2)^2}
\end{bmatrix}.
\]

Let \( \nabla^2 f(x) \) be denoted by the matrix \( \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \) given as in Lemma 2.2 (here \( A \) is a scalar). Then, we have

\[
AC - B^T B = (x_1^2 + \|x_2\|^2) \left( (x_1^2 - \|x_2\|^2)I + 2x_2x_2^T \right) - 4x_1^2x_2x_2^T
\]

\[
= (x_1^4 - \|x_2\|^4)I - 2(x_1^2 - \|x_2\|^2)x_2x_2^T
\]

\[
= (x_1^2 - \|x_2\|^2) \left( (x_1^2 + \|x_2\|^2)I - 2x_2x_2^T \right)
\]

\[
= (x_1^2 - \|x_2\|^2) \cdot M,
\]

where we denote the whole matrix in the big parenthesis of the last second equality by \( M \). From Lemma 2.1, we know that \( x_2x_2^T \) is a p.s.d. with only one nonzero eigenvalue \( \|x_2\|^2 \). Hence, all the eigenvalues of the matrix \( M \) are \( (x_1^2 + \|x_2\|^2) - 2\|x_2\|^2 = x_1^2 - \|x_2\|^2 \) and
From Lemma 2.1, we know that
where we denote the whole matrix in the big parenthesis of the last second equality by
a scalar). Then, we have
Again, let
(d) The idea for proving this is the same as the one for part (a). Since int(K^n) is a convex set, it is enough to show that \( \nabla^2 f(x) \) is positive semidefinite. Note that \( f(x) = \text{tr} (x^{-1}) = \frac{2x_1}{x_1^2 - \|x_2\|^2} \). Thus, from direct computations, we have
\[
\nabla^2 f(x) = \frac{2}{(x_1^2 - \|x_2\|^2)^3} \begin{bmatrix}
2x_1^3 + 6x_1\|x_2\|^2, & -(6x_1^2 + 2\|x_2\|^2)x_2^T \\
-(6x_1^2 + 2\|x_2\|^2)x_2, & 2x_1\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right)
\end{bmatrix}.
\]
Again, let \( \nabla^2 f(x) \) be denoted by the matrix \( \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \) given as in Lemma 2.2 (here \( A \) is a scalar). Then, we have
\[
AC - B^TB = 2x_1\left(2x_1^3 + 6x_1\|x_2\|^2\right)\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right) - \left(6x_1^2 + 2\|x_2\|^2\right)^2x_2x_2^T
\] 
\[
= \left(4x_1^4 + 12x_1^2\|x_2\|^2\right)\left(x_1^2 - \|x_2\|^2\right)I - \left(20x_1^4 - 24x_1^2\|x_2\|^2 + 4\|x_2\|^4\right)x_2x_2^T
\] 
\[
= \left(4x_1^4 + 12x_1^2\|x_2\|^2\right)\left(x_1^2 - \|x_2\|^2\right)I - 4\left(5x_1^2 - \|x_2\|^2\right)^2x_2x_2^T
\] 
\[
= \left(x_1^2 - \|x_2\|^2\right)\left(4x_1^4 + 12x_1^2\|x_2\|^2\right)I - 4\left(5x_1^2 - \|x_2\|^2\right)x_2x_2^T
\] 
\[
= \left(x_1^2 - \|x_2\|^2\right)\cdot M,
\]
where we denote the whole matrix in the big parenthesis of the last second equality by \( M \).
From Lemma 2.1, we know that \( x_2x_2^T \) is a p.s.d. with only one nonzero eigenvalue \( \|x_2\|^2 \).
Hence, all the eigenvalues of the matrix \( M \) are \( (4x_1^4 + 12x_1^2\|x_2\|^2) - 20x_1^4\|x_2\|^2 + 4\|x_2\|^4) \) and
4x_1^4 + 12x_1^2\|x_2\|^2 \text{ with multiplicity of } n - 2, \text{ which are all positive since }
4x_1^4 + 12x_1^2\|x_2\|^2 - 20x_1\|x_2\|^2 + 4\|x_2\|^4
= 4x_1^4 - 8x_1^2\|x_2\|^2 + 4\|x_2\|^4
= 4\left(x_1^2 - \|x_2\|^2\right)
> 0.

Thus, by Lemma 2.2, we obtain that $\nabla^2 f(x)$ is positive definite and hence is positive semidefinite. Therefore, $f$ is convex on $\text{int}(K^n)$.

3 \ SOC-convex function and SOC-monotone function

In this section, we define the SOC-convexity and SOC-monotonicity and study some examples of such functions.

Definition 3.1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Then
\begin{enumerate}[(a)]
\item $f$ is said to be SOC-monotone of order $n$ if the corresponding vector-valued function $f_{soc}$ satisfies the following:
\[ x \succeq_{K^n} y \implies f_{soc}(x) \succeq_{K^n} f_{soc}(y). \]
\item $f$ is said to be SOC-convex of order $n$ if the corresponding vector-valued function $f_{soc}$ satisfies the following:
\[ f_{soc}\left((1 - \lambda)x + \lambda y\right) \preceq_{K^n} (1 - \lambda)f_{soc}(x) + \lambda f_{soc}(y), \tag{9} \]
for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$.
\end{enumerate}

We say $f$ is SOC-monotone (respectively, SOC-convex) if $f$ is SOC-monotone of all order $n$ (respectively, SOC-convex of all order $n$). If $f$ is continuous, then the condition (9) can be replaced by the more special condition:
\[ f_{soc}\left(\frac{x + y}{2}\right) \preceq_{K^n} \frac{1}{2}\left(f_{soc}(x) + f_{soc}(y)\right). \tag{10} \]

It is clear that the set of SOC-monotone functions and the set of SOC-convex functions are both closed under positive linear combinations and under point-wise limits.

Proposition 3.1 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $f(t) = \alpha + \beta t$, then
(a) \( f \) is SOC-monotone on \( \mathbb{R} \) for every \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \).

(b) \( f \) is SOC-convex on \( \mathbb{R} \) for all \( \alpha, \beta \in \mathbb{R} \).

**Proof.** The proof is straightforward by checking that definition 3.1 is satisfied. \( \square \)

**Proposition 3.2 (a)** Let \( f : \mathbb{R} \to \mathbb{R} \) be \( f(t) = t^2 \), then \( f \) is SOC-convex on \( \mathbb{R} \).

(b) Hence, the function \( g(t) = \alpha + \beta t + \gamma t^2 \) is SOC-convex on \( \mathbb{R} \) for all \( \alpha, \beta \in \mathbb{R} \) and \( \gamma \geq 0 \).

**Proof.** (a) For any \( x, y \in \mathbb{R}^n \), we have
\[
\frac{1}{2} \left( f(x) + f(y) \right) - f \left( \frac{x + y}{2} \right) = \frac{x^2 + y^2}{2} - \left( \frac{x + y}{2} \right)^2 = \frac{1}{4} (x - y)^2 \succeq_{K^n} 0.
\]
Since \( f \) is continuous, the above implies that \( f \) is SOC-convex.

(b) It’s an immediate consequence of (a). \( \square \)

**Example 3.1** The function \( f(t) = t^2 \) is not SOC-monotone on \( \mathbb{R} \).

To see this, let \( x = (1,0), y = (-2,0) \), then \( x - y = (3,0) \succeq_{K^n} 0 \). But, \( x^2 - y^2 = (1,0) - (4,0) = (-3,0) \not\succeq_{K^n} 0 \). \( \square \)

It is clear that \( f(t) = t^2 \) is also SOC-convex on the smaller interval \([0, \infty)\) by Prop. 3.2(a). We may ask a natural question: Is \( f(t) = t^2 \) SOC-monotone on the interval \([0, \infty)\)? The answer is: it’s true only for \( n = 2 \), however, false for general \( n \geq 3 \). We show this in the next example.

**Example 3.2 (a)** The function \( f(t) = t^2 \) is SOC-monotone on \([0, \infty)\) for \( n = 2 \).

(b) However, \( f(t) = t^2 \) is not SOC-monotone on \([0, \infty)\) for \( n \geq 3 \).

(a) Let \( x = (x_1, x_2) \succeq_{K^2} y = (y_1, y_2) \succeq_{K^2} 0 \). Then we have the following inequalities:
\[
|x_2| \leq x_1, \quad |y_2| \leq y_1, \quad |x_2 - y_2| \leq x_1 - y_1,
\]
which implies
\[
\begin{aligned}
x_1 - x_2 & \geq y_1 - y_2 \geq 0, \\
x_1 + x_2 & \geq y_1 + y_2 \geq 0,
\end{aligned}
\]
(11)
We want to prove that $f(x) - f(y) = (x_1^2 + x_2^2 - y_1^2 - y_2^2, 2x_1x_2 - 2y_1y_2) \succeq_{K^2} 0$, which is enough to verify that $x_1^2 + x_2^2 - y_1^2 - y_2^2 \geq |2x_1x_2 - 2y_1y_2|$. This can be seen by

$$
\begin{align*}
x_1^2 + x_2^2 - y_1^2 - y_2^2 - |2x_1x_2 - 2y_1y_2| &= \begin{cases} 
    x_1^2 + x_2^2 - y_1^2 - y_2^2 - (2x_1x_2 - 2y_1y_2), & \text{if } x_1x_2 - y_1y_2 \geq 0 \\
    x_1^2 + x_2^2 - y_1^2 - y_2^2 - (2y_1y_2 - 2x_1x_2), & \text{if } x_1x_2 - y_1y_2 \leq 0 
\end{cases} \\
\geq 0,
\end{align*}
$$

where the inequalities are true due to the inequalities (11).

(b) For $n \geq 3$, we give a counterexample to show that $f(t) = t^2$ is not SOC-monotone on the interval $[0, \infty)$. Let $x = (3, 1, -2) \in K^3$ and $y = (1, 1, 0) \in K^3$. It is clear that $x - y = (2, 0, -2) \succeq_{K^3} 0$. But, $x^2 - y^2 = (14, 6, -12) - (2, 2, 0) = (12, 4, -12) \not\succeq_{K^3} 0$. □

Now we look at the function $f(t) = t^3$. As expected, $f(t) = t^3$ is not SOC-convex. However, it is true that $f(t) = t^3$ is SOC-convex on $[0, \infty)$ for $n = 2$, whereas false for $n \geq 3$. Besides, we will see $f(t) = t^3$ is neither SOC-monotone on $\mathbb{R}$ nor SOC-monotone on the interval $[0, \infty)$ in general. Nonetheless, it is true that it is SOC-monotone on the interval $[0, \infty)$, for $n = 2$. The following two examples show what we have just said.

**Example 3.3**

(a) The function $f(t) = t^3$ is not SOC-convex on $\mathbb{R}$.

(b) Moreover, $f(t) = t^3$ is not SOC-convex on $[0, \infty)$ for $n \geq 3$.

(c) However, $f(t) = t^3$ is SOC-convex on $[0, \infty)$ for $n = 2$.

To see (a), let $x = (0, -2), y = (1, 0)$. It can be verified that $\frac{1}{2} \left( f(x) + f(y) \right) - f \left( \frac{x+y}{2} \right) = \left( -\frac{9}{8}, -\frac{9}{4} \right) \not\succeq_{K^2} 0$, which says $f(t) = t^3$ is not SOC-convex on $\mathbb{R}$.

To see (b), let $x = (2, 1, -1), y = (1, 1, 0) \succeq_{K^3} 0$, then we have $\frac{1}{2} \left( f(x) + f(y) \right) - f \left( \frac{x+y}{2} \right) = (3, 1, -3) \not\succeq_{K^3} 0$, which implies $f(t) = t^3$ is not even SOC-convex on the interval $[0, \infty)$.

To see (c), it is enough to show that $f \left( \frac{x+y}{2} \right) \succeq_{K^2} \frac{1}{2} \left( f(x) + f(y) \right)$, for any $x, y \succeq_{K^2} 0$.

Let $x = (x_1, x_2) \succeq_{K^2} 0$ and $y = (y_1, y_2) \succeq_{K^2} 0$, then we have

$$
\begin{align*}
x^3 &= (x_1^3 + 3x_1x_2^2, 3x_1^2x_2 + x_2^3), \\
y^3 &= (y_1^3 + 3y_1y_2^2, 3y_1^2y_2 + y_2^3),
\end{align*}
$$

which yields

$$
\begin{align*}
f \left( \frac{x+y}{2} \right) &= \frac{1}{8} \left( (x_1 + y_1)^3 + 3(x_1 + y_1)(x_2 + y_2)^2, 3(x_1 + y_1)^2(x_2 + y_2) + (x_2 + y_2)^3 \right), \\
\frac{1}{2} \left( f(x) + f(y) \right) &= \frac{1}{2} \left( x_1^3 + y_1^3 + 3x_1x_2^2 + 3y_1y_2^2, x_2^3 + y_2^3 + 3x_1^2x_2 + 3y_1^2y_2 \right).
\end{align*}
$$
After simplifications, we denote \( \frac{1}{2} \left( f(x) + f(y) \right) - f\left( \frac{x+y}{2} \right) := \frac{1}{8} (\Xi_1, \Xi_2) \), where

\[
\begin{align*}
\Xi_1 &= 4x^3 + 4y^3 + 12x_1x_2^2 + 12x_1^2x_2 - (x_1 + y_1)^3 - 3(x_1 + y_1)(x_2 + y_2)^2, \\
\Xi_2 &= 4x^3 + 4y^3 + 12x_1^2x_2 + 12y_1^2y_2 - (x_2 + y_2)^3 - 3(x_1 + y_1)^2(x_2 + y_2).
\end{align*}
\]

We want to show that \( \Xi_1 \geq |\Xi_2| \) for which we discuss two cases. First, if \( \Xi_2 \geq 0 \), then

\[
\Xi_1 - |\Xi_2| = (4x^3 + 12x_1x_2^2 - 12x_1^2x_2 - 4x_2^3) + (4y^3 + 12y_1y_2^2 - 12y_1^2y_2 - 4y_2^3)
- \left( (x_1 + y_1)^3 + 3(x_1 + y_1)(x_2 + y_2)^2 - 3(x_1 + y_1)^2(x_2 + y_2) - (x_2 + y_2)^3 \right)
= 4(x_1 - x_2)^3 + 4(y_1 - y_2)^3 - \left( (x_1 + y_1) - (x_2 + y_2) \right)^3
= 4(x_1 - x_2)^3 + 4(y_1 - y_2)^3 - \left( (x_1 - x_2) + (y_1 - y_2) \right)^3
= 3(x_1 - x_2)^3 + 3(y_1 - y_2)^3 - 3(x_1 - x_2) (y_1 - y_2) - 3(x_1 - x_2)(y_1 - y_2)^2
= 3 \left( (x_1 - x_2) + (y_1 - y_2) \right) \left( (x_1 - x_2)^2 - (x_1 - x_2) (y_1 - y_2) + (y_1 - y_2)^2 \right)
- 3(x_1 - x_2)(y_1 - y_2) \left( (x_1 - x_2) + (y_1 - y_2) \right)
= 3 \left( (x_1 - x_2) + (y_1 - y_2) \right) \left( (x_1 - x_2) - (y_1 - y_2) \right)^2
\geq 0,
\]

where the inequality is true since \( x, y \in \mathbb{K}^2 \). Similarly, if \( \Xi_2 \leq 0 \), we also have,

\[
\Xi_1 - |\Xi_2| = (4x^3 + 12x_1x_2^2 + 12x_1^2x_2 + 4x_2^3) + (4y^3 + 12y_1y_2^2 + 12y_1^2y_2 + 4y_2^3)
- \left( (x_1 + y_1)^3 + 3(x_1 + y_1)(x_2 + y_2)^2 + 3(x_1 + y_1)^2(x_2 + y_2) + (x_2 + y_2)^3 \right)
= 4(x_1 + x_2)^3 + 4(y_1 + y_2)^3 - \left( (x_1 + y_1) + (x_2 + y_2) \right)^3
= 4(x_1 + x_2)^3 + 4(y_1 + y_2)^3 - \left( (x_1 + x_2) + (y_1 + y_2) \right)^3
= 3(x_1 + x_2)^3 + 3(y_1 + y_2)^3 - 3(x_1 + x_2) (y_1 + y_2) - 3(x_1 + x_2)(y_1 + y_2)^2
= 3 \left( (x_1 + x_2) + (y_1 + y_2) \right) \left( (x_1 + x_2)^2 - (x_1 + x_2) (y_1 + y_2) + (y_1 + y_2)^2 \right)
- 3(x_1 + x_2)(y_1 + y_2) \left( (x_1 + x_2) + (y_1 + y_2) \right)
= 3 \left( (x_1 + x_2) + (y_1 + y_2) \right) \left( (x_1 + x_2) - (y_1 + y_2) \right)^2
\geq 0,
\]

where the inequality is true since \( x, y \in \mathbb{K}^2 \). Thus, we have verified that \( f(t) = t^3 \) is SOC-convex on \([0, \infty)\) for \( n = 2 \). \( \square \)
Example 3.4 (a) The function \( f(t) = t^3 \) is not SOC-monotone on \( \mathbb{R} \).

(b) Moreover, \( f(t) = t^3 \) is not SOC-monotone on \([0, \infty)\) for \( n \geq 3 \).

(c) However, \( f(t) = t^3 \) is SOC-monotone on \([0, \infty)\) for \( n = 2 \).

To see (a) and (b), let \( x = (2, 1, -1) \succeq_{K^3} 0 \) and \( y = (1, 1, 0) \succeq_{K^3} 0 \). It is clear that \( x \succeq_{K^3} y \). But, we have \( f(x) = x^3 = (20, 14, -14) \) and \( f(y) = y^3 = (4, 4, 0) \), which gives \( f(x) - f(y) = (16, 10, -14) \nless_{K^3} 0 \). Thus, we show that \( f(t) = t^3 \) is not even SOC-monotone on the interval \([0, \infty)\).

To see (c), let \( x = (x_1, x_2) \succeq_{K^2} y = (y_1, y_2) \succeq_{K^2} 0 \). Again, we have the following inequalities:

\[
|x_2| \leq x_1, \quad |y_2| \leq y_1, \quad |x_2 - y_2| \leq x_1 - y_1,
\]

which leads to the inequalities (11). In addition, we know

\[
\begin{align*}
f(x) &= x^3 = \left( x_1^3 + 3x_1x_2^2, 3x_1^2x_2 + x_2^3 \right), \\
f(y) &= y^3 = \left( y_1^3 + 3y_1y_2^2, 3y_1^2y_2 + y_2^3 \right).
\end{align*}
\]

We denote \( f(x) - f(y) := (\Xi_1, \Xi_2) \), where

\[
\begin{align*}
\Xi_1 &= x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2, \\
\Xi_2 &= x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2.
\end{align*}
\]

We wish to prove that \( f(x) - f(y) = x^3 - y^3 \succeq_{K^2} 0 \), which is enough to show \( \Xi_1 \geq |\Xi_2| \).

This is true because

\[
\begin{align*}
&x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2 \\
= &\begin{cases} 
  x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2), & \text{if } \Xi_2 \geq 0 \\
  x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 + (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2), & \text{if } \Xi_2 \leq 0 
\end{cases} \\
= &\begin{cases} 
  (x_1 - x_2)^3 - (y_1 - y_2)^3, & \text{if } \Xi_2 \geq 0 \\
  (x_1 + x_2)^3 - (y_1 + y_2)^3, & \text{if } \Xi_2 \leq 0 
\end{cases} \\
\geq & 0,
\end{align*}
\]

where the inequalities are true by the inequalities (11).

Hence, we complete the verification. \( \Box \)

Now, we move to another simple function \( f(t) = 1/t \). We will prove that \(-1/t\) is SOC-monotone on the interval \((0, \infty)\) and \(1/t\) is SOC-convex on the interval \((0, \infty)\) as well. For the proof, we need the following technical lemmas.
Lemma 3.1 For any \(a \geq b > 0\) and \(c \geq d > 0\), we always have
\[
\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) \geq \frac{a + c}{b + d} \tag{12}
\]

Proof. The proof is followed by \(ac(b + d) - bd(a + c) = ab(c - d) + cd(a - b) \geq 0\). \(\square\)

Lemma 3.2 For any \(x = (x_1, x_2), y = (y_1, y_2) \in K^n\), we have

(a) \((x_1 + y_1)^2 - \|y_2\|^2 \geq 4x_1\sqrt{y_1^2 - \|y_2\|^2}\).

(b) \(\left(x_1 + y_1 - \|y_2\|\right)^2 \geq 4x_1\left(y_1 - \|y_2\|\right)\).

(c) \(\left(x_1 + y_1 + \|y_2\|\right)^2 \geq 4x_1\left(y_1 + \|y_2\|\right)\).

(d) \(x_1y_1 - \langle x_2, y_2 \rangle \geq \sqrt{x_1^2 - \|x_2\|^2} \cdot \sqrt{y_1^2 - \|y_2\|^2}\).

(e) \((x_1 + y_1)^2 - \|x_2 + y_2\|^2 \geq 4\sqrt{x_1^2 - \|x_2\|^2} \cdot \sqrt{y_1^2 - \|y_2\|^2}\).

Proof. (a) The proof follows from
\[
(x_1 + y_1)^2 - \|y_2\|^2 = x_1^2 + (y_1^2 - \|y_2\|^2) + 2x_1y_1 \\
\geq 2x_1\sqrt{y_1^2 - \|y_2\|^2} + 2x_1y_1 \\
\geq 2x_1\sqrt{y_1^2 - \|y_2\|^2} + 2x_1\sqrt{y_1^2 - \|y_2\|^2} \\
= 4x_1\sqrt{y_1^2 - \|y_2\|^2},
\]
where the first inequality is true due to the fact that \(a + b \geq 2\sqrt{ab}\) for any positive numbers \(a\) and \(b\).

(b) The proof follows from
\[
\left(x_1 + y_1 - \|y_2\|\right)^2 - 4x_1\left(y_1 - \|y_2\|\right) \\
= x_1^2 + y_1^2 + \|y_2\|^2 - 2x_1y_1 - 2y_1\|y_2\| + 2x_1\|y_2\| \\
= \left(x_1 - y_1 + \|y_2\|\right)^2 \geq 0.
\]

(c) Similarly, the proof follows from
\[
\left(x_1 + y_1 + \|y_2\|\right)^2 - 4x_1\left(y_1 + \|y_2\|\right) \\
= x_1^2 + y_1^2 + \|y_2\|^2 - 2x_1y_1 + 2y_1\|y_2\| - 2x_1\|y_2\| \\
= \left(x_1 - y_1 - \|y_2\|\right)^2 \geq 0.
\]

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(d) We know that \( x_1 y_1 - \langle x_2, y_2 \rangle \geq x_1 y_1 - \|x_2\| \cdot \|y_2\| \geq 0, \) and

\[
\begin{align*}
(x_1 y_1 - \|x_2\| \cdot \|y_2\|)^2 - (x_1^2 - \|x_2\|^2)(x_1 y_1 - \|x_2\|)^2
&= x_2^2 \|y_2\|^2 + y_2^2 \|x_2\|^2 - 2 x_1 y_1 \|x_2\| \cdot \|y_2\| \\
&= \left( x_1 \|y_2\| - y_1 \|x_2\| \right)^2 \geq 0.
\end{align*}
\]

Hence, we obtain \( x_1 y_1 - \langle x_2, y_2 \rangle \geq x_1 y_1 - \|x_2\| \cdot \|y_2\| \geq \sqrt{x_1^2 - \|x_2\|^2} \cdot \sqrt{y_1^2 - \|y_2\|^2}, \)

(e) The proof follows from

\[
\begin{align*}
(x_1 + y_1)^2 - \|x_2 + y_2\|^2
&= \left( x_1^2 - \|x_2\|^2 \right) + \left( y_1^2 - \|y_2\|^2 \right) + 2 \left( x_1 y_1 - \langle x_2, y_2 \rangle \right) \\
&\geq 2 \sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} + 2 \left( x_1 y_1 - \langle x_2, y_2 \rangle \right) \\
&\geq 2 \sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} + 2 \sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} \\
&= 4 \sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)},
\end{align*}
\]

where the first inequality is true since \( a + b \geq 2 \sqrt{ab} \) for all positive \( a, b. \) and the second inequality is from part (d). \( \square \)

**Proposition 3.3** Let \( f : (0, \infty) \rightarrow (0, \infty) \) be \( f(t) = 1/t. \) Then

(a) \(-f\) is SOC-monotone on \((0, \infty).\)

(b) \( f \) is SOC-convex on \((0, \infty).\)

**Proof.** (a) It suffices to show that \( x \succeq_{\mathbb{K}^n} y \succeq_{\mathbb{K}^n} 0 \) implies \( x^{-1} \succeq_{\mathbb{K}^n} y^{-1}. \) For any \( x, y \in \mathbb{K}^n, \) we know that \( y^{-1} = \frac{1}{\det(y)}(y_1, -y_2), x^{-1} = \frac{1}{\det(x)}(x_1, -x_2). \) Thus,

\[
\begin{align*}
y^{-1} - x^{-1} &= \left( \frac{y_1}{\det(y)} x_1 - \frac{x_1}{\det(x)} y_1 - \frac{x_2}{\det(x)} \right) \\
&= \frac{1}{\det(x) \det(y)} \left( \det(x) y_1 - \det(y) x_1, \det(y) x_2 - \det(x) y_2 \right).
\end{align*}
\]

To complete the proof, we need to verify two things.

(1) First, we have to show that \( \det(x) y_1 - \det(y) x_1 \geq 0. \) Applying Lemma 3.1 yields

\[
\frac{\det(x)}{\det(y)} = \frac{x_1^2 - \|x_2\|^2}{y_1^2 - \|y_2\|^2} = \left( \frac{x_1 + \|x_2\|}{y_1 + \|y_2\|} \right) \left( \frac{x_1 - \|x_2\|}{y_1 - \|y_2\|} \right) \geq \frac{2 x_1}{2 y_1} = \frac{x_1}{y_1}.
\]

Then cross multiplying gives \( \det(x) y_1 \geq \det(y) x_1, \) i.e., \( \det(x) y_1 - \det(y) x_1 \geq 0. \)
(2) Secondly, we show that \( \|\det(y)x_2 - \det(x)y_2\| \leq \det(x)y_1 - \det(y)x_1 \). This is true by

\[
\left( \det(x)y_1 - \det(y)x_1 \right)^2 - \|\det(y)x_2 - \det(x)y_2\|^2
\]

\[
= (\det(x))^2 y_1^2 - 2\det(x)\det(y)x_1 y_1 + (\det(y))^2 x_1^2
\]

\[- \left( (\det(y))^2 \|x_2\|^2 - 2\det(x)\det(y)x_2 y_2 + (\det(x))^2 \|y_2\|^2 \right)
\]

\[
= (\det(x))^2 (y_1^2 - \|y_2\|^2) + (\det(y))^2 (x_1^2 - \|x_2\|^2)
\]

\[- 2\det(x)\det(y)(x_1 y_1 - \langle x_2, y_2 \rangle)
\]

\[
= (\det(x))^2 \det(y) + (\det(y))^2 \det(x) - 2\det(x)\det(y)(x_1 y_1 - \langle x_2, y_2 \rangle)
\]

\[
= \det(x)\det(y) \left( \det(x) + \det(y) - 2x_1 y_1 + 2\langle x_2, y_2 \rangle \right)
\]

\[
= \det(x)\det(y) \left( (x_1 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) - 2x_1 y_1 + 2\langle x_2, y_2 \rangle \right)
\]

\[
= \det(x)\det(y) \left( (x_1 - y_1)^2 - (\|x_2\|^2 + \|y_2\|^2 - 2\langle x_2, y_2 \rangle) \right)
\]

\[
= \det(x)\det(y) \left( (x_1 - y_1)^2 - (\|x_2 - y_2\|^2) \right)
\]

\[
\geq 0,
\]

where the last step holds by the inequality (8) given as in the proof of Prop. 2.1(a).

Thus, from all the above, we proved \( y^{-1} - x^{-1} \in \mathcal{K}_n \), i.e., \( y^{-1} \succeq_x x^{-1} \).

(b) For any \( x \succeq_x 0 \) and \( y \succeq_x 0 \), we have

\[
\begin{align*}
\mathcal{F}_x = \left\{ \begin{array}{l}
x_1 - \|x_2\| > 0 \\
y_1 - \|y_2\| > 0 \\
\langle x_2, y_2 \rangle \leq \|x_2\| \cdot \|y_2\| \leq x_1 y_1
\end{array} \right.
\end{align*}
\]

(13)

From \( x^{-1} = \frac{1}{\det(x)}(x_1, -x_2) \) and \( y^{-1} = \frac{1}{\det(y)}(y_1, -y_2) \), we also have

\[
\frac{1}{2} \left( f(x) + f(y) \right) = \frac{1}{2} \left( \frac{x_1}{\det(x)} + \frac{y_1}{\det(y)} , - \frac{x_2}{\det(x)} - \frac{y_2}{\det(y)} \right),
\]

and

\[
f \left( \frac{x + y}{2} \right) = \left( \frac{x + y}{2} \right)^{-1} = \frac{2}{\det(x + y)} \left( x_1 + y_1 , -(x_2 + y_2) \right).
\]

We denote \( \frac{1}{2} \left( f(x) + f(y) \right) - f \left( \frac{x + y}{2} \right) = \frac{1}{2}(\Xi_1, \Xi_2) \), where \( \Xi_1 \in \mathbb{R} \) and \( \Xi_2 \in \mathbb{R}^{n-1} \) are given by

\[
\begin{align*}
\Xi_1 &= \left( \frac{x_1}{\det(x)} + \frac{y_1}{\det(y)} \right) - \frac{4(x_1 + y_1)}{\det(x + y)},
\Xi_2 &= \frac{4(x_2 + y_2)}{\det(x + y)} - \left( \frac{x_2}{\det(x)} + \frac{y_2}{\det(y)} \right).
\end{align*}
\]

Again, to prove \( f \) is SOC-convex, it suffices to verify two things: \( \Xi_1 \geq 0 \) and \( \|\Xi_2\| \leq \Xi_1 \).
(1) First, we verify that $\Xi_1 \geq 0$. In fact, if we define the function

$$ g(x) := \frac{x_1}{x_1^2 - \|x_2\|^2} = \frac{x_1}{det(x)} , $$

then we observe that

$$ g\left(\frac{x + y}{2}\right) \leq \frac{1}{2} \left( g(x) + g(y) \right) \iff \Xi_1 \geq 0. $$

Hence, to prove $\Xi_1 \geq 0$, it is equivalent to show $g$ is convex on $\text{int}(\mathcal{K}^n)$. Since $\text{int}(\mathcal{K}^n)$ is a convex set, it is sufficient to show that $\nabla^2 g(x)$ is a positive semidefinite matrix. From direct computations, we have

$$ \nabla^2 g(x) = \frac{1}{(x_1^2 - \|x_2\|^2)^3} \begin{bmatrix} 2x_1^3 + 6x_1\|x_2\|^2 & -(6x_1^2 + 2\|x_2\|^2)x_2^T \\ -(6x_1^2 + 2\|x_2\|^2)x_2 & 2x_1\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right) \end{bmatrix}. $$

Let $\nabla^2 g(x)$ be denoted by the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ given as in Lemma 2.2 (here $A$ is a scalar). Then, we have

$$ AC - B^TB = 2x_1\left(2x_1^3 + 6x_1\|x_2\|^2\right)\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right) - \left(6x_1^2 + 2\|x_2\|^2\right)x_2x_2^T $$
$$ = (4x_1^4 + 12x_1^2\|x_2\|^2)\left(x_1^2 - \|x_2\|^2\right)I - \left(20x_1^3 - 24x_1^2\|x_2\|^2 + 4\|x_2\|^4\right)x_2x_2^T $$
$$ = (4x_1^4 + 12x_1^2\|x_2\|^2)\left(x_1^2 - \|x_2\|^2\right)I - 4\left(5x_1^2 - \|x_2\|^2\right)x_2x_2^T $$
$$ = \left(x_1^2 - \|x_2\|^2\right)\left((4x_1^4 + 12x_1^2\|x_2\|^2)I - 4\left(5x_1^2 - \|x_2\|^2\right)x_2x_2^T\right) $$
$$ = \left(x_1^2 - \|x_2\|^2\right) \cdot M, $$

where we denote the whole matrix in the big parenthesis of the last second equality by $M$. From Lemma 2.1, we know that $x_2x_2^T$ is p.s.d. with only one nonzero eigenvalue $\|x_2\|^2$. Hence, all the eigenvalues of the matrix $M$ are $(4x_1^4 + 12x_1^2\|x_2\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4)$ and $4x_1^4 + 12x_1^2\|x_2\|^2$ with multiplicity of $n - 2$, which are all positive since

$$ 4x_1^4 + 12x_1^2\|x_2\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4 $$
$$ = 4x_1^4 - 8x_1^2\|x_2\|^2 + 4\|x_2\|^4 $$
$$ = 4\left(x_1^2 - \|x_2\|^2\right) $$
$$ > 0. $$

Thus, by Lemma 2.2, we obtain that $\nabla^2 g(x)$ is positive definite and hence is positive semidefinite. It follows $g$ is convex on $\text{int}(\mathcal{K}^n)$ which says $\Xi_1 \geq 0$. 

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(2) It remains to show that \( \Xi_1^2 - \|\Xi_2\|^2 \geq 0 \):

\[
\Xi_1^2 - \|\Xi_2\|^2 = \left[ \left( \frac{x_1^2}{\text{det}(x)^2} + \frac{2x_1y_1}{\text{det}(x)\text{det}(y)} + \frac{y_1^2}{\text{det}(y)^2} \right) - \frac{8(x_1 + y_1)}{\text{det}(x+y)} \left( \frac{x_1}{\text{det}(x)} + \frac{y_1}{\text{det}(y)} \right) \right] + \left[ \left( \frac{x_2^2}{\text{det}(x)^2} + \frac{2x_1y_1}{\text{det}(x)\text{det}(y)} + \frac{y_2^2}{\text{det}(y)^2} \right) - \frac{8(x_1 + y_1)}{\text{det}(x+y)} \left( \frac{x_2}{\text{det}(x)} + \frac{y_2}{\text{det}(y)} \right) \right] + \frac{16}{\text{det}(x+y)^2} \left[ x_2^2 - \|x_2\|^2 + 2\langle x_2, y_2 \rangle + (y_2^2 - \|y_2\|^2) \right] - 8 \left( \frac{y_2^2}{\text{det}(x+y)^2} - \frac{y_1^2}{\text{det}(x+y)^2} \right) \left( \frac{d\text{det}(x+y)}{\text{det}(x+y)^2} \right) \left( \frac{x_2}{\text{det}(x)} + \frac{y_2}{\text{det}(y)} \right)
\]

Now apply the fact that \( \text{det}(x) = x_1^2 - \|x_2\|^2 \), \( \text{det}(y) = y_1^2 - \|y_2\|^2 \), and \( \text{det}(x+y) = 2(x_1y_1 - \langle x_2, y_2 \rangle) \), we can simplify the last equality (after a lot of algebra simplifications) and obtain

\[
\Xi_1^2 - \|\Xi_2\|^2 = \left[ \frac{\text{det}(x+y) - 2\text{det}(x) - 2\text{det}(y)}{\text{det}(x)\text{det}(y)\text{det}(x+y)} \right]^2 \geq 0.
\]

Hence, we proved that \( f \left( \frac{x+y}{2} \right) \leq \frac{1}{2} \left( f(x) + f(y) \right) \) which says the function \( f(t) = \frac{1}{t} \) is SOC-convex on the interval \((0, \infty)\). \( \square \)
Proposition 3.4 (a) The function $f(t) = \frac{t}{1+t}$ is SOC-monotone on $(0, \infty)$.

(b) For any $\lambda > 0$, the function $f(t) = \frac{t}{\lambda + t}$ is SOC-monotone on $(0, \infty)$.

Proof. (a) Let $g(t) = -\frac{1}{t}$ and $h(t) = 1 + t$, then both functions are SOC-monotone from Prop. 3.2 and 3.3. Since $f(t) = 1 - \frac{1}{1 + t} = h(g(1 + t))$, the result follows from that the composition of two SOC-monotone functions is also SOC-monotone.

(b) Similarly, let $g(t) = \frac{t}{1 + t}$ and $h(t) = t / \lambda$, then both functions are SOC-monotone by part(a). Since $f(t) = g(h(t))$, the result is true by the same reason as in part(a).

Proposition 3.5 For any $x \succ_{Kn} 0$ and $y \succ_{Kn} 0$, we have

$$L_x \succ_{Kn} L_y \iff L_y^{-1} \succeq_{Kn} L_x^{-1} \iff L_y^{-1} \succeq_{Kn} L_{x^{-1}}$$

Proof. By the property of $L_x$ that $x \succeq_{Kn} y \iff L_x \succeq_{Kn} L_y$, and Prop. 3.3(a), then proof follows.

Next, we examine another simple function $f(t) = \sqrt{t}$. We will see that it is SOC-monotone on the interval $[0, \infty)$, and $-\sqrt{t}$ is SOC-convex on $[0, \infty)$.

Proposition 3.6 Let $f : [0, \infty) \to [0, \infty)$ be $f(t) = t^{1/2}$. Then

(a) $f$ is SOC-monotone on $[0, \infty)$.

(b) $-f$ is SOC-convex on $[0, \infty)$.

Proof. (a) This is a consequence of Property 2.3(b).

(b) To show $-f$ is SOC-convex, it is enough to prove that $f\left(\frac{x+y}{2}\right) \succeq_{Kn} \frac{f(x) + f(y)}{2}$, which is equivalent to verify that $\left(\frac{x+y}{2}\right)^{1/2} \succeq_{Kn} \frac{\sqrt{x} + \sqrt{y}}{2}$, $\forall x, y \in K^n$. Since $x+y \succeq_{Kn} 0$, by Property 2.3(c), it is sufficient to show that $\left(\frac{x+y}{2}\right) \succeq_{Kn} \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2$. This can be seen by $\left(\frac{x+y}{2}\right) - \left(\frac{\sqrt{x} + \sqrt{y}}{2}\right)^2 = \frac{(\sqrt{x} - \sqrt{y})^2}{4} \succeq_{Kn} 0$. Thus, we complete the proof.

Proposition 3.7 Let $f : [0, \infty) \to [0, \infty)$ be $f(t) = t^r$, $0 \leq r \leq 1$. Then

(a) $f$ is SOC-monotone on $[0, \infty)$.
(b) \( f \) is SOC-convex on \([0, \infty)\).

**Proof.** (a) Let \( r \) be a dyadic rational, i.e., a number of the form \( r = \frac{m}{2^n} \), where \( n \) is any positive integer and \( 1 \leq m \leq 2^n \). It is enough to prove the assertion is true for such \( r \) since such \( r \) are dense in \([0, 1]\). We will show this by induction on \( n \). Let \( x, y \in K^n \) with \( x \succeq_{K^n} y \), then by Property 2.3(b) we have \( x^{1/2} \succeq_{K^n} y^{1/2} \). Therefore, part (a) is true when \( n = 1 \). Suppose it is also true for all dyadic rational \( x = \frac{m}{2^n} \), in which \( 1 \leq j \leq n - 1 \). Now let \( r = \frac{m}{2^n} \) with \( m \leq 2^n \). By induction hypothesis, we know \( x^{\frac{m}{2^n}} \succeq_{K^n} y^{\frac{m}{2^n}} \). Then, by applying Property 2.3(b), we obtain \( (x^{\frac{m}{2^n}})^{1/2} \succeq_{K^n} (y^{\frac{m}{2^n}})^{1/2} \), which says \( x^{\frac{m}{2^n}} \succeq_{K^n} y^{\frac{m}{2^n}} \). Thus, we have shown that \( x \succeq_{K^n} y \succeq_{K^n} 0 \) implies \( x^r \succeq_{K^n} y^r \), for all dyadic rational \( r \) in \([0, 1] \). Such \( r \) are dense in \([0, 1]\), which says part (a) is true.

(b) The proof for part (b) is very similar to the above arguments. First, we observe that

\[
\left( \frac{x + y}{2} \right)^{\frac{m}{2^n}} - \left( \frac{x^{\frac{m}{2^n}} + y^{\frac{m}{2^n}}}{2} \right)^2 \succeq_{K^n} \left( \frac{x^{\frac{m}{2^{n-1}}} + y^{\frac{m}{2^{n-1}}}}{2} \right)^2 - \left( \frac{x^{\frac{m}{2^n}} - y^{\frac{m}{2^n}}}{2} \right)^2 \succeq_{K^n} 0,
\]

which implies \( \left( \frac{x + y}{2} \right)^{\frac{m}{2^n}} \succeq_{K^n} \left( \frac{x^{\frac{m}{2^{n-1}}} + y^{\frac{m}{2^{n-1}}}}{2} \right) \) by Property 2.3(b). Hence, we show that the assertion is true when \( n = 1 \). By induction hypothesis, suppose \( \left( \frac{x + y}{2} \right)^{\frac{m}{2^n}} \succeq_{K^n} \left( \frac{x^{\frac{m}{2^{n-1}}} + y^{\frac{m}{2^{n-1}}}}{2} \right) \). Then we have

\[
\left( \frac{x + y}{2} \right)^{\frac{m}{2^n}} - \left( \frac{x^{\frac{m}{2^n}} + y^{\frac{m}{2^n}}}{2} \right)^2 \succeq_{K^n} 0,
\]

which implies \( \left( \frac{x + y}{2} \right)^{\frac{m}{2^n}} \succeq_{K^n} \left( \frac{x^{\frac{m}{2^{n-1}}} + y^{\frac{m}{2^{n-1}}}}{2} \right) \) by Property 2.3(b). Following the same arguments about dyadic rational in part (a) yields the desired result. \( \square \)

From all the above examples, we know that \( f \) being monotone does not imply \( f^{soc} \) is SOC-monotone. Similarly, \( f \) being convex does not imply \( f^{soc} \) is SOC-convex. Now, we move onto some famous functions which are used very often for NCP (nonlinear complementarity problem), SDCP, and SOCCP. It would be interesting to know about the SOC-convexity and SOC-monotonicity of these functions. First, we will look at the Fischer-Burmeister function, \( \phi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \), given by

\[
\phi(x, y) = (x^2 + y^2)^{1/2} - (x + y), \quad \text{(14)}
\]

which is a well known merit function for complementarity problem, see [13, 18]. For SOCCP, it has been shown that squared norm of \( \phi \), i.e.,

\[
\psi(x, y) = \|\phi(x, y)\|^2; \quad \text{(15)}
\]

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is continuously differentiable (see [7]) whereas \( \psi \) is only shown differentiable for SDCP (see [21]). In addition, \( \phi \) is proved to have semismoothness and Lipschitz continuity in the recent paper [20] for both cases of SOCCP and SDCP. In NCP, \( \phi \) is a convex function, so we may wish to have an analogy for SOCCP. Unfortunately, as shown below, it is not a SOC-convex function.

**Example 3.5** Let \( \phi \) be defined as in (14) and \( \psi \) defined as in (15).

(a) The function \( \rho(x, y) = (x^2 + y^2)^{1/2} \) is not SOC-convex.

(b) The Fischer-Burmeister function \( \phi \) is not SOC-convex.

(c) The function \( \psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) is not convex.

To see (a), a counterexample occurs when \( x = (1, 1) \) and \( y = (1, 0) \).

To see (b), suppose it is SOC-convex. Then we will have \( \rho \) is SOC-convex by \( \rho(x, y) = \phi(x, y) + (x + y) \), which contradicts to (a). Thus, \( \phi \) is not SOC-convex.

To see (c), let \( x = (1, -2) \), \( y = (1, -1) \) and \( u = (0, -1) \), \( v = (1, -1) \). Then we have

\[
\phi(x, y) = \left( \frac{-3 + \sqrt{13}}{2}, \frac{7 - \sqrt{13}}{2} \right) \implies \psi(x, y) = \|\phi(x, y)\|^2 = 21 - 5\sqrt{13}.
\]

\[
\phi(u, v) = \left( \frac{-1 + \sqrt{5}}{2}, \frac{5 - \sqrt{5}}{2} \right) \implies \psi(u, v) = \|\phi(u, v)\|^2 = 9 - 3\sqrt{5}.
\]

Thus, \( \frac{1}{2}(\psi(x, y) + \psi(u, v)) = \frac{1}{2}(30 - 5\sqrt{13} - 3\sqrt{5}) \approx 2.632. \)

On the other hand, let \( (a, b) := \frac{1}{2}(x, y) + \frac{1}{2}(u, v) \), that is, \( a = \left( \frac{1}{2}, -\frac{3}{2} \right) \) and \( b = (1, -1) \). Indeed, we have \( a^2 + b^2 = \left( \frac{9}{2}, -\frac{7}{2} \right) \) and hence \( (a^2 + b^2)^{1/2} = \left( \frac{1+2\sqrt{2}}{2}, \frac{1-2\sqrt{2}}{2} \right) \), which implies \( \psi(a, b) = \|\phi(a, b)\|^2 = 14 - 8\sqrt{2} \approx 2.686. \) Therefore, we obtain

\[
\psi\left( \frac{1}{2}(x, y) + \frac{1}{2}(u, v) \right) > \frac{1}{2}\psi(x, y) + \frac{1}{2}\psi(u, v),
\]

which shows \( \psi \) is not convex. \( \square \)

Another function based on the Fischer-Burmeister function is \( \psi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), given by

\[
\psi_1(x, y) := \|\phi(x, y)\|^2,
\]

where \( \phi \) is the Fischer-Burmeister function given as in (14). In the NCP case, it is known that \( \psi_1 \) is convex. It has been an open question whether this is still true for SDCP and SOCCP (see Q3 on page 182 of [21]). In fact, Qi and Chen [16] gave the negative answer for the SDCP case. Here we provide an answer to the question for SOCCP: \( \psi_1 \) is not convex in the SOCCP case.
**Example 3.6** Let $\phi$ be defined as in (14) and $\psi_1$ defined as in (16).

(a) The function $[\phi(x, y)]_+ = [(x^2 + y^2)^{1/2} - (x + y)]_+$ is not SOC-convex.

(b) The function $\psi_1$ is not convex.

To see (a), let $x = (2, 1, -1), y = (1, 1, 0)$ and $u = (1, -2, 5), v = (-1, 5, 0)$. Also, we denote $\phi_1(x, y) := [\phi(x, y)]_+$. Then, by direct computations, we obtain

$$\frac{1}{2}\phi_1(x, y) + \frac{1}{2}\phi_1(u, v) - \phi_1\left(\frac{1}{2}(x, y) + \frac{1}{2}(u, v)\right) = \left(1.0794, 0.4071, -1.0563\right) \not\succeq_{K^3} 0,$$

which says $\phi_1$ is not SOC-convex.

To see (b), let $x = (17, 5, 16), y = (20, -3, 15)$ and $u = (2, 3, 3), v = (9, -7, 2)$. It can be easily verified that $\frac{1}{2}\psi_1(x, y) + \frac{1}{2}\psi_1(u, v) - \psi_1\left(\frac{1}{2}(x, y) + \frac{1}{2}(u, v)\right) < 0$, which implies $\psi_1$ is not convex.

**Example 3.7** (a) The function $f(t) = |t|$ is not SOC-monotone.

(b) The function $f(t) = |t|$ is not SOC-convex.

(c) The function $f(t) = [t]_+$ is not SOC-monotone.

(d) The function $f(t) = [t]_+$ is not SOC-convex.

To see (a), let $x = (1, 0), y = (-2, 0)$. It is clear that $x \succeq_{K^2} y$. Besides, we have $x^2 = (1, 0), y^2 = (4, 0)$ which yields $|x| = (1, 0)$ and $|y| = (2, 0)$. But, $|x| - |y| = (1, 0) \not\succeq_{K^2} 0$.

To see (b), let $x = (1, 1), y = (-1, 1, 0)$. In fact, we have $|x| = (\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}), |y| = (1, -1, 0)$, and $|x + y| = (\sqrt{5}, 0, 0)$. Therefore, $|x| + |y| - |x + y| = (\sqrt{5} - 1 + 1/\sqrt{2}, 1/\sqrt{2}) \not\succeq_{K^3} 0$. Thus, $f\left(\frac{x + y}{2}\right) \not\succeq_{K^3} \frac{1}{2}\left(f(x) + f(y)\right)$, which shows $f(t) = |t|$ is not SOC-convex.

To see (c) and (d), just follows (a) and (b) and the facts that $[t]_+ = \frac{1}{2}(t + |t|)$ where $t \in \mathbb{R}$, and $[x]_+ = \frac{1}{2}(x + |x|)$ where $x \in \mathbb{R}^n$.

To close this section, we check with one popular smoothing function. It is the function, $f(t) = \frac{1}{2}(\sqrt{t^2 + 4} + t)$, proposed by Chen and Harker [4], Kanzow [12], and Smale [17], and is called the CHKS function. The associated SOC-function is defined by

$$f(x) = \frac{1}{2}(\sqrt{x^2 + 4e} + x),$$

where $e = (1, 0, \cdots, 0)^T$. The function $f(t)$ is convex and monotone functions, so we may also wish to know whether the SOC-function is SOC-convex or SOC-monotone. Unfortunately, it is neither SOC-convex nor SOC-monotone for $n \geq 3$, though it is both SOC-convex and SOC-monotone for $n = 2$. The following example shows what we have just said.
Example 3.8 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be $f(t) = \frac{\sqrt{t^2 + 4} + t}{2}$. Then

(a) $f$ is not SOC-monotone for general $n \geq 3$.

(b) however, $f$ is SOC-monotone for $n = 2$.

(c) $f$ is not SOC-convex for general $n \geq 3$.

(d) however, $f$ is SOC-convex for $n = 2$.

Again, let $x = (2, 1, -1), y = (1, 1, 0)$, then it is the counterexample for both (a) and (c). To see (b) and (d), by direct verifications as we have done before. □

4 Characterization of SOC-convexity and SOC-monotonicity

Based on all the results in the previous section, one may expect some certain relation between SOC-convex function and SOC-monotone function. One may also like to know under what conditions a function is SOC-convex. The same question arises for SOC-monotone. In this section, we explore these relations. In fact, there already have some analogous results for matrix-functions (see Chapter V of [2]). However, not much for this kind of vector-valued SOC-functions, so further study on these topics are definitely necessary.

Proposition 4.1 Let $f : [0, \infty) \rightarrow [0, \infty)$ be continuous. If $f$ is SOC-concave, then $f$ is SOC-monotone.

Proof. For any $0 < \lambda < 1$, we can write $\lambda x = \lambda y + (1 - \lambda) \frac{\lambda}{1 - \lambda} (x - y)$. Then the SOC-concavity of $f$ yields that

$$f^{soc}(\lambda x) \succeq_{\mathbb{R}_n} \lambda f^{soc}(y) + (1 - \lambda) f^{soc}\left(\frac{\lambda}{1 - \lambda} (x - y)\right) \succeq_{\mathbb{R}_n} 0,$$

where the second inequality is true since $f$ is from $[0, \infty)$ into itself and $x - y \succeq_{\mathbb{R}_n} 0$. Letting $\lambda \rightarrow 1$, we obtain that $f^{soc}(x) \succeq_{\mathbb{R}_n} f^{soc}(y)$, which says that $f$ is SOC-monotone. □

We notice that if $f$ is not a function from $[0, \infty)$ into itself, the above proposition is false. For instance, $f(t) = -t^2$ is SOC-concave, but not SOC-monotone.

Proposition 4.2 A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is SOC-convex if and only if the real-valued function $g(x) := \langle f^{soc}(x), z \rangle$ is a convex function $\forall z \succeq_{\mathbb{R}_n} 0$. 25
Proof. Suppose $f$ is SOC-convex and let $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$. Then, we have

$$f^{\text{soc}}((1 - \lambda)x + \lambda y) \succeq_{K^n} (1 - \lambda) f^{\text{soc}}(x) + \lambda f^{\text{soc}}(y).$$

Hence,

$$g((1 - \lambda)x + \lambda y) = \langle f^{\text{soc}}((1 - \lambda)x + \lambda y), z \rangle$$

$$\leq \langle (1 - \lambda) f^{\text{soc}}(x) + \lambda f^{\text{soc}}(y), z \rangle$$

$$= (1 - \lambda) \langle f^{\text{soc}}(x), z \rangle + \langle f^{\text{soc}}(y), z \rangle$$

$$= (1 - \lambda) g(x) + \lambda g(y),$$

where the inequality holds by Property 2.3(d). Thus, $g$ is a convex function.

For the other direction, from $g$ is convex, we obtain

$$\langle f^{\text{soc}}((1 - \lambda)x + \lambda y), z \rangle \leq \langle (1 - \lambda) f^{\text{soc}}(x) + \lambda f^{\text{soc}}(y), z \rangle.$$

Since $z \succeq_{K^n} 0$, by Property (2.3)(d) again, the above yields

$$f^{\text{soc}}((1 - \lambda)x + \lambda y) \succeq_{K^n} (1 - \lambda) f^{\text{soc}}(x) + \lambda f^{\text{soc}}(y),$$

which says $f$ is SOC-convex. \qed

Proposition 4.3 A differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is SOC-convex if and only if

$$f^{\text{soc}}(y) \succeq_{K^n} f^{\text{soc}}(x) + \nabla f^{\text{soc}}(x)(y - x),$$

for all $x, y \in \mathbb{R}^n$.

Proof. By [8, Prop. 5.3], we know that $f$ is differentiable if and only if $f^{\text{soc}}$ is differentiable. Using the gradient formula given therein and following the arguments as in [1, Prop. B.3] or [3, Theorem 2.3.5], the proof can be done easily. We omit the details. \qed

At last, we state two conjectures based on observing all the results and examples discussed in this paper. The conjectures describe the relationship between SOC-convex and SOC-monotone functions. We are not able to complete the proof right now. Nonetheless, we notice that some interesting results related to the trace of $x$, for example, [15, Prop. 6.2.9], might help towards proving our conjectures. Further study is certainly desirable. On the other hand, this paper is just an initial start of research on SOC-convex and SOC-monotone functions. There are many more properties to be investigated and studied. For instance, there is a useful property [2, Theorem V3.6] for matrix-valued function that says every matrix-monotone function $f^{\text{mat}}$ on an interval $I$ is smooth. Similarly, we can ask whether the extension of this theorem to SOC functions is true or not? We leave it as the future research.
Conjecture 4.1 If $f : (0, \infty) \to \mathbb{R}$ is continuous, convex, and nonincreasing, then

(a) $f_{soc}$ is SOC-convex.
(b) $-f_{soc}$ is SOC-monotone.

Conjecture 4.2 If $f : [0, \infty) \to [0, \infty)$ is continuous, then

$-f_{soc}$ is SOC-convex $\iff f_{soc}$ is SOC-monotone.

References


