Boundary-Value Problems for Ordinary Differential Equations

NTNU

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The two-point boundary-value problems (BVP) considered in this chapter involve a second-order differential equation together with boundary condition in the following form:

\[
\begin{cases}
y'' = f(x, y, y') \\
y(a) = \alpha, \quad y(b) = \beta
\end{cases}
\]  

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The numerical procedures for finding approximate solutions to the initial-value problems can not be adapted to the solution of this problem since the specification of conditions involve two different points, \(x = a\) and \(x = b\).
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The numerical procedures for finding approximate solutions to the initial-value problems can not be adapted to the solution of this problem since the specification of conditions involve two different points, \(x = a\) and \(x = b\). New techniques are introduced in this chapter for handling problems (1) in which the conditions imposed are of a boundary-value rather than an initial-value type.
Before considering numerical methods, a few mathematical theories about the two-point boundary-value problem (1), such as the existence and uniqueness of solution, shall be discussed in this section.

**Theorem 1** Suppose that $f$ in (1) is continuous on the set

\[ D = \{(x, y, y') | a \leq x \leq b, -\infty < y < \infty, -\infty < y' < \infty\}, \]

and that $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial y'}$ are also continuous on $D$. If

1. $\frac{\partial f}{\partial y}(x, y, y') > 0$ for all $(x, y, y') \in D$, and

2. a constant $M$ exists, with $\left| \frac{\partial f}{\partial y'}(x, y, y') \right| \leq M$, $\forall (x, y, y') \in D$,

then (1) has a unique solution.
When the function $f(x, y, y')$ has the special form

$$f(x, y, y') = p(x)y' + q(x)y + r(x),$$

the differential equation become a so-called linear problem. The previous theorem can be simplified for this case.

**Corollary 1** If the linear two-point boundary-value problem

$$\begin{cases}
y'' = p(x)y' + q(x)y + r(x) \\
y(a) = \alpha, \quad y(b) = \beta
\end{cases}$$

satisfies

1. $p(x), q(x),$ and $r(x)$ are continuous on $[a, b],$ and

2. $q(x) > 0$ on $[a, b],$

then the problem has a unique solution.
Many theories and application models consider the boundary-value problem in a special form as follows.

$$\begin{cases} y'' = f(x, y) \\ y(0) = 0, \quad y(1) = 0 \end{cases}$$

We will show that this simple form can be derived from the original problem by simple techniques such as changes of variables and linear transformation. To do this, we begin by changing the variable. Suppose that the original problem is

$$\begin{cases} y'' = f(x, y) \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

where $y = y(x)$. Now let $\lambda = b - a$ and define a new variable

$$t = \frac{x - a}{b - a} = \frac{1}{\lambda}(x - a).$$
That is, \( x = a + \lambda t \). Notice that \( t = 0 \) corresponds to \( x = a \), and \( t = 1 \) corresponds to \( x = b \). Then we define

\[
z(t) = y(a + \lambda t) = y(x)
\]

with \( \lambda = b - a \). This gives

\[
z'(t) = \frac{d}{dt}z(t) = \frac{d}{dt}y(a + \lambda t) = \left[ \frac{d}{dx}y(x) \right] \left[ \frac{d}{dt}(a + \lambda t) \right] = \lambda y'(x)
\]

and, analogously,

\[
z''(t) = \frac{d}{dt}z'(t) = \lambda^2 y''(x) = \lambda^2 f(x, y(x)) = \lambda^2 f(a + \lambda t, z(t)).
\]

Likewise the boundary conditions are changed to

\[
z(0) = y(a) = \alpha \quad \text{and} \quad z(1) = y(b) = \beta.
\]
With all these together, the problem (2) is transformed into

\[
\begin{align*}
  z''(t) &= \lambda^2 f(a + \lambda t, z(t)) \\
  z(0) &= \alpha, \quad z(1) = \beta
\end{align*}
\] (3)

Thus, if \( y(x) \) is a solution for (2), then \( z(t) = y(a + \lambda t) \) is a solution for the boundary-value problem (3). Conversely, if \( z(t) \) is a solution for (3), then \( y(x) = z\left(\frac{1}{\lambda}(x - a)\right) \) is a solution for (2).

**Example 1** Simplify the boundary conditions of the following equation by use of changing variables.

\[
\begin{align*}
  y'' &= \sin(xy) + y^2 \\
  y(1) &= 3, \quad y(4) = 7
\end{align*}
\]

**Solution:** In this problem \( a = 1, b = 4 \), hence \( \lambda = 3 \). Now define the new variable \( t = \frac{1}{3}(x - 1) \), hence \( x = 1 + 3t \), and let \( z(t) = y(x) = y(1 + 3t) \). Then
\[ \lambda^2 f(a + \lambda t, z) = 9 \left[ \sin((1 + 3t)z) + z^2 \right], \]

and the original equation is reduced to

\[
\begin{cases}
  z''(t) = 9 \sin((1 + 3t)z) + 9z^2 \\
  z(0) = 3, \quad z(1) = 7
\end{cases}
\]

To reduce a two-point boundary-value problem

\[
\begin{cases}
  z''(t) = g(t, z) \\
  z(0) = \alpha, \quad z(1) = \beta
\end{cases}
\]

to a homogeneous system, let

\[ u(t) = z(t) - [\alpha + (\beta - \alpha)t] \]

then \[ u''(t) = z''(t) \], and

\[ u(0) = z(0) - \alpha = 0 \quad \text{and} \quad u(1) = z(1) - \beta = 0 \]
Moreover,

\[ g(t, z) = g(t, u + \alpha + (\beta - \alpha)t) \equiv h(t, u). \]

The system is now transformed into

\[
\begin{align*}
  u''(t) &= h(t, u) \\
  u(0) &= 0, \quad u(1) = 0
\end{align*}
\]

**Example 2** Reduce the system

\[
\begin{align*}
  z'' &= [5z - 10t + 35 + \sin(3z - 6t + 21)]e^t \\
  z(0) &= -7, \quad z(1) = -5
\end{align*}
\]

to a homogeneous problem by linear transformation technique.

**Solution:** Let

\[ u(t) = z(t) - [-7 + (-5 + 7)t] = z(t) - 2t + 7. \]
Then \( z(t) = u(t) + 2t - 7 \), and

\[
\begin{align*}
  u'' &= z'' = [5z - 10t + 35 + \sin(3z - 6t + 21)]e^t \\
  &= [5(u + 2t - 7) - 10t + 35 + \sin(3(u + 2t - 7) - 6t + 21)]e^t \\
  &= [5u + \sin(3u)]e^t
\end{align*}
\]

The system is transformed to

\[
\begin{cases}
  u''(t) = [5u + \sin(3u)]e^t \\
  u(0) = u(1) = 0
\end{cases}
\]

**Example 3** Reduce the following two-point boundary-value problem

\[
\begin{cases}
  y'' = y^2 + 3 - x^2 + xy \\
  y(3) = 7, \quad y(5) = 9
\end{cases}
\]

to a homogeneous system.
**Solution:** In the original system, \(a = 3, b = 5, \alpha = 7, \beta = 9\). Let \(\lambda = b - a = 2\) and define a new variable

\[
t = \frac{1}{2}(x - 3) \implies x = 2t + 3.
\]

Let the function \(z(t) = y(x) = y(2t + 3)\). Then

\[
z''(t) = \lambda^2 y''(2t + 3) = \lambda^2 f(2t + 3, u)
\]

\[
= 4[z^2 + 3 - (2t + 3)^2 + (2t + 3)z]
\]

\[
= 4[z^2 + 3z + 2tz - 4t^2 - 12t - 6]
\]

The original problem is first transformed into

\[
\begin{aligned}
z''(t) &= 4[z^2 + 3z + 2tz - 4t^2 - 12t - 6] \\
z(0) &= 7, \quad z(1) = 9
\end{aligned}
\]

Next let

\[
u(t) = z(t) - [7 + 2t], \quad \text{or equivalently, } z(t) = u(t) + 2t + 7.
\]
Then
\[
    u''(t) = 4[(z + 2t + 7)^2 + 3(u + 2t + 7) + 2t(u + 2t + 7) - 4t^2 - 12t - 6] \\
    = 4[u^2 + 6tu + 17u + 4t^2 + 36t + 64].
\]

The original problem is transformed into the following homogeneous system
\[
\begin{cases}
    u''(t) = 4[u^2 + 6tu + 17u + 4t^2 + 36t + 64] \\
    u(0) = u(1) = 0
\end{cases}
\]

**Theorem 2** The boundary-value problem
\[
\begin{cases}
    y'' = f(x, y) \\
    y(0) = 0, \quad y(1) = 0
\end{cases}
\]
has a unique solution if \( \frac{\partial f}{\partial y} \) is continuous, non-negative, and bounded in the strip \( 0 \leq x \leq 1 \) and \( -\infty < y < \infty \).
Theorem 3  If $f$ is a continuous function of $(s, t)$ in the domain $0 \leq s \leq 1$ and $-\infty < t < \infty$ such that

$$|f(s, t_1) - f(s, t_2)| \leq K|t_1 - t_2|, \quad (K < 8).$$

Then the boundary-value problem

$$\begin{cases} y'' = f(x, y) \\ y(0) = 0, \quad y(1) = 0 \end{cases}$$

has a unique solution in $C[0, 1]$. 
We consider finite difference method for solving the linear two-point boundary-value problem of the form

$$\begin{aligned}
y'' &= p(x)y' + q(x)y + r(x) \\
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Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation.
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Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation.

2.1 – The Finite Difference Formulation
We consider finite difference method for solving the linear two-point boundary-value problem of the form

\[
\begin{align*}
\frac{d^2y}{dx^2} &= p(x)y' + q(x)y + r(x) \\
y(a) &= \alpha, \quad y(b) = \beta.
\end{align*}
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Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation.

2.1 – The Finite Difference Formulation

First, partition the interval \([a, b]\) into \(n\) equally-spaced subintervals by points

\[a = x_0 < x_1 < \ldots < x_n < x_n = b.\]
We consider finite difference method for solving the linear two-point boundary-value problem of the form

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Methods involving finite differences for solving boundary-value problems replace each of the derivatives in the differential equation by an appropriate difference-quotient approximation.

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First, partition the interval \([a, b]\) into \(n\) equally-spaced subintervals by points

\[a = x_0 < x_1 < \ldots < x_n < x_n = b.\]

Each mesh point \(x_i\) can be computed by

\[x_i = a + i \times h, \quad i = 0, 1, \ldots, n, \quad \text{with} \quad h = \frac{b - a}{n}\]

where \(h\) is called the mesh size.
At the interior mesh points, \( x_i \), for \( i = 1, 2, \ldots, n - 1 \), the differential equation to be approximated satisfies

\[
y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i). \tag{5}
\]
At the interior mesh points, $x_i$, for $i = 1, 2, \ldots, n - 1$, the differential equation to be approximated satisfies

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i).$$  (5)

The central finite difference formulae

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y^{(3)}(\eta_i),$$  (6)

for some $\eta_i$ in the interval $(x_{i-1}, x_{i+1})$, for some $\eta_i$ in the interval $(x_{i-1}, x_{i+1})$,
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for some \( \eta_i \) in the interval \( (x_{i-1}, x_{i+1}) \), and

\[
y''(x_i) = \frac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i), \tag{7}
\]

for some \( \xi_i \) in the interval \( (x_{i-1}, x_{i+1}) \), can be derived from Taylor’s theorem by expanding \( y \) about \( x_i \).
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for some \( \xi_i \) in the interval \((x_{i-1}, x_{i+1})\), can be derived from Taylor’s theorem by expanding \( y \) about \( x_i \).

Let \( u_i \) denote the approximate value of \( y_i = y(x_i) \).
At the interior mesh points, $x_i$, for $i = 1, 2, \ldots, n - 1$, the differential equation to be approximated satisfies

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i).$$

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for some $\eta_i$ in the interval $(x_{i-1}, x_{i+1})$, and

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for some $\xi_i$ in the interval $(x_{i-1}, x_{i+1})$, can be derived from Taylor’s theorem by expanding $y$ about $x_i$.

Let $u_i$ denote the approximate value of $y_i = y(x_i)$. If $y \in C^4[a, b]$, then a finite difference method with truncation error of order $O(h^2)$ can be obtained by using the approximations
\( y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \)

and

\( y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \)

for \( y'(x_i) \) and \( y''(x_i) \), respectively.
\[ y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and} \quad y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \]

for \( y'(x_i) \) and \( y''(x_i) \), respectively. Furthermore, let

\[ p_i = p(x_i), \quad q_i = q(x_i), \quad r_i = r(x_i). \]
\[ y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and} \quad y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \]

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The discrete version of equation (4) is then

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i + r_i, \quad i = 1, 2, \ldots, n - 1, \tag{8} \]

together with boundary conditions \( u_0 = \alpha \) and \( u_n = \beta \).
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\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i + r_i, \quad i = 1, 2, \ldots, n - 1,
\]

(8)
together with boundary conditions \(u_0 = \alpha\) and \(u_n = \beta\). Equation (8) can be written in the form

\[
\left(1 + \frac{h}{2} p_i \right) u_{i-1} - (2 + h^2 q_i) u_i + \left(1 - \frac{h}{2} p_i \right) u_{i+1} = h^2 r_i,
\]

(9)
for \(i = 1, 2, \ldots, n - 1\).
BVP of ODE

\[ y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and} \quad y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \]

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\[ \left( 1 + \frac{h}{2}p_i \right) u_{i-1} - \left( 2 + h^2 q_i \right) u_i + \left( 1 - \frac{h}{2}p_i \right) u_{i+1} = h^2 r_i, \quad (9) \]

for \( i = 1, 2, \ldots, n - 1 \). In (8), \( u_1, u_2, \ldots, u_{n-1} \) are the unknown, and there are \( n - 1 \) linear equations to be solved.
\[ y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and} \quad y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \]

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together with boundary conditions \( u_0 = \alpha \) and \( u_n = \beta \). Equation (8) can be written in the form

\[ \left( 1 + \frac{h}{2} p_i \right) u_{i-1} - (2 + h^2 q_i) u_i + \left( 1 - \frac{h}{2} p_i \right) u_{i+1} = h^2 r_i, \quad (9) \]

for \( i = 1, 2, \ldots, n - 1 \). In (8), \( u_1, u_2, \ldots, u_{n-1} \) are the unknown, and there are \( n - 1 \) linear equations to be solved. The resulting system of linear equations can be expressed in the matrix form

\[ Au = f, \quad (10) \]
where

\[ A = \begin{bmatrix}
-2 - h^2 q_1 & 1 - \frac{h}{2} p_1 \\
1 + \frac{h}{2} p_2 & -2 - h^2 q_2 & 1 - \frac{h}{2} p_2 \\
& \ddots & \ddots & \ddots \\
& & 1 + \frac{h}{2} p_{n-2} & -2 - h^2 q_{n-2} & 1 - \frac{h}{2} p_{n-2} \\
& & & 1 + \frac{h}{2} p_{n-1} & -2 - h^2 q_{n-1}
\end{bmatrix}, \]

\[ u = \begin{bmatrix} u_1 \\
u_2 \\
\vdots \\
u_{n-2} \\
u_{n-1} \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} h^2 r_1 - (1 + \frac{h}{2} p_1) \alpha \\
h^2 r_2 \\
\vdots \\
h^2 r_{n-2} \\
h^2 r_{n-1} - (1 - \frac{h}{2} p_{n-1}) \beta \end{bmatrix}. \]
Since the matrix $A$ is tridiagonal, this system can be solved by a special Gaussian elimination in $O(n^2)$ flops.
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**Theorem 4** Suppose that $p(x)$, $q(x)$, and $r(x)$ in (4) are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$. Then (10) has a unique solution provided that $h < 2/L$, where $L = \max_{a \leq x \leq b} |p(x)|$. 
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### 2.2 – Convergence Analysis

We shall analyze that when $h$ converges to zero, the solution $u_i$ of the discrete problem (8) converges to the solution $y_i$ of the original continuous problem (5).
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**Theorem 4** Suppose that $p(x)$, $q(x)$, and $r(x)$ in (4) are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$. Then (10) has a unique solution provided that $h < 2/\max_{a \leq x \leq b} |p(x)|$.

### 2.2 – Convergence Analysis

We shall analyze that when $h$ converges to zero, the solution $u_i$ of the discrete problem (8) converges to the solution $y_i$ of the original continuous problem (5).

$y_i$ satisfies the following system of equations

$$
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i) = p_i \left( \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y^{(3)}(\eta_i) \right) + q_i y_i + r_i,
$$

for $i = 1, 2, \ldots, n - 1$. 

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Since the matrix $A$ is tridiagonal, this system can be solved by a special Gaussian elimination in $O(n^2)$ flops.

**Theorem 4** Suppose that $p(x)$, $q(x)$, and $r(x)$ in (4) are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$. Then (10) has a unique solution provided that $h < 2/L$, where $L = \max_{a \leq x \leq b} |p(x)|$.

### 2.2 – Convergence Analysis

We shall analyze that when $h$ converges to zero, the solution $u_i$ of the discrete problem (8) converges to the solution $y_i$ of the original continuous problem (5).

$y_i$ satisfies the following system of equations

\[
\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i) = p_i \left( \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y^{(3)}(\eta_i) \right) + q_i y_i + r_i,
\]

for $i = 1, 2, \ldots, n - 1$. Subtract (8) from (11) and let $e_i = y_i - u_i$, the result is

\[
\frac{e_{i+1} - 2e_i + e_{i-1}}{h^2} = p_i \frac{e_{i+1} - e_{i-1}}{2h} + q_i e_i + h^2 g_i, \quad i = 1, 2, \ldots, n - 1,
\]
where

\[ g_i = \frac{1}{12}y^{(4)}(\xi_i) - \frac{1}{6}p_i y^{(3)}(\eta_i). \]
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\[ g_i = \frac{1}{12} y^{(4)}(\xi_i) - \frac{1}{6} p_i y^{(3)}(\eta_i). \]

After collecting terms and multiplying by \( h^2 \), we have

\[
\left(1 + \frac{h}{2} p_i\right) e_{i-1} - (2 + h^2 q_i) e_i + \left(1 - \frac{h}{2} p_i\right) e_{i+1} = h^4 g_i, i = 1, 2, \ldots, n - 1.
\]
where

\[ g_i = \frac{1}{12} y^{(4)}(\xi_i) - \frac{1}{6} p_i y^{(3)}(\eta_i). \]

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\]

Let \( e = [e_1, e_2, \ldots, e_{n-1}]^T \) and \( |e_k| = \|e\|_\infty \).
where
\[ g_i = \frac{1}{12} y^{(4)}(\xi_i) - \frac{1}{6} p_i y^{(3)}(\eta_i). \]

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\left( 1 + \frac{h}{2} p_i \right) e_{i-1} - (2 + h^2 q_i) e_i + \left( 1 - \frac{h}{2} p_i \right) e_{i+1} = h^4 g_i, \, i = 1, 2, \ldots, n - 1. 
\]

Let \( e = [e_1, e_2, \ldots, e_{n-1}]^T \) and \( |e_k| = \|e\|_\infty \). Then
\[
(2 + h^2 q_k) e_k = \left( 1 + \frac{h}{2} p_k \right) e_{k-1} + \left( 1 - \frac{h}{2} p_k \right) e_{k+1} - h^4 g_k, 
\]
where
\[ g_i = \frac{1}{12} y^{(4)}(\xi_i) - \frac{1}{6} p_i y^{(3)}(\eta_i). \]

After collecting terms and multiplying by \( h^2 \), we have
\[
\left(1 + \frac{h^2}{2} p_i\right) e_{i-1} - (2 + h^2 q_i) e_i + \left(1 - \frac{h^2}{2} p_i\right) e_{i+1} = h^4 g_i, i = 1, 2, \ldots, n - 1.
\]

Let \( e = [e_1, e_2, \ldots, e_{n-1}]^T \) and \(|e_k| = \|e\|_\infty\). Then
\[
(2 + h^2 q_k) e_k = \left(1 + \frac{h^2}{2} p_k\right) e_{k-1} + \left(1 - \frac{h^2}{2} p_k\right) e_{k+1} - h^4 g_k,
\]
and, hence
\[
|2 + h^2 q_k| |e_k| \leq \left|1 + \frac{h^2}{2} p_k\right| |e_{k-1}| + \left|1 - \frac{h^2}{2} p_k\right| |e_{k+1}| + h^4 |g_k|
\]
\[
\leq \left|1 + \frac{h^2}{2} p_k\right| \|e\|_\infty + \left|1 - \frac{h^2}{2} p_k\right| \|e\|_\infty + h^4 \|g\|_\infty.
\]
When \( q(x) > 0, \forall x \in [a, b] \) and \( h \) is chosen small enough so that \( \left| \frac{h}{2} p_i \right| < 1, \forall i \), then

the above inequality induces

\[
h^2 q_k \|e\|_\infty \leq h^4 \|g\|_\infty.
\]
When \( q(x) > 0 \), \( \forall x \in [a, b] \) and \( h \) is chosen small enough so that \( \frac{h}{2} p_i < 1 \), \( \forall i \), then the above inequality induces

\[
h^2 q_k \| e \|_\infty \leq h^4 \| g \|_\infty.
\]

Therefore, we derive an upper bound for \( \| e \|_\infty \)

\[
\| e \|_\infty \leq h^2 \left( \frac{\| g \|_\infty}{\inf q(x)} \right).
\]
When \( q(x) > 0, \forall x \in [a, b] \) and \( h \) is chosen small enough so that \( \left| \frac{h}{2} p_i \right| < 1, \forall i \), then the above inequality induces

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By the definition of \( g_i \), we have

\[
\| g \|_\infty \leq \frac{1}{12} \| y^{(4)}(x) \|_\infty + \frac{1}{6} \| p(x) \|_\infty \| y^{(3)}(x) \|_\infty.
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\]

Hence \( \frac{\| g \|_\infty}{\inf q(x)} \) is a bound independent of \( h \).
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Therefore, we derive an upper bound for \( \| e \|_\infty \)

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\| g \|_\infty \leq \frac{1}{12} \| y^{(4)}(x) \|_\infty + \frac{1}{6} \| p(x) \|_\infty \| y^{(3)}(x) \|_\infty.
\]

Hence \( \frac{\| g \|_\infty}{\inf q(x)} \) is a bound independent of \( h \). Thus we can conclude that \( \| e \|_\infty \) is \( O(h^2) \) as \( h \to 0 \).
3 – Shooting Methods

We consider solving the following 2-point boundary-value problem:

\[
\begin{align*}
  y'' &= f(x, y, y') \\
  y(a) &= \alpha, \quad y(b) = \beta
\end{align*}
\] (12)
We consider solving the following 2-point boundary-value problem:

\[
\begin{cases}
y'' = f(x, y, y') \\
y(a) = \alpha, \quad y(b) = \beta
\end{cases}
\]  \hspace{1cm} (12)

The idea of shooting method for (12) is to solve a related initial-value problem with a guess for \( y'(a) \), say \( z \).
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\[
\begin{cases}
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\end{cases}
\]  

(12)

The idea of shooting method for (12) is to solve a related initial-value problem with a guess for \( y'(a) \), say \( z \). The corresponding IVP

\[
\begin{cases}
  y'' = f(x, y, y') \\
  y(a) = \alpha, \quad y'(a) = z
\end{cases}
\]  

(13)

can then be solved by, for example, Runge-Kutta method.
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can then be solved by, for example, Runge-Kutta method. We denote this approximate solution \( y_z \) and hope \( y_z(b) = \beta \). If not, we use another guess for \( y'(a) \), and try to solve an altered IVP (13) again.
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can then be solved by, for example, Runge-Kutta method. We denote this approximate solution \(y_z\) and hope \(y_z(b) = \beta\). If not, we use another guess for \(y'(a)\), and try to solve an altered IVP (13) again. This process is repeated and can be done systematically.

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Objective: select $z$, so that $y_z(b) = \beta$. 

Let $(z) = y_z(b)$.

Now our objective is simply to solve the equation $(z) = 0$. Hence secant method can be used.

How to compute $z$?

Suppose we have solutions $y_{z_1}; y_{z_2}$ with guesses $z_1; z_2$ and obtain $(z_1)$ and $(z_2)$.

If these guesses cannot generate satisfactory solutions, we can obtain another guess $z_3$ by the secant method

$$z_3 = z_2 - \frac{(z_2)(z_1)}{(z_2) - (z_1)}.$$

In general $z_{k+1} = z_k - \frac{(z_k)(z_{k-1})}{(z_k) - (z_{k-1})}$. 

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Tsung-Min Hwang December 20, 2003
Objective: select $z$, so that $y_z(b) = \beta$.

Let

$$\phi(z) = y_z(b) - \beta.$$
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\]

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Suppose we have solutions $y_{z_1}, y_{z_2}$ with guesses $z_1, z_2$ and obtain $\phi(z_1)$ and $\phi(z_2)$. 
Objective: select \( z \), so that \( y_z(b) = \beta \).

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\phi(z) = y_z(b) - \beta.
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Now our objective is simply to solve the equation \( \phi(z) = 0 \). Hence secant method can be used.

How to compute \( z \)?

Suppose we have solutions \( y_{z_1}, y_{z_2} \) with guesses \( z_1, z_2 \) and obtain \( \phi(z_1) \) and \( \phi(z_2) \). If these guesses can not generate satisfactory solutions, we can obtain another guess \( z_3 \) by the secant method

\[
z_3 = z_2 - \phi(z_2) \frac{z_2 - z_1}{\phi(z_2) - \phi(z_1)}.
\]
Objective: select $z$, so that $y_z(b) = \beta$.

Let

$$\phi(z) = y_z(b) - \beta.$$  

Now our objective is simply to solve the equation $\phi(z) = 0$. Hence secant method can be used.

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Suppose we have solutions $y_{z_1}, y_{z_2}$ with guesses $z_1, z_2$ and obtain $\phi(z_1)$ and $\phi(z_2)$. If these guesses can not generate satisfactory solutions, we can obtain another guess $z_3$ by the secant method

$$z_3 = z_2 - \phi(z_2) \frac{z_2 - z_1}{\phi(z_2) - \phi(z_1)}.$$  

In general

$$z_{k+1} = z_k - \phi(z_k) \frac{z_k - z_{k-1}}{\phi(z_k) - \phi(z_{k-1})}.$$
Special BVP:

\[
\begin{aligned}
&y'' = u(x) + v(x)y + w(x)y' \\
y(a) = \alpha, &\quad y(b) = \beta
\end{aligned}
\]  \tag{14}

where \( u(x) \), \( v(x) \), \( w(x) \) are continuous in \([a, b]\).
Special BVP:

\[
\begin{align*}
    \begin{cases}
        y'' &= u(x) + v(x)y + w(x)y' \\
        y(a) &= \alpha, \quad y(b) = \beta
    \end{cases}
\end{align*}
\]  

(14)

where \( u(x), v(x), w(x) \) are continuous in \([a, b]\).

Suppose we have solved (14) twice with initial guesses \( z_1 \) and \( z_2 \), and obtain approximate solutions \( y_1 \) and \( y_2 \).
Special BVP:

\[
\begin{align*}
\begin{cases}
y'' &= u(x) + v(x)y + w(x)y' \\
y(a) &= \alpha, \quad y(b) = \beta
\end{cases}
\end{align*}
\]  

(14)

where \(u(x), v(x), w(x)\) are continuous in \([a, b]\).

Suppose we have solved (14) twice with initial guesses \(z_1\) and \(z_2\), and obtain approximate solutions \(y_1\) and \(y_2\), hence

\[
\begin{align*}
\begin{cases}
y_1'' &= u + vy_1 + wy_1' \\
y_1(a) = \alpha, \quad y_1'(a) = z_1
\end{cases}
\end{align*}
\quad \text{and} \quad
\begin{align*}
\begin{cases}
y_2'' &= u + vy_2 + wy_2' \\
y_2(a) = \alpha, \quad y_2'(a) = z_2
\end{cases}
\end{align*}
\]
Special BVP:

\[
\begin{cases}
    y'' = u(x) + v(x)y + w(x)y' \\
    y(a) = \alpha, \quad y(b) = \beta
\end{cases}
\]  \quad (14)

where \(u(x), v(x), w(x)\) are continuous in \([a, b]\).

Suppose we have solved (14) twice with initial guesses \(z_1\) and \(z_2\), and obtain approximate solutions \(y_1\) and \(y_2\), hence

\[
\begin{cases}
    y_1'' = u + vy_1 + wy_1' \\
    y_1(a) = \alpha, \quad y_1'(a) = z_1
\end{cases}
\] \quad and \quad \begin{cases}
    y_2'' = u + vy_2 + wy_2' \\
    y_2(a) = \alpha, \quad y_2'(a) = z_2
\end{cases}

Now let

\[ y(x) = \lambda y_1(x) + (1 - \lambda)y_2(x) \]

for some parameter \(\lambda\),
Special BVP:

\[
\begin{cases}
  y'' = u(x) + v(x)y + w(x)y' \\
y(a) = \alpha, \quad y(b) = \beta
\end{cases}
\]  

(14)

where \(u(x), v(x), w(x)\) are continuous in \([a, b]\).

Suppose we have solved (14) twice with initial guesses \(z_1\) and \(z_2\), and obtain approximate solutions \(y_1\) and \(y_2\), hence

\[
\begin{cases}
  y_1'' = u + vy_1 + wy_1' \\
y_1(a) = \alpha, \quad y_1'(a) = z_1
\end{cases}
\]

and

\[
\begin{cases}
  y_2'' = u + vy_2 + wy_2' \\
y_2(a) = \alpha, \quad y_2'(a) = z_2
\end{cases}
\]

Now let

\[
y(x) = \lambda y_1(x) + (1 - \lambda)y_2(x)
\]

for some parameter \(\lambda\), we can show

\[
y'' = u + vy + wy'
\]
and

\[ y(a) = \lambda y_1(a) + (1 - \lambda)y_2(a) = \alpha \]
and

\[ y(a) = \lambda y_1(a) + (1 - \lambda) y_2(a) = \alpha \]

We can select \( \lambda \) so that \( y(b) = \beta \).
and

\[ y(a) = \lambda y_1(a) + (1 - \lambda)y_2(a) = \alpha \]

We can select \( \lambda \) so that \( y(b) = \beta \).

\[ \beta = y(b) = \lambda y_1(b) + (1 - \lambda)y_2(b) \]

\[ = \lambda(y_1(b) - y_2(b)) + y_2(b) \]
and

\[ y(a) = \lambda y_1(a) + (1 - \lambda) y_2(a) = \alpha \]

We can select \( \lambda \) so that \( y(b) = \beta \).

\[
\begin{align*}
\beta &= y(b) = \lambda y_1(b) + (1 - \lambda) y_2(b) \\
&= \lambda (y_1(b) - y_2(b)) + y_2(b) \\
\Rightarrow \lambda &= \frac{\beta - y_2(b)}{(y_1(b) - y_2(b))}
\end{align*}
\]
and

\[ y(a) = \lambda y_1(a) + (1 - \lambda)y_2(a) = \alpha \]

We can select \( \lambda \) so that \( y(b) = \beta \).

\[
\beta = y(b) = \lambda y_1(b) + (1 - \lambda)y_2(b) \\
= \lambda (y_1(b) - y_2(b)) + y_2(b)
\]

\[ \Rightarrow \lambda = \frac{\beta - y_2(b)}{(y_1(b) - y_2(b))} \]

In practice, we can solve the following two IVPs (in parallel)

\[
\begin{cases}
  y'' = u(x) + v(x)y + w(x)y' \\
  y(a) = \alpha, \quad y'(a) = 0
\end{cases}
\]

and

\[
\begin{cases}
  y'' = u(x) + v(x)y + w(x)y' \\
  y(a) = \alpha, \quad y'(a) = 1
\end{cases}
\]
i.e.,

\[
\begin{align*}
    y_1' &= y_3 \\
    y_3' &= y_1'' = u + vy_1 + wy_1' = u + vy_1 + wy_3 \\
    y_1(a) &= \alpha, \quad y_3(a) = y_1'(a) = 0
\end{align*}
\]

and

\[
\begin{align*}
    y_2' &= y_4 \\
    y_4' &= u + vy_2 + wy_4 \\
    y_2(a) &= \alpha, \quad y_4(a) = 1
\end{align*}
\]

to obtain approximate solutions \( y_1 \) and \( y_2 \), then compute \( \lambda \) and form the solution \( y \).