Lecture note on Complex Variables

1 Complex numbers

In this section, we will survey the algebraic and geometric properties of the complex number field, which is denoted by \( C \) throughout this course. Properties of the real number field (denoted by \( R \)) have been introduced in the course of advanced calculus and are assumed here.

**Definition 1.1.** The complex number field \( C \) is the field extension of degree 2 of the real number field \( R \). There is a basis \( \{1, i\} \) for \( C \) which satisfies \( i^2 = -1 \). An element in \( C \) is called a complex number.

In the course of modern algebra, one can show that there is only one finite field extension of \( R \), and it is of degree 2. We find an element \( i \) in that field such that \( i^2 = -1 \), and then show that \( \{1, i\} \) is a basis for this field, which can be regarded as a vector space over \( R \). Every complex number can then be uniquely expressed as

\[ z = x + iy, \quad x, y \in \mathbb{R}. \]

Under this expression, the numbers

\[ x = \text{Re} z, \quad y = \text{Im} z \]

are called the real part and the imaginary part of the complex number \( z \) respectively. Sometimes it is convenient to represent a complex number \( z = x + iy \) by the point \((x, y)\) with rectangular coordinates on the complex plane.

The sum and the product of two complex numbers are given by:

\[
\begin{align*}
(x_1 + i y_1) + (x_2 + i y_2) &= (x_1 + x_2) + i (y_1 + y_2), \\
(x_1 + i y_1) \cdot (x_2 + i y_2) &= (x_1 y_1 - x_2 y_2) + i (x_1 y_2 + x_2 y_1).
\end{align*}
\]

The multiplicative inverse of a non-zero complex number \( z = x + iy \) is

\[ z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}. \] (Check it: \( z \cdot z^{-1} = 1 \).)

The (complex) conjugate of a complex number \( z = x + iy \) is defined as

\[ \bar{z} = x - iy; \]

and the modulus, or the absolute value of \( z \) is

\[ |z| = \sqrt{x^2 + y^2}. \]

(The square root is conventionally defined in \( \mathbb{R}_+ \).) It is straightforward to check that \( |z| = |\bar{z}| \) and \( z \bar{z} = |z|^2 \). Furthermore, \((C, |\cdot|)\) becomes a normed space, that is, it satisfies the following conditions:

1. \( |z| \geq 0 \) for all \( z \in C \), and \( |z| = 0 \) if and only if \( z = 0 \).
2. \( |\lambda z| = |\lambda| \cdot |z| \) for \( z \in C \) and \( \lambda \in \mathbb{R} \).
3. \( |z_1 + z_2| \leq |z_1| + |z_2| \) (triangle inequality.)

(Proof of the triangle inequality:

\[
|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + 2 \text{Re}(z_1 \bar{z}_2) + |z_2|^2 \\
\leq |z_1|^2 + 2|z_1 z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.
\]

Since both bases are non-negative, we take the square roots and the inequality is proved.)
The space $\mathbb{C}$ is then equipped with the topology endowed by the norm structure. Indeed, it is the same as that of $\mathbb{R}^2$ under the Euclidean metric. One might check that
\[ |z_1 z_2| = |z_1| \cdot |z_2|, \quad z_1, z_2 \in \mathbb{C}. \]

We may define a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}$ by
\[ \langle z_1, z_2 \rangle = z_1 \bar{z}_2, \quad z_1, z_2 \in \mathbb{C}. \]

Clearly $\langle z, z \rangle = |z|^2$. It follows that $\langle \cdot, \cdot \rangle$ defines an inner product structure on $\mathbb{C}$. With the completeness of $\mathbb{R}$, $\mathbb{C}$ becomes a Hilbert space, that is, a complete inner product space.

There is another way to express a complex number, called the polar coordinate. Take
\[ r = |z|, \quad x = r \cos \theta, \quad y = r \sin \theta, \]
then $z$ is written as
\[ z = r(\cos \theta + i \sin \theta). \]

When $z = 0$, the coordinate $\theta$ is undefined.

As we identify complex numbers with vectors on the complex plane, the number $r$ is the distance between the origin and the point $z$, while $\theta$ is the oriented angle between the positive real axis ($x$-axis) and the radial vector representing $z$. Technically speaking, there are infinitely many choices for $\theta$, for we may add any multiple of $2\pi$ to the oriented angle; each choice of $\theta$ is called an argument of $z$, and the set of all arguments of $z$ is denoted by $\text{Arg } z$. Among these values, there is exactly one $\Theta$ such that $-\pi < \Theta \leq \pi$; it is called the principal argument of $z$ and denoted by Arg $z$. Note that
\[ \text{arg } z = \text{Arg } z + 2n\pi, \quad n \in \mathbb{Z}. \]

As a consequence, two non-zero complex numbers $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ are equal if and only if
\[ r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2n\pi \]
for some integer $n$.

Using this expression, the multiplication of complex numbers are easy to write down. If $z_j = r_j(\cos \theta_j + i \sin \theta_j)$, $j = 1, 2$, then
\[ z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \]

This leads to de Moivre’s formula by mathematical induction and inversion:
\[ z^n = r^n (\cos n\theta + i \sin n\theta), \quad n \in \mathbb{Z}. \]

If $n$ is a positive integer and $z_0 = r_0(\cos \theta_0 + i \sin \theta_0)$ is a non-zero complex number, the equation $z^n = z_0$ has $n$ distinct solutions, which are called the $n^{\text{th}}$ roots of $z_0$. Using de Moivre’s formula, the solutions are
\[ z = \sqrt[n]{r_0} \left[ \cos \left( \frac{\theta_0 + 2k\pi}{n} \right) + i \sin \left( \frac{\theta_0 + 2k\pi}{n} \right) \right], \quad k = 0, 1, \ldots, n - 1. \]

When $n \geq 3$, the roots lie at the vertices of a regular polygon of $n$ sides inscribed in the circle, centered at the origin with radius $\sqrt[n]{r_0}$.

## 2 Lines and circles on the complex plane

As in algebraic geometry, we would like to describe some geometric objects like lines and circles by equations. Keep in mind that some variables we will be considering are complex-valued.

The easiest equation is for a circle, because we have a notion of distance in hand. Coming from the definition, the circle centered at a point $z_0$ with radius $r > 0$ is defined by the equation
\[ |z - z_0| = r. \quad (2.1) \]
Squaring and expanding Equation (2.1), we reach an equivalent form

\[ z \bar{z} - z_0 \bar{z} - \bar{z}_0 z + (|z_0|^2 - r^2) = 0. \]

Similarly one can try to write down the equations for other conic curves. For instance, an equivalent definition for an ellipse is that the sum of the distances between a point to two fixed ones is constant; translating this sentence into an equation of a complex variable yields

\[ |z - z_1| + |z - z_2| = 2a. \]

For straight lines, we use the following strategy: a line on the \( xy \)-plane is defined by the equation

\[ ax + by = c, \quad a, b, c \in \mathbb{R}. \]  \hfill (2.2)

(Some technical assumptions are omitted.) For \( x \) (resp. \( y \)) being the real part (resp. imaginary part) of the complex variable \( z \), we have the identifications

\[ x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}). \]

So plug in Equation (2.2) and we see that

\[ \left( \frac{a}{2} - \frac{ib}{2} \right) z + \left( \frac{a}{2} + \frac{ib}{2} \right) \bar{z} = c. \]  \hfill (2.3)

If we let \( w_0 = (a + ib)/2 \), then Equation (2.3) becomes

\[ \bar{w}_0 z + w_0 \bar{z} = c. \]  \hfill (2.4)

**Exercise.** What happens if \( c \) in Equation (2.4) is a complex number but not a real one?

### 3 Topological jargons

We now describe the topology in \( \mathbb{C} \), defined by the norm \( |\cdot| \). An \( \varepsilon \) neighborhood of a point \( z_0 \in \mathbb{C} \) is the set

\[ B(z_0, \varepsilon) := \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}. \]

It consists of all points \( z \) lying inside but not on a circle centered at \( z_0 \) with the positive radius \( \varepsilon \). When the value of \( \varepsilon \) is understood or is immaterial in the discussion, we often refer the set as just a neighborhood. On the complex plane, a neighborhood looks like a circular disk without the circle that bounds it. Occasionally, it is convenient to speak of a deleted neighborhood

\[ B'(z_0, \varepsilon) := \{ z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon \} \]

consisting of all points \( z \) in an \( \varepsilon \) neighborhood of \( z_0 \) except for the point \( z_0 \) itself.

**Definition 3.1.** Let \( S \) be a subset of \( \mathbb{C} \). A point \( z_0 \) is said to be an interior point of \( S \) if there is some neighborhood of \( z_0 \) lying completely in \( S \); it is called an exterior point of \( S \) if there is some neighborhood of \( z_0 \) lying completely outside of \( S \). If \( z_0 \) is neither an interior point nor an exterior point of \( S \), it is called a boundary point of \( S \). The collection of all boundary points of \( S \), denoted by \( \partial S \), is called the boundary of \( S \).

**Exercise.** A point \( z_0 \) is a boundary point of \( S \) if and only if every neighborhood of \( z_0 \) contains a point in \( S \) and another point not in \( S \).

**Exercise.** Let \( A \) be the set \( \{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \} \). What is the boundary \( \partial A \) of \( A \) in \( \mathbb{C} \)? (Be careful!)

**Definition 3.2.** A set is open when it contains none of its boundary points. A set is closed when it contains all of its boundary points. The closure \( \bar{S} \) of a set \( S \) is the union of the set \( S \) itself with its boundary \( \partial S \), hence the closure is always closed.
Remark. Under this definition, the empty set $\emptyset$ is both open and closed!

Exercise. A set is open if and only if all of its points are interior points.

Exercise. A set $S$ is open if and only if its complement $\mathbb{C} \setminus S$ is closed. (Caution: $\partial(\mathbb{C} \setminus S)$ needs to be identified.)

Definition 3.3. An open set $S$ is connected if it cannot be written as a disjoint union of two non-empty open subsets. A connected open set is called a domain. A domain together with some, none, or all of its boundary points is referred to as a region.

Remark. A set $S$ is path connected if any pair of points in $S$ can be joined by a path completely lying in $S$. Hence when a set is open in $\mathbb{C}$, it is connected if and only if it is path connected. A detailed discussion about this notion will be more suited in a course of point set topology.

Definition 3.4. A set $S$ is bounded if every point of $S$ lies inside some circle $|z| = R$; otherwise $S$ is called unbounded.

Definition 3.5. A point $z_0$ is said to be an accumulation point of a set $S$ if each deleted neighborhood of $z_0$ contains at least one point of $S$. If a point $z_0 \in S$ is not an accumulation point of $S$, it is called an isolated point of $S$.

Exercise. What is the set of all accumulation points of the set $\{1/n \mid n \in \mathbb{N}\}$ in $\mathbb{C}$?
4 Functions of a complex variable

In this section, we will define functions whose variables could assume complex values. It should be noted first that not all functions that were encountered before can be done in this way. One has to be very careful on those functions.

We shall emphasize on the concept of domains of functions of a complex variable. The domain of a function is the set where the function is defined. When the domain of definition is not mentioned, we agree that the largest possible set is to be taken. Two functions are considered the same only when they possess the same domain and their values are identical at every point in that domain. This distinction will be important when we discuss analytic continuation later this year.

Suppose \( w = u + iv \) is the value of a function \( f \) at \( z = x + iy \), that is,
\[
u + iv = f(x + iy).
\]
Using the identification \( \mathbb{C} \simeq \mathbb{R}^2 \), we can sometimes think of \( f \) as a function from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \):
\[
(u, v) = f(x, y) = (u(x, y), v(x, y)),
\]
or equivalently,
\[
f(z) = f(x + iy) = u(x, y) + iv(x, y).
\]

Example. Take \( f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy \). Then the above identification produces two functions \( u, v \) of two real variables as
\[
u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy.
\]

At this stage, it is clear that we can consider polynomials in the variables \( z \) and \( \bar{z} \). Furthermore, rational functions, which are quotients of two such polynomials, are defined where the denominator does not vanish. But other elementary functions like trigonometric functions and exponential functions cannot be defined in the conventional way (ask yourself: what is \( \sin i \)? We don’t have the Euler formula at hand yet!) The answer to this question will be postponed until we have enough tools.

5 Limits and continuity

Definition 5.1. A function \( f(z) \) of a complex variable \( z \) is said to have the limit \( A \) as \( z \) tends to \( z_0 \), denoted by
\[
\lim_{z \to z_0} f(z) = A, \quad (5.1)
\]
if and only if for every \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that \( |f(z) - A| < \varepsilon \) whenever \( z \) lies in the deleted \( \delta \)-neighborhood \( B'(z_0, \delta) \) of \( z_0 \), that is, \( 0 < |z - z_0| < \delta \). A function \( f(z) \) is said to be continuous at \( z_0 \) if and only if
\[
\lim_{z \to z_0} f(z) = f(z_0). \quad (5.2)
\]

A continuous function is a function which is continuous at all points where it is defined.

With the complex plane \( \mathbb{C} \) equipped with the topology induced by the norm \( |\cdot| \) introduced in Section 1, it is an easy exercise to prove the following theorem.

Theorem 5.2. Suppose that
\[
f(z) = u(x, y) + iv(x, y), \quad z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0.
\]
Then
\[
\lim_{z \to z_0} f(z) = w_0
\]
if and only if
\[
\lim_{(x, y) \to (x_0, y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x, y) \to (x_0, y_0)} v(x, y) = v_0.
\]
Proof. The topology on \( \mathbb{C} \) is the same as the product topology on \( \mathbb{R}^2 \).

As one can expect, there are properties concerning limits which are just the same as those in real-valued functions.

**Theorem 5.3.** Suppose that
\[
\lim_{z \to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z \to z_0} F(z) = W_0.
\]

Then,
\[
\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0, \\
\lim_{z \to z_0} [f(z) \cdot F(z)] = w_0 \cdot W_0,
\]
and if \( W_0 \neq 0 \),
\[
\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0}.
\]

The concept of continuity is the same as that in advanced calculus. We will simply state the definition.

**Definition 5.4.** A function \( f \) of a complex variable is **continuous** at a point \( z = a \) when
\[
\lim_{z \to a} f(z) = f(a).
\]

Let \( \Omega \) be an open subset of \( \mathbb{C} \). We say that \( f \) is continuous in \( \Omega \) if and only if it is continuous at every point in \( \Omega \).

There is a concept of infinity in the course of complex variables. In the space of real variables, we have two such notions, \( +\infty \) and \( -\infty \), denoting the infinities for the positive and negative directions, respectively. On the contrary, there is only one infinity \( \infty \) in the extended complex plane, which is obtained by the so-called **stereographic projection**. Consider the unit sphere \( \partial B(0,1) \) in \( \mathbb{R}^3 \) and think of the complex plane as the plane passing through the equator of the sphere. Every point \( z \) in the complex plane can be joined with the north pole of the sphere by a straight line, which intersects at another point on the sphere. The correspondence can be reversed from any point on the sphere other than the north pole to a point in the complex plane. The exception, which is thenorth pole, then corresponds to what we think of the **point at infinity** \( \infty \) on the extended complex plane. Since the space of the unit sphere is homogeneous, the topology near \( \infty \) is no other than that near any other points.

**Definition 5.5.** We say that a complex-valued function \( f \) has the limit \( \infty \) near the point \( z = a \), denoted by
\[
\lim_{z \to a} f(z) = \infty,
\]
if, for any positive number \( R > 0 \), there is a \( \delta > 0 \) such that
\[
0 < |z - a| < \delta \quad \implies \quad |f(z)| > R.
\]

**Definition 5.6.** We say that a complex-valued function \( f \) has the limit \( L \) near the point of infinity \( z = \infty \), denoted by
\[
\lim_{z \to \infty} f(z) = L,
\]
if, for any \( \varepsilon > 0 \), there is an \( R > 0 \) such that
\[
|z| < R \quad \implies \quad |f(z) - L| < \varepsilon.
\]

**Exercise.** Formulate the precise definition of the following notion
\[
\lim_{z \to \infty} f(z) = \infty.
\]
Lecture note on Complex Variables

6 Derivatives

The derivative of a function of a complex variable is defined in the usual way, but there is some tricky part as we investigate the limit.

Definition 6.1. The derivative of a function \( f(z) \) of a complex variable at the point \( z = a \) is defined as

\[
f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}, \tag{6.1}
\]

provided that the limit exists, and we say that \( f \) is differentiable at the point \( a \). When \( \Omega \) is an open subset of \( \mathbb{C} \) and \( f \) is differentiable at every point in \( \Omega \), we will say that \( f \) is analytic on \( \Omega \).

Example. The derivative of a monomial \( f(z) = z^n \) for \( n \in \mathbb{N} \) is

\[
f'(z) = nz^{n-1}.
\]

In the case of a real variable, a point \( x \) can approach some fixed point \( a \) on the real line only from two directions, left and right. However this is not true anymore in the case of a complex variable. To illustrate what really happens, we make a change of variable to reach an equivalent definition of Equation (6.1):

\[
f'(a) := \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{6.2}
\]

We also write the real and the imaginary parts of \( f \) as \( f = u + iv \) since the operation of taking limits is linear over \( \mathbb{C} \). If \( h \) is real, then

\[
\lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \tag{6.3}
\]

However, if \( ik \) is purely imaginary, then

\[
\lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{ik} = -i \left( \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{6.4}
\]

Comparing the real and the imaginary parts of Equations (6.3) and (6.4), we see that a necessary condition for a function \( f \) to be differentiable is

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6.5}
\]

These equations are called the Cauchy-Riemann equations, abbreviated as the CR equations.

Example. The monomial \( g(z) = \bar{z}^n \) can only be differentiated at \( z = 0 \) when \( n \geq 2 \). The function \( h(z) = \bar{z} \) cannot be differentiated anywhere.

On the other hand, with a stronger assumption we can now prove its converse.

Proposition 6.2. If \( u(x, y) \) and \( v(x, y) \) have continuous first-order partial derivatives which satisfy Equation (6.5), then \( f(z) = u(z) + iv(z) \) is analytic with continuous derivative \( f'(z) \).

Proof. For sufficiently small \( h, k > 0 \), we can write

\[
u(x + h, y + k) = \frac{\partial u}{\partial x} h + \frac{\partial u}{\partial y} k + \varepsilon_1,
\]

\[
v(x + h, y + k) = \frac{\partial v}{\partial x} h + \frac{\partial v}{\partial y} k + \varepsilon_2,
\]

where \( \varepsilon_1, \varepsilon_2 \) are negligible.
where the remainders $\varepsilon_1$, $\varepsilon_2$ tend to zero more rapidly than $h + i k$ in the sense that

$$\lim_{h + i k \to 0} \frac{\varepsilon_1}{\sqrt{h^2 + k^2}} = 0 = \lim_{h + i k \to 0} \frac{\varepsilon_2}{\sqrt{h^2 + k^2}}.$$ 

With the notion $f(z) = u(x, y) + i v(x, y)$, we obtain by virtue of Equation (6.5)

$$f(z + h + i k) - f(z) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right)(h + i k) + (\varepsilon_1 + i \varepsilon_2)$$

and hence

$$\lim_{h + i k \to 0} \frac{f(z + h + i k) - f(z)}{h + i k} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$ 

We then conclude $f(z)$ is analytic.

It should be noted that the hypothesis in Proposition 6.2 can be weakened, but it will not be pursued here. We shall prove later that the derivative of an analytic function is itself analytic, that is, it can be differentiated again. Hence $u$ and $v$ will have continuous partial derivatives of all orders, and in particular the mixed derivatives are equal. If we let $\Delta = \partial^2_{xx} + \partial^2_{yy}$ denote the Laplace operator in $\mathbb{R}^2$, we obtain by the Cauchy-Riemann equations (6.5):

$$\Delta u = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{\partial v}{\partial x} \right) = 0.$$ 

Similarly $\Delta v = 0$ as well. A function $u$ that satisfies the Laplace’s equation $\Delta u = 0$ is said to be harmonic. Hence the previous argument shows that the real and the imaginary parts of an analytic function are harmonic. If two harmonic functions $u$ and $v$ satisfy the Cauchy-Riemann equations (6.5), then $v$ is said to be a harmonic conjugate of $u$.

**Exercise.** What happens if $v$ is a harmonic conjugate of $u$ and also $u$ is a harmonic conjugate of $v$?

Recall that if $z = x + iy$, then

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}.$$ 

Motivated by these expressions, we define the differential operator

$$\partial_z = \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

One then checks that a function $f(z) = u(x, y) + i v(x, y)$ satisfies the Cauchy-Riemann equations if and only if $f$ lies in the kernel of the operator $\partial_z$, that is,

$$\partial_z f = \frac{\partial f}{\partial z} = \frac{1}{2} \left[ (u_x - v_y) + i(v_x + u_y) \right] = 0.$$ 

**Exercise.** Define another differential operator

$$\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$ 

Show by direct computation that the Laplace operator $\Delta$ can be written as

$$\Delta f = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f = 4 \frac{\partial^2}{\partial z \partial \bar{z}} f$$

when $f$ is a function with continuous derivatives up to order 2.
Lecture note on Complex Variables

7 The exponential function, the trigonometric functions and the hyperbolic functions

Consider the function \( f(x) = \exp(x) = e^x \) for the real variable \( x \). This function is smooth and satisfies the property:

\[
f'(x) = f(x), \quad f(0) = 1.
\]

By the existence and uniqueness in the theory of ODE, we know that this is the unique function with the above property. Motivated from the real-variable case, we seek a function in a complex variable with the same property:

\[
f'(z) = f(z) \quad \text{for all } z, \quad f(0) = 1.
\] (7.1)

**Definition 7.1.** Define the **exponential function**

\[
f(z) = f(x + iy) = e^x (\cos y + i \sin y).
\] (7.2)

Then \( f(z) \) is the unique function which satisfies (7.1).

In short, the exponential function will be denoted by \( \exp z \), or simply \( e^z \). Note that the exponent function has a period of \( 2\pi i \), that is, \( \exp(z) = \exp(z + 2n\pi i) \) for every integer \( n \in \mathbb{Z} \).

Obviously \( f(z) \) satisfies the initial condition \( f(0) = 1 \). Now we check the differentiability: set \( u(x,y) = e^x \cos y \) and \( v(x,y) = e^x \sin y \), we obtain

\[
\begin{align*}
u_x &= v_y = e^x \cos y, \\
u_y &= -v_x = -e^x \sin y.
\end{align*}
\]

Since all of first-order partial derivatives are continuous, we conclude that \( f \) is analytic everywhere by Proposition 6.2. And its derivative is

\[
f'(z) = u_x + iv_x = (e^x \cos y) + i(e^x \sin y) = f(z).
\]

Suppose there is another function \( g(z) \) satisfying (7.1). Then the derivative of \( f(z) - g(z) \) is zero, hence \( f(z) - g(z) \) must be a constant by the mean value theorem. Using the initial condition at \( z = 0 \) we conclude that \( g(z) \) must be identical to \( f(z) \), which proves the uniqueness.

The expression has the advantage of expressing the multiplication in polar coordinates. As suggested, we have

\[
e^{z+w} = e^z \cdot e^w.
\]

One can easily verify this identity under the polar coordinates.

It is now not surprising to reach the following definition.

**Definition 7.2 (Euler’s formula).** For a real number \( \theta \), the value \( e^{i\theta} \) is defined as

\[
e^{i\theta} = \cos \theta + i \sin \theta.
\]

Now it is time to define the trigonometric functions for a complex variable. Using some elementary algebra, we have the following definition.

**Definition 7.3.** The **trigonometric functions** are defined by

\[
\begin{align*}
\cos z &= \frac{e^{iz} + e^{-iz}}{2}, \\
\sin z &= \frac{e^{iz} - e^{-iz}}{2i}.
\end{align*}
\]

It is then an easy exercise to find the derivatives of these functions. We state here a proposition as a reference.

**Proposition 7.4.** The derivatives of \( \cos z \) and \( \sin z \) are

\[
D(\cos z) = -\sin z, \quad D(\sin z) = \cos z.
\] (7.3)
The other trigonometric functions \( \tan z, \cot z, \sec z \) and \( \csc z \) are defined in the same way as before, and their derivatives can be found likewise.

**Exercise.** Use Equation (7.3) to show that \( \sin^2 z + \cos^2 z = 1 \) for all \( z \in \mathbb{C} \).

**Definition 7.5.** The hyperbolic functions are defined as

\[
\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}.
\]

We list a few identities whose verifications are left as exercise.

**Exercise.**

1. \( D(\cosh z) = \sinh z \); \( D(\sinh z) = \cosh z \).
2. \( \cosh^2 z - \sinh^2 z = 1 \).
3. \( \sinh z = \sinh x \cos y + i \cosh x \sin y \); \( \cosh z = \cosh x \cos y + i \sinh x \sin y \).
4. \( |\sinh z|^2 = \sinh^2 x + \sin^2 y \); \( |\cosh z|^2 = \sinh^2 x + \cos^2 y \).

Also the other hyperbolic functions are defined as usual, which will be omitted here.

### 8 The logarithmic function

The logarithmic function is supposed to be the inverse function of the exponential function. However from the discussion earlier, we realize that the equation \( e^w = z \) has infinitely many complex solutions \( w \) for any non-zero \( z \). Since we do not have any preference from one to another, we define the logarithmic “function” as follows.

**Definition 8.1.** The logarithmic function is defined as the set

\[
\log z := \{ w \in \mathbb{C} \mid e^w = z \}
\]

for any nonzero complex number \( z \).

Indeed, this is not a function under the traditional sense. We call such things multi-valued functions, and the study of multi-valued functions in complex analysis will lead to the concept of Riemann surfaces. Basically, one of the motivations for Riemann surfaces is to find the right domains and ranges such that a multi-valued holomorphic function becomes single-valued over them.

To solve the equation \( e^w = z \), we first write the non-zero complex number \( z \) under the polar coordinates:

\[
z = re^{i\Theta}.
\]

Then we see that the solution \( w \) can be expressed as

\[
w = \ln r + i(\Theta + 2n\pi) \quad \text{for any } n \in \mathbb{Z}
\]

\[
= \ln |z| + i \arg z.
\]

Therefore we write

\[
\log z = \ln |z| + i \arg z. \tag{8.1}
\]

The principal value of \( \log z \) is obtained from replacing \( \arg z \) by \( \text{Arg} z \) in Equation (8.1), hence

\[
\text{Log } z = \ln |z| + i \text{Arg } z.
\]

From this we know that \( 0 \leq \text{Im } \text{Log } z < 2\pi \). Furthermore, the map

\[
\text{Log} : \mathbb{C} \setminus \{0\} \to \{ w \in \mathbb{C} \mid 0 \leq \text{Im } w < 2\pi \}
\]

becomes a single-valued function. Sometimes we call this function the principal branch of the logarithmic function.
Now we ask whether the logarithmic function is differentiable or not. Since differentiability is a local property, we can restrict the range of \( \log z \) to any branch: \( \alpha < \arg z < \alpha + 2\pi \). First we call the Cauchy-Riemann equations under the polar coordinates.

**Exercise.** If \( f(z) = f(r e^{i\theta}) = u(r, \theta) + i v(r, \theta) \), then the Cauchy-Riemann equations are given by

\[
\begin{align*}
    u_r &= \frac{1}{r} v_{\theta}, \\
    \frac{1}{r} u_{\theta} &= -v_r.
\end{align*}
\]  

(8.2)

It is then an easy verification that \( u(r, \theta) = \ln r \), \( v(r, \theta) = \theta \) for \( \log z \) satisfy the Cauchy-Riemann equations (8.2).

We now use the chain rule to compute the derivative of \( \log z \):

\[
\exp(\log z) = z \exp(\log z) \cdot D(\log z) = 1 \quad \text{(take } D \text{ on both sides)}
\]

\[
D(\log z) = \frac{1}{\exp(\log z)} = \frac{1}{z} \quad (|z| > 0, \alpha < \arg z < \alpha + 2\pi).
\]

**9 Complex exponents**

When \( z \neq 0 \) and \( c \) is any complex number, the function \( z^c \) is defined by means of the equation

\[
z^c := \exp(c \log z). \tag{9.1}
\]

Note that \( z^c \) is now a multi-valued function, for we have many choices for \( \log z \).

**Example.** We compute

\[
i^{-2i} = \exp(-2i \log i) = \exp(-2i(2n + \frac{1}{2})\pi i) = \exp((4n + 1)\pi)
\]

for all \( n \in \mathbb{Z} \). If we specify a branch for \( z^c \), then its derivative can be found as

\[
\frac{d}{dz}(z^c) = \frac{d}{dz}(\exp(c \log z)) = \exp(c \log z) \cdot \frac{c}{z} = c \cdot \frac{\exp(c \log z)}{\exp(\log z)} = c \exp((c - 1) \log z) = cz^{c-1}.
\]

Note that everything appearing above is a single-valued function once we choose a branch once and for all. Likewise, the principal branch of \( z^c \) is obtained by choosing the principal branch \( \log z \) in Equation (9.1). One should also note that the \( n \)th root of a non-zero complex number \( z \) is defined in this way, that is,

\[
\sqrt[n]{z} = z^{\frac{1}{n}} = \exp\left(\frac{1}{n} \log z\right).
\]

It can be verified that \( \sqrt[n]{z} \) is an \( n \)-to-1 function.

According to Equation (9.1), the exponential function with base \( c \), where \( c \) is any non-zero complex constant, is written as

\[
c^z := \exp(z \log c). \tag{9.2}
\]

Again \( c^z \) is a multi-valued function. If we specify a branch for \( \log c \), the derivative of \( c^z \) with respect to \( z \) is computed as

\[
\frac{d}{dz}(c^z) = \frac{d}{dz}(\exp(z \log c)) = \exp(z \log c) \cdot \log c = c^z \log c.
\]

**10 Inverse trigonometric and hyperbolic functions**

Inverses of the trigonometric and hyperbolic functions are described in terms of logarithms. We will illustrate them by an example.

**Example.** We want to find \( \sin^{-1} z \). By definition for the inverse function, \( \sin^{-1} z \) are those complex numbers \( w \) satisfying \( \sin w = z \). Hence

\[
\frac{e^{iw} - e^{-iw}}{2i} = z. \tag{10.1}
\]
The equation (10.1) is in fact quadratic in $e^{iw}$, since multiplying $e^{iw}$ on the both sides of the equation yields

$$(e^{iw})^2 - 2iz e^{iw} - 1 = 0.$$ 

Solving for $e^{iw}$ by using the quadratic formula, we see that

$$e^{iw} = iz + \sqrt{-z^2 + 1}$$

so that

$$\sin^{-1} z = w = -i \log(iz + \sqrt{1 - z^2}).$$

Here, of course, both $\sqrt{1 - z^2}$ and the logarithm are multi-valued functions.

Other inverse trigonometric and hyperbolic functions are found in similar ways, also one can work out their derivatives once a branch has been chosen. We will leave them for interested readers.
Lecture note on Complex Variables, Appendix A

We make up the proof for the uniqueness of the exponential function here. Suppose $g$ is a function satisfying $g'(z) = g(z)$ and $g(0) = 1$. Define another function

$$h(z) = \exp(-z) \cdot g(z).$$

The initial condition is $h(0) = \exp(0) \cdot g(0) = 1$. Since $h$ is a product of two differentiable functions, $h$ itself is differentiable as well. We find

$$h'(z) = -\exp'(-z) \cdot g(z) + \exp(-z) \cdot g'(z) = 0.$$

Therefore $h$ is a constant function, which is 1 from the initial condition. At last,

$$g(z) = \frac{1}{\exp(-z)} = \exp(z).$$

So the uniqueness is proved.
11 Contour integrals

In the following sections we will be discussing integrals of complex-valued functions over contours. Before the formal definition of contour integrals, we start with defining integrals of complex-valued functions of a real variable. Suppose \( w : [a, b] \rightarrow \mathbb{C} \) is a complex-valued function. Write

\[
w(t) = u(t) + iv(t), \quad u, v : [a, b] \rightarrow \mathbb{R}.
\]

Then we define the integral of \( w(t) \) over the interval \( [a, b] \) as

\[
\int_{a}^{b} w(t) \, dt := \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt
\]

provided that both integrals on the right-hand side exist. Indeed, if one performs either Riemann sums or Lebesgue integrals formally to the function \( w(t) \), the operation is actually linear on the integrand over \( \mathbb{C} \). Therefore this definition is desired.

**Example.** As an illustration,

\[
\int_{0}^{1} (1 + it)^2 \, dt = \int_{0}^{1} (1 - t^2) \, dt + i \int_{0}^{1} 2t \, dt = \frac{2}{3} + i.
\]

The fundamental theorem of calculus, involving antiderivatives, can be extended so as to apply the integrals defined here. Namely, suppose that the functions \( w(t) = u(t) + iv(t) \) and \( W(t) = U(t) + iV(t) \) are continuous on the interval \( [a, b] \). If \( W'(t) = w(t) \) when \( a \leq t \leq b \), then \( U'(t) = u(t) \) and \( V'(t) = v(t) \) (beware of the differentiation here!) Hence,

\[
\int_{a}^{b} w(t) \, dt = \int_{a}^{b} u(t) \, dt + i \int_{a}^{b} v(t) \, dt = [U(b) - U(a)] + i[V(b) - V(a)] = W(b) - W(a).
\]

**Proposition 11.1.** Let \( w(t) = u(t) + iv(t) \) be a complex-valued function which is integrable over the interval \([a, b]\). Then,

\[
\left| \int_{a}^{b} w(t) \, dt \right| \leq \int_{a}^{b} |w(t)| \, dt. \tag{11.1}
\]

**Proof.** The inequality (11.1) clearly holds when the left-hand side is zero. Thus, in the verification, we may assume that \( \int_{a}^{b} w(t) \, dt = r_0 e^{i\theta_0} \) for some \( r_0 > 0 \). We rewrite as

\[
\frac{r_0}{e^{-i\theta_0}} = \int_{a}^{b} e^{-i\theta_0} w(t) \, dt.
\]

The left-hand side of this equation is a real number, hence so is the right-hand side. Furthermore,

\[
\frac{r_0}{e^{-i\theta_0}} = \text{Re} \left( \int_{a}^{b} e^{-i\theta_0} w(t) \, dt \right) = \int_{a}^{b} \text{Re} \left( e^{-i\theta_0} w(t) \right) \, dt.
\]

Now we use the inequality on the complex number \( e^{-i\theta_0} w(t) \):

\[
\text{Re} \left( e^{-i\theta_0} w(t) \right) \leq |e^{-i\theta_0} w(t)| = |w(t)|,
\]
Notice that the value the latter notation often being used when the value of the integral is dependent only on the endpoints \(z\) in terms of the values \(f\) in Definition 11.2.

Jordan curve theorem, \(C\) of two distinct domains, one of which is the interior of \(C\), is well-defined with angle of inclination \(\arg z\) as geometrically evident. The proof, a deep result in the course of topology, will not be pursued here.

We now turn to the integrals of complex-valued functions \(f\) of a complex variable \(z\). Such an integral is defined in terms of the values \(f(z)\) along a given contour \(C\), therefore it is a \textit{line integral}. The integral is written as

\[
\int_C f(z) \, dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z) \, dz,
\]

the latter notation often being used when the value of the integral is dependent only on the endpoints \(z_1, z_2\) of the contour \(C\).

**Definition 11.3.** \textit{The contour integral} of a complex-valued \(f\) over a contour \(C\) is obtained by

\[
\int_C f(z) \, dz := \int_a^b f(z(t)) \, d[z(t)], \quad (11.2)
\]

where \(z: [a, b] \to \mathbb{C}\) is a parametrization of the contour \(C\), and \(d[z(t)] = z'(t) \, dt\).

Again the value of the contour integral is independent of parametrization of \(C\) by the discussion above.
Proposition 11.4.  
(1) Let $-C$ denote the orientation-reversed contour of a contour $C$. Then
\[ \int_{-C} f(z) \, dz = - \int_C f(z) \, dz. \]

(2) If a contour $C$ consists of a contour $C_1$ from $z_0$ to $z_1$ followed by another contour $C_2$ from $z_1$ to $z_2$, then the contour integral of $f$ over $C$ is the sum of those over $C_1$ and $C_2$, that is,
\[ \int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz. \]

Proof. (2) is clear. For (1), suppose that $z : [a, b] \to \mathbb{C}$ is a parametrization of $C$. Then $-C$ can be parameterized by $w(t) := z(a + b - t)$ over the interval $[a, b]$. Hence
\[ \int_{-C} f(z) \, dz = \int_a^b f(w(t)) w'(t) \, dt = - \int_a^b f(z(a + b - t)) z'(a + b - t) \, dt = \int_a^b f(z(u)) z'(u) \, du = - \int_C f(z) \, dz. \]

Finally, we assume that for some fixed positive number $M$, $|f(z)| \leq M$ for all points $z$ in the contour $C$. According to (11.1), we have
\[ \left| \int_C f(z) \, dz \right| \leq \int_a^b |f(z(t))z'(t)| \, dt \leq M \int_a^b |z'(t)| \, dt = ML, \tag{11.3} \]
where $L$ is the length of the contour $C$ (provided it is finite.) This inequality will be of great use when we estimate various integrals later on.

Example. Let $C$ be the contour defined by the semicircle from $z = -2i$ to $z = 2i$ along the circle $|z| = 2$, going counterclockwise (we will say the orientation of a circle is positive if it is going counterclockwise.) Let us find the value of the contour integral
\[ I = \int_C \bar{z} \, dz. \]
We first choose a parametrization of $C$ by $z = 2e^{it}, -\pi/2 \leq t \leq \pi/2$. Then the contour integral is computed as follows:
\[ I = \int_{-\pi/2}^{\pi/2} 2e^{it} \, d(2e^{it}) = \int_{-\pi/2}^{\pi/2} 2e^{-it} \cdot 2ie^{it} \, dt = \int_{-\pi/2}^{\pi/2} 4i \, dt = 4\pi i. \]

Example. Let $C_R$ be the semicircular path $z = Re^{i\theta}, \theta \in [0, \pi]$, and $z^{1/2}$ denote the branch of the square root when we fix $\text{arg} z \in (-\pi/2, 3\pi/2)$. We want to show that
\[ \lim_{R \to \infty} \int_{C_R} \frac{z^{1/2}}{z^2 + 1} \, dz = 0. \tag{11.4} \]
For $z = Re^{i\theta}$, $|z| = R$ which we assume to be greater than, say 1029. Then $|z^{1/2}| = \sqrt{R}$, and $|z^2 + 1| \geq R^2/2$. Consequently, at points on $C_R$ where the integrand is defined,
\[ \left| \frac{z^{1/2}}{z^2 + 1} \right| \leq \frac{\sqrt{R}}{R^2/2} = \frac{2}{R^{3/2}} =: M_R. \]
Since the length of the contour $C_R$ is $L_R = \pi R$, it follows from (11.3) that
\[ \left| \int_{C_R} \frac{z^{1/2}}{z^2 + 1} \, dz \right| \leq M_R L_R = \frac{2\pi}{\sqrt{R}} \to 0 \]
as $R \to \infty$. Hence the limit (11.4) is proved.
Lecture note on Complex Variables

12 Antiderivatives

As the notation suggests, the contour integral
\[ \int_{z_1}^{z_2} f(z) \, dz \]
might be independent of the path once the endpoints \( z_1, z_2 \) are fixed. This does not always happen, but it holds under certain circumstances. We will investigate the situations in this section.

**Definition 12.1.** Let \( D \) be a domain in \( \mathbb{C} \) and \( f \) be a function in \( D \). We say that \( F \) is an antiderivative of \( f \) when \( F'(z) = f(z) \) for all \( z \in D \).

Notice that an antiderivative is not unique if it exists. For if \( F_1 \) and \( F_2 \) are antiderivatives of a function \( f \), then the derivative \( (F_1 - F_2)'(z) = F_1'(z) - F_2'(z) = f(z) - f(z) = 0 \) for all \( z \in D \), hence \( F_1(z) - F_2(z) \) is a constant. This property is the same as that in elementary calculus.

**Theorem 12.2.** Suppose that a function \( f \) is continuous on a domain \( D \) in \( \mathbb{C} \). The following are equivalent:

(a) \( f \) has an antiderivative \( F \) in \( D \).

(b) The integrals of \( f(z) \) along contours lying entirely in \( D \) and extending from any fixed point \( z_1 \) to another fixed point \( z_2 \) all have the same value.

(c) The integrals of \( f(z) \) around closed contours lying entirely in \( D \) all have value 0.

Notice that this theorem does not claim that any of these statements is true for a given function \( f \) in a given domain \( D \). It only says that if any one of the statements is true, so are the others.

**Proof.** (a) \( \Rightarrow \) (b): We suppose that \( F'(z) = f(z) \) for all points \( z \in D \). If a contour \( C \) from \( z_1 \) to \( z_2 \) is differentiable, then it can be parametrized by a differentiable function \( z: [a, b] \rightarrow D \) with \( z(a) = z_1 \) and \( z(b) = z_2 \). According to the chain rule, we have
\[ \frac{d}{dt} [F(z(t))] = F'(z(t)) z'(t) = f(z(t)) z'(t). \]

Therefore the contour integral of \( f \) over \( C \) is computed as
\[ \int_C f(z) \, dz = \int_a^b f(z(t)) z'(t) \, dt = \int_a^b \frac{d}{dt} [F(z(t))] \, dt \]
\[ = F(z(t)) \bigg|_a^b = F(z(b)) - F(z(a)) = F(z_2) - F(z_1) \]
by the fundamental theorem of calculus. Notice that this value is evidently independent of the contour \( C \) as long as \( C \) connects \( z_1 \) to \( z_2 \) and lies entirely in \( D \).

In general, let \( C \) be a piecewise differentiable contour from \( z_1 \) to \( z_2 \). Then \( C \) consists of a finite number of differentiable contours \( C_k \) \((k = 1, \ldots, n)\), each \( C_k \) extending from \( w_{k-1} \) to \( w_k \) \((w_0 = z_1, w_n = z_2)\). Then
\[ \int_C f(z) \, dz = \sum_{k=1}^n \int_{C_k} f(z) \, dz = \sum_{k=1}^n (F(w_k) - F(w_{k-1})) = F(w_n) - F(w_0) = F(z_2) - F(z_1). \]

Again, this value does not depend on how \( C \) connects \( z_1 \) and \( z_2 \).

(b) \( \Leftrightarrow \) (c): Suppose (b) holds first. We let \( z_1 \) and \( z_2 \) be any two points on a closed contour \( C \) in \( D \) and form two paths \( C_1, C_2 \), each with the initial point \( z_1 \) and the final point \( z_2 \), such that \( C = C_1 - C_2 \). By (b), we have
\[ \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \]
because they have the same starting and end points. Therefore
\[ \int_C f(z) \, dz = \int_{C_1-C_2} f(z) \, dz = \int_{C_1} f(z) \, dz - \int_{C_2} f(z) \, dz = 0. \]

On the other hand, if (c) holds, then for any pair of points \( z_1 \) and \( z_2 \) in \( D \), we may choose two contours \( C_1 \) and \( C_2 \) in \( D \) connecting \( z_1 \) to \( z_2 \). Then \( C_1 - C_2 \) is a closed contour that lies entirely in \( D \), hence \( \int_{C_1-C_2} f(z) \, dz = 0 \), or \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \). Therefore (b) is proved.

(b) \( \Rightarrow \) (a): Now suppose (b) is true. We first fix a point \( z_0 \) in \( D \), and define
\[ F(z) := \int_{z_0}^z f(s) \, ds, \quad z \in D. \]

Notice that \( F \) is well-defined because the integral is independent of path by (b). Now we show that \( F'(z) = f(z) \). Suppose \( z + \Delta z \) is any point, distinct from \( z \), lying in some neighborhood of \( z \) that is sufficiently small to be contained in \( D \). Then
\[ F(z + \Delta z) - F(z) = \int_{z_0}^{z + \Delta z} f(s) \, ds - \int_{z_0}^{z} f(s) \, ds = \int_{z}^{z + \Delta z} f(s) \, ds \]
where the path of integration from \( z \) to \( z + \Delta z \) can be chosen as a line segment. Since
\[ \int_{z}^{z + \Delta z} ds = \Delta z, \]
it follows that
\[ \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z + \Delta z} [f(s) - f(z)] \, ds. \]

By hypothesis \( f \) is continuous at the point \( z \). Hence, for each positive number \( \varepsilon \), there is a positive number \( \delta \) such that
\[ |f(s) - f(z)| < \varepsilon \]
whenever \( |s - z| < \delta \). Consequently, if the point \( z + \Delta z \) lies in the \( \delta \)-neighborhood of \( z \) (which can be shrunk to be contained in \( D \) if necessary), then
\[ \left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| < \frac{1}{|\Delta z|} \cdot \varepsilon |\Delta z| = \varepsilon. \]

This is exactly the condition for the limit
\[ \lim_{\Delta z \to 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = f(z), \]
or, \( F'(z) = f(z) \).

**Example.** The function \( 1/z^2 \), which is continuous everywhere except at the origin, has an antiderivative \(-1/z\) in the domain \( |z| > 0 \). Consequently,
\[ \int_{z_1}^{z_2} \frac{dz}{z^2} = -\frac{1}{z} \bigg|_{z_1}^{z_2} = \frac{1}{z_2} - \frac{1}{z_1} \quad (z_1 \neq 0, z_2 \neq 0) \]
for any contour from \( z_1 \) to \( z_2 \) that does not pass through the origin. In particular,
\[ \int_C \frac{dz}{z^2} = 0 \]
when \( C \) is any circle \( z = re^{i\theta} \) \( (r > 0, -\pi \leq \theta \leq \pi) \) about the origin.

**Example.** Let \( f(z) = 1/z \) on \( \mathbb{C} \setminus \{0\} \) and \( C_R \) be the closed contour \( |z| = R \) with positive orientation. To compute \( \int_{C_R} f(z) \, dz \), we parametrize \( C_R \) as \( z(t) = Re^{i\theta}, 0 \leq \theta \leq 2\pi \). Then the contour integral is
\[ \int_{C_R} f(z) \, dz = \int_0^{2\pi} \frac{Re^{i\theta}}{Re^{i\theta}} \, d\theta = \int_0^{2\pi} i \, d\theta = 2\pi i. \]
In particular this is a nonzero number. Hence the function \( f \) does not have an antiderivative on \( \mathbb{C} \setminus \{0\} \). However, if we remove a ray emitting from the origin, \( f \) does have an antiderivative which is a branch of the logarithmic function we have seen in Section 8.

**Example.** Let us use an antiderivative to evaluate the integral
\[
\int_{C_1} z^{1/2} \, dz,
\]
where the integrand is the branch
\[
z^{1/2} = \sqrt{r} e^{i\theta/2} \quad (r > 0, 0 < \theta < 2\pi)
\]
of the square root function and where \( C_1 \) is any contour from \( z = -3 \) to \( z = 3 \) that, except for its endpoints, lies above the real axis. Although the integrand is not defined on the ray \( \theta = 0 \), it is still piecewise continuous on \( C_1 \), and the integral therefore exists. Indeed, we can use another branch
\[
f_1(z) = \sqrt{r} e^{i\theta/2} \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}),
\]
which is defined and continuous everywhere on \( C_1 \). The values of \( f_1(z) \) at all points on \( C_1 \) except \( z = 3 \) coincide with those of our integrand (12.2); so the integrand can be replaced by \( f_1(z) \). Since an antiderivative of \( f_1(z) \) is the function
\[
F_1(z) = \frac{2}{3} r^{3/2} e^{i3\theta/2}, \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}),
\]
we can now write
\[
\int_{C_1} z^{1/2} \, dz = \int_{-3}^{3} f_1(z) \, dz = F_1(z) \bigg|_{-3}^{3} = 2\sqrt{3}(1 + i).
\]

Integral (12.1) over any contour \( C_2 \) that extends from \( z = -3 \) to \( z = 3 \) below the real axis has another value. In this case, we can replace the integrand by the branch
\[
f_2(z) = \sqrt{r} e^{i\theta/2} \quad (r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}),
\]
whose values coincide with those of the branch (12.2) in the lower half plane. The analytic function
\[
F_2(z) = \frac{2}{3} r^{3/2} e^{i3\theta/2}, \quad (r > 0, \frac{\pi}{2} < \theta < \frac{5\pi}{2}),
\]
is an antiderivative of \( f_2(z) \). Thus,
\[
\int_{C_2} z^{1/2} \, dz = \int_{-3}^{3} f_2(z) \, dz = F_2(z) \bigg|_{-3}^{3} = 2\sqrt{3}(-1 + i).
\]
Notice how it follows that the integral of the function (12.2) around the closed contour \( C_2 - C_1 \) has the value
\[
2\sqrt{3}(-1 + i) - 2\sqrt{3}(1 + i) = -4\sqrt{3}.
\]

### 13 Cauchy-Goursat theorem

Let \( C \) be a simple closed contour \( z = z(t) \) (\( a \leq t \leq b \)), positively oriented (counterclockwise,) and we assume that \( f \) is analytic at any point interior to and on \( C \). Then the contour integral is computed as
\[
\int_C f(z) \, dz = \int_a^b f[z(t)] z'(t) \, dt.
\]
If we write \( f(z) = u(x, y) + iv(x, y) \) and \( z(t) = x(t) + iy(t) \), then Equation (13.1) can be written as
\[
\int_C f(z) \, dz = \int_a^b (ux' - vy') \, dt + i \int_a^b (vx' + uy') \, dt.
\]
In terms of line integrals of real-valued functions of two real variables, then

\[
\int_C f(z) \, dz = \int_C (u \, dx - v \, dy) + i \int_C (v \, dx + u \, dy).
\]  

(13.3)

Let \( R \) be the region bounded by \( C \). If \( f \) is analytic in some neighborhood of \( R \) and \( f' \) is continuous there, the Green's theorem implies that

\[
\int_C f(z) \, dz = \iint_R (-v_x - u_y) \, dA + i \iint_R (u_x - v_y) \, dA.
\]

But, in view of the Cauchy-Riemann equations \( u_x = v_y, \; u_y = -v_x \), Cauchy deduced that \( \int_C f(z) \, dz = 0 \) in this case.
Goursat was the first to prove that the condition of continuity of \( f' \) can be omitted. Its removal is important and will allow us to show, for example, that the derivative \( f' \) of an analytic function \( f \) is again analytic without having to assume the continuity of \( f' \), which follows as a consequence. We now state the revised form of Cauchy’s result:

**Theorem 13.1 (Cauchy-Goursat theorem).** If a function \( f \) is analytic at all points interior to and on a simple closed contour \( C \), then

\[
\int_C f(z) \, dz = 0.
\]

**Proof.** There are several forms of the Cauchy-Goursat theorem, but they differ in their topological rather than in their analytical content. It is natural to begin with with a case in which the topological considerations are trivial.

We assume first that the simple closed contour \( C \) is the boundary of a rectangle \( R = [a,b] \times [c,d] \), positively oriented. For any rectangle \( S \) contained in \( R \), define

\[
\eta(S) = \int_{\partial S} f(z) \, dz.
\]

Subdividing \( R \) into four identical rectangles \( R^{(j)}, j = 1, 2, 3, 4 \), we find

\[
\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}),
\]

for integrals over the common sides cancel each other. It then follows that at least one of the rectangles \( R^{(j)}, j = 1, 2, 3, 4 \), must satisfy the condition

\[
|\eta(R^{(j)})| \geq \frac{1}{4} |\eta(R)|.
\]

Denote this rectangle by \( R_1 \); if several \( R^{(j)} \)'s have this property, the choice shall be made according to some definite rule.

This process can be repeated indefinitely, and we obtain a sequence of nested rectangles \( R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots \) with the property

\[
|\eta(R_n)| \geq \frac{1}{4} |\eta(R_{n-1})|
\]

and hence

\[
|\eta(R_n)| \geq \frac{1}{4^n} |\eta(R)|. \tag{13.4}
\]

The rectangles \( R_n \)'s converge to a point \( z^* \in R \) in the sense that \( R_n \) will be contained in a prescribed neighborhood \( B(z^*, \delta) \) as soon as \( n \) is sufficiently large. First of all, we choose \( \delta \) so small that \( f(z) \) is defined and analytic in \( B(z^*, \delta) \). Secondly, if \( \varepsilon > 0 \) is given, we can choose \( \delta \) so that

\[
\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \varepsilon
\]

or

\[
|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \varepsilon |z - z^*| \tag{13.5}
\]

for all \( z \in B(z^*, \delta) \) by analyticity. We assume that \( \delta \) satisfies both conditions and that \( R_n \) is contained in \( B(z^*, \delta) \).

We make now the observation that

\[
\int_{\partial R_n} dz = 0 = \int_{\partial R_n} z \, dz,
\]

for both functions 1 and \( z \) have antiderivatives and \( R_n \) is a closed contour. By virtue of these equations we are able to write

\[
\eta(R_n) = \int_{\partial R_n} [f(z) - f(z^*) - f'(z^*)(z - z^*)] \, dz,
\]
and it follows from Inequality (13.5) that

\[ |\eta(R_n)| \leq \varepsilon \int_{\partial R_n} |z - z^*| \cdot |dz|. \quad (13.6) \]

In the last integral \(|z - z^*|\) is at most equal to the length \(d_n\) of the diagonal of \(R_n\). If \(L_n\) denotes the length of the perimeter of \(R_n\), the integral is hence at most \(d_n L_n\). But if \(d\) and \(L\) are the corresponding quantities for the original rectangle \(R\), it is clear that \(d_n = 2^{-n}d\) and \(L_n = 2^{-n}L\). By Inequality (13.6) we have hence

\[ |\eta(R_n)| \leq 4^{-n}dL \varepsilon, \]

and comparison with Inequality (13.4) yields

\[ |\eta(R)| \leq dL \varepsilon. \]

Since \(\varepsilon\) is arbitrary, we must have \(\eta(R) = 0\).

Now we assume the next scenario that the simple closed contour \(C\) is contained in a circular disk \(\Delta\) in which \(f\) is analytic everywhere. We define a function \(F(z)\) by

\[ F(z) = \int_{\sigma} f(z) \, dz \quad (13.7) \]

where \(\sigma\) consists of the horizontal line segment from the center \((x_0, y_0)\) to \((x, y)\) and the vertical segment from \((x, y_0)\) to \((x, y)\); it is immediately seen that \(\partial F/\partial y = if(z)\). On the other hand, by the previous case for rectangles \(\sigma\) can be replaced by a path consisting of a vertical segment followed by a horizontal segment. This choice defines the same function \(F(z)\), and we obtain that \(\partial F/\partial x = f(z)\). Hence by the Cauchy-Riemann equations \(F(z)\) is analytic in \(\Delta\) with the derivative \(f(z)\), and by Theorem 12.2 \(\int_{\sigma} f(z) \, dz = 0\).

For the most general case, we can choose a triangulization of the region enclosed by the simple closed contour \(C\) so that each triangle lies in an open disk in which \(f\) is analytic. With the orientations of each triangle assigned consistently, the original contour integral \(\int_C f(z) \, dz\) can be written as a sum of integrals over these triangles, each of which is zero by the argument in the previous paragraph. Hence the original integral \(\int_C f(z) \, dz = 0\) as well.

**Definition 13.2.** A domain \(D\) is said to be simply connected if every simple closed contour within it enclosed only points in \(D\). A domain which is not simply connected is called multiply connected.

With this definition, there is an immediate corollary for the Cauchy-Goursat theorem:

**Corollary 13.3.** If a function \(f\) is analytic throughout a simply connected domain \(D\), then \(\int_C f(z) \, dz = 0\) for every closed contour \(C\) lying in \(D\).

The proof is easy if the closed contour \(C\) itself is simple or intersects itself a finite number of times. For, if \(C\) is simple and lies in \(D\), the function \(f\) is analytic at each point interior to and on \(C\); so we apply the Cauchy-Goursat theorem directly. On the other hand, if \(C\) is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours. By apply the Cauchy-Goursat theorem to each of those simple closed contours, we obtain the desired result for \(C\). Subtleties arise if the closed contour has infinitely many self-intersecting points. Still the corollary is true under this circumstance and we will not elaborate a complete proof here.

**Corollary 13.4.** A function \(f\) which is analytic throughout a simply connected domain \(D\) must have an antiderivative in \(D\).

The Cauchy-Goursat theorem can also be extended in a way that involves integrals along the boundary of a multiply connected domain:

**Theorem 13.5.** Let \(C\) and \(C_k\) \((k = 1, 2, \ldots, n)\) be simple closed contours such that \(C\) is described in the counterclockwise direction, \(C_k\)'s are in the clockwise direction, interior to \(C\) and their interior points are disjoint. If a function \(f\) is analytic throughout the closed region consisting of all points interior to and on \(C\) except for points interior to any of \(C_k\), then

\[ \int_C f(z) \, dz + \sum_{k=1}^n \int_{C_k} f(z) \, dz = 0. \quad (13.8) \]
To prove this theorem, we will draw a finite number of auxiliary lines so that the sum of the contours $C + C_1 + C_2 + \cdots + C_n$ can be decomposed as the sum of finite simple closed contours. Then we apply the Cauchy-Goursat theorem to each of the simple closed contours and conclude that Equation (13.8) holds.

**Example.** When $C$ is any positively oriented simple closed contour surrounding the origin, we must have

$$\int_C \frac{dz}{z} = 2\pi i.$$ 

To show this, we need only to construct a positively oriented circle $C_0$ with center at the origin and radius so small that $C_0$ lies entirely inside $C$. Then the theorem implies that

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z} = 2\pi i$$

by earlier computation.
Lecture note on Complex Variables

14 Cauchy integral formula

Now it is time to establish one of the most fundamental formulae in the complex analysis.

**Theorem 14.1 (Cauchy integral formula).** Let \( f \) be analytic everywhere within and on a simple closed contour \( C \), taken in the positive sense. If \( z_0 \) is any interior point to \( C \), then

\[
f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz.
\]

(14.1)

**Proof.** Since \( f \) is continuous at \( z_0 \), we know that for every \( \varepsilon > 0 \) there is a positive number \( \delta > 0 \) such that

\[
|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta.
\]

Let us now choose a positive number \( \rho \) that is less than \( \delta \) and is so small that the positively oriented circle \( |z - z_0| = \rho \), denoted by \( C_0 \), is interior to \( C \).

Since the function \( \frac{f(z)}{z - z_0} \) is analytic in the closed region consisting of the contours \( C \) and \( C_0 \) and all points between them, we know from Theorem 13.5, the principle of deformation of paths, that

\[
\oint_C \frac{f(z)}{z - z_0} \, dz = \oint_{C_0} \frac{f(z)}{z - z_0} \, dz.
\]

It has been established before that

\[
\int_{C_0} \frac{dz}{z - z_0} = 2\pi i,
\]

therefore,

\[
\oint_{C_0} \frac{f(z)}{z - z_0} \, dz = \oint_{C_0} \frac{f(z)}{z - z_0} \, dz - 2\pi if(z_0).
\]

(14.2)

It remains to estimate the last integral in Equation (14.2). Combining all of our assumptions, we know that

\[
\left| \oint_{C_0} \frac{f(z)}{z - z_0} \, dz \right| < \varepsilon \cdot 2\pi \rho = 2\pi \varepsilon,
\]

because \( |z - z_0| = \rho < \delta \) for all \( z \in \partial B(z_0, \rho) \), and the arc length of \( C_0 \) is \( 2\pi \rho \). Hence

\[
\left| \oint_{C} \frac{f(z)}{z - z_0} \, dz - 2\pi if(z_0) \right| < 2\pi \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we conclude that the difference inside the absolute value must be zero, and the theorem is proved.

The Cauchy integral formula tells us that if a function is to be analytic within and on a simple closed contour \( C \), then the values of \( f \) interior to \( C \) are completely determined by those on \( C \).

**Example.** Let \( C \) be the positively oriented circle \( |z| = 2 \). Since the function

\[
f(z) = \frac{z}{9 - z^2}
\]

is analytic within and on \( C \) and since the point \( z = -i \) is interior to \( C \), the Cauchy integral formula tells us that

\[
\int_{C} \frac{z}{(9 - z^2)(z + i)} \, dz = \int_{C} \frac{f(z)}{z - (-i)} \, dz = 2\pi i f(-i) = 2\pi i \cdot \frac{-i}{10} = \frac{\pi}{5}.
\]

An important consequence of the Cauchy integral formula which we are about to prove is that if a function is analytic in a neighborhood of a point, its derivatives of all orders exist around that point and are themselves analytic there. This is done by the following theorem.
Theorem 14.2 (Cauchy integral formula for derivatives). Let $f$ be analytic within and on a positively oriented, simple closed contour $C$, and $z_0$ be any point interior to $C$. Then the derivatives of all orders of $f$ exist and are analytic at $z_0$, and they are given by the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz, \quad n = 0, 1, 2, \ldots \tag{14.3}$$

Proof. We will first establish the case when $n = 1$, that is,

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} \, dz. \tag{14.4}$$

Notice that Equation (14.4) can be obtained formally by differentiating the integrand in Equation (14.1) with respect to $z_0$.

To prove Equation (14.4), we observe that,

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \int_C \left( \frac{1}{z - z_0} - \frac{1}{z - z_0 - \Delta z} \right) \frac{f(z)}{\Delta z} \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} \, dz$$

when $0 < |\Delta z| < d$, where $d$ is the smallest distance from $z_0$ to points $z$ on $C$. Since on the contour $C$ the integrand $(z - z_0 - \Delta z)(z - z_0)$ converges uniformly to $(z - z_0)^2$ as $\Delta z \to 0$, the limit of the integral exists and equals $f'(z_0)$.

For the general cases, suppose Equation (14.3) holds for a positive integer $m$. We compute

$$\frac{f^{(m)}(z_0 + \Delta z) - f^{(m)}(z_0)}{\Delta z} = \frac{m!}{2\pi i} \int_C \left( \frac{1}{(z - z_0)^{m+1}} - \frac{1}{(z - z_0)^{m+1}} \right) \frac{f(z)}{\Delta z} \, dz$$

$$= \frac{m!}{2\pi i} \int_C \left[ \frac{(m+1)(z - z_0)^m + (\Delta z)(g(z, z, \Delta z))f(z)}{(z - z_0 - \Delta z)(z - z_0)^{m+1}} \right] \, dz \quad (0 < |\Delta z| < d),$$

where $g(z, z, \Delta z)$ is a polynomial in these three variables. One then sees that the integrand converges uniformly to $(m+1)f(z)/(z - z_0)^{m+2}$ as $\Delta z$ goes to zero, and the theorem is proved by mathematical induction. \hfill \Box

Example. If $C$ is the positively oriented unit circle $|z| = 1$, and $f(z) = \exp(2z)$ which is analytic everywhere, then

$$\int_C \frac{e^{2z}}{z^4} \, dz = \int_C \frac{f(z)}{(z - 0)^3 + 1} \, dz = \frac{2\pi i}{3!} f'''(0) = \frac{8\pi i}{3}.$$

We conclude this section with a theorem due to Morera.

Theorem 14.3 (Morera). If a function $f$ is continuous throughout a domain $D$ and if

$$\int_C f(z) \, dz = 0$$

for every closed contour $C$ lying in $D$, then $f$ is analytic throughout $D$.

Proof. If a function $f$ satisfies the hypothesis, it has an antiderivative $F$ in the domain $D$ by Theorem 12.2. The function $F$ is then analytic, therefore our original function $f$, being the derivative of $F$, is also analytic in $D$ by the Cauchy integral formula for derivatives. \hfill \Box

15 Liouville’s theorem and the fundamental theorem of algebra

Suppose $C$ is the positively oriented circular contour $|z - z_0| = R, R > 0$, and $f(z)$ is a function analytic within and on the contour $C$. Then the basic estimate of contour integrals shows that

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right| \leq \frac{n! M_R}{R^n} \quad n = 0, 1, 2, \ldots, \tag{15.1}$$

for any constant $M_R$ which is an upper bound for $|f(z)|$ on the circle $C$. Formula (15.1) is called the Cauchy inequality. Usually the constant $M_R$ is difficult to be determined, but with stronger assumption we have a beautiful theorem due to Liouville.
Theorem 15.1 (Liouville). If $f$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof. Since $f$ is entire, the Inequality (15.1) holds for any choices of $z_0$ and $R$. The boundedness condition tells us that a non-negative constant $M$ exists such that $|f(z)| \leq M$ for all $z$. Hence the constant $M_R$ can always be chosen as $M$, and we have

$$|f'(z_0)| \leq \frac{M}{R}$$

for estimate of the first derivative of $f$. Since $R$ can be arbitrarily large, we see that $f'(z_0)$ must be zero for every $z_0$. Therefore $f$ is a constant function. $\square$

With the Liouville’s theorem, we can give a beautiful proof for the fundamental theorem of algebra, which says that the complex number field $\mathbb{C}$ is algebraically closed.

Theorem 15.2 (Fundamental theorem of algebra). Any polynomial

$$P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n \quad (a_n \neq 0)$$
of degree $n$ ($n \geq 1$) with complex coefficients $a_0, \ldots, a_n$ has at least one zero in $\mathbb{C}$.

Proof. The proof goes by contradiction. Suppose $P(z)$ does not vanish everywhere in $\mathbb{C}$, then the reciprocal

$$f(z) = \frac{1}{P(z)}$$

must be an entire function. Now we are going to show that it is also bounded.

We write

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + \cdots + \frac{a_{n-1}}{z},$$

so that $P(z) = (a_n + w)z^n$. Choose

$$R = \frac{2n}{|a_n|} \cdot \max\{|a_0|, |a_1|, |a_2|, \ldots, |a_{n-1}|\} + 1.$$

Then for every $|z| > R$, each term in the sum of $w$ is less than $|a_n|/2n$, hence $|w|$ is less than $|a_n|/2$. Hence

$$|P(z)| = |(a_n + w)\cdot |z|^n \geq \frac{|a_n|}{2} \cdot R^n$$

for all $|z| > R$, which implies that its reciprocal $f(z)$ is bounded outside the circle $|z| > R$. Since $f$ is continuous on the compact set $B(0, R)$, it is also bounded there. Hence by choosing the larger bound from the two mentioned above, we see that $f$ is bounded.

Now according to the Liouville’s theorem, $f$ must be constant, and so must $P$, which is clearly a contradiction. $\square$
Lemma 16.1. Suppose that \( f(z) \) is analytic throughout a neighborhood \( B(z_0, \varepsilon) \) of a point \( z_0 \). If \( |f(z)| \leq |f(z_0)| \) for each point \( z \) in that neighborhood, then \( f(z) \) has the constant value \( f(z_0) \) throughout the neighborhood.

Proof. Let \( f \) be such a function and \( z_1 \) be any point other than \( z_0 \) in the given neighborhood. We let \( \rho \) be the distance between \( z_1 \) and \( z_0 \). If \( C_\rho \) denotes the positively oriented circle \( |z - z_0| = \rho \), centered at \( z_0 \) and passing through \( z_1 \), the Cauchy integral formula tells us that

\[
 f(z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(z)}{z - z_0} \, dz.
\]

The parametric representation

\[
 z = z_0 + \rho e^{i\theta} \quad (0 \leq \theta \leq 2\pi)
\]

for \( C_\rho \) enables us to write the above contour integral as

\[
 f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta. \tag{16.1}
\]

We note from Equation (34.3) that when a function is analytic within and on a given circle, its value at the center is the arithmetic mean of its values on the circle. This result is called Gauss’s mean value theorem.

From Equation (34.3), we obtain the inequality

\[
 |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta. \tag{16.2}
\]

On the other hand, since from the hypothesis

\[
 |f(z_0 + \rho e^{i\theta})| \leq |f(z_0)| \quad (0 \leq \theta \leq 2\pi), \tag{16.3}
\]

we find that

\[
 \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, d\theta = |f(z_0)|. \tag{16.4}
\]

Combining Inequalities (16.2) and (16.4), we see that

\[
 |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta,
\]

or

\[
 \int_0^{2\pi} \left[ |f(z_0)| - |f(z_0 + \rho e^{i\theta})| \right] \, d\theta = 0.
\]

The integrand in this last integral is continuous in the variable \( \theta \) and always non-negative, but the integral is zero. Therefore the integrand must be identically zero, that is,

\[
 |f(z_0 + \rho e^{i\theta})| = |f(z_0)| \quad (0 \leq \theta \leq 2\pi).
\]

This shows that \( |f(z)| = |f(z_0)| \) for all points on the circle \( |z - z_0| = \rho \).

Finally, since \( z_1 \) is an arbitrary chosen point in the deleted neighborhood \( B'(z_0, \varepsilon) \), we see that the equation \( |f(z)| = |f(z_0)| \) is, in fact, satisfied by all points everywhere in the circle \( B(z_0, \varepsilon) \). But any analytic function with constant modulus must be constant in a domain, therefore \( f \) must be constant in that neighborhood.

This lemma can be used to prove the following theorem, which is known as the maximum modulus principle.
Theorem 16.2 (Maximum principle). If a function $f$ is analytic and not constant in a given domain $D$, then $|f(z)|$ has no maximum value in $D$. That is, there is no point $z_0$ in the domain such that $|f(z)| \leq |f(z_0)|$ for all points $z \in D$.

Proof. Assume the contrary, that is, there does exist a point $z_0 \in D$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. We shall show that $f$ must be a constant function on $D$.

Consider the subset $E := \{z \in D \mid f(z) = f(z_0)\}$ of $D$. Clearly $E$ is not empty because $z_0 \in E$. Since $f$ is a continuous function, $E$ is also closed. If $z \in E$, then there is a neighborhood $U = B(z, \varepsilon) \subseteq D$ such that $|f(\zeta)| \leq |f(z_0)| = |f(z)|$ for all $\zeta \in U$; by the previous lemma $f(\zeta) = f(z) = f(z_0)$ for all $\zeta \in U$. Hence $U \subseteq E$ and $E$ is open. Because $D$ is connected, the only nonempty open and closed subset of $D$ is $D$ itself. Therefore $E = D$ and $f$ is a constant function on $D$, which is a contradiction.

Corollary 16.3. Suppose that a function $f$ is continuous in a closed bounded region $R$ and that it is analytic and not constant in the interior of $R$. Then the maximum value of $|f(z)|$ in $R$, which is always reached, occurs somewhere on the boundary of $R$ and never in the interior. □

When the function $f$ in the corollary is written $f(z) = u(x, y) + iv(x, y)$, the component function $u(x, y)$ also has a maximum value in $R$ which is assumed on the boundary of $R$ and never in the interior, where it is harmonic. For the composite function $g(z) = \exp[f(z)]$ is continuous in $R$ and analytic and not constant in the interior. Consequently, its modulus $|g(z)| = \exp[u(x, y)]$, which is continuous in $R$, must assume its maximum value in $R$ on the boundary. Because of the increasing nature of the exponential function, it follows that the maximum value of $u(x, y)$ also occurs on the boundary.
Lecture note on Complex Variables

17 Sequences and series

**Definition 17.1.** An infinite sequence of complex numbers is a function \( z : \mathbb{N} \rightarrow \mathbb{C} \). Traditionally we will denote \( z(n) \) be \( z_n \). Such a sequence has a limit \( Z \) if, for every positive number \( \varepsilon \), there is a positive integer \( N \) such that

\[
z_n \in B(Z, \varepsilon) \quad \text{whenever } n > N.
\]

Any sequence can only have at most one limit (Exercise.) When that limit \( Z \) exists, the sequence is said to converge to \( Z \), and we write

\[
\lim_{n \to \infty} z_n = Z.
\]

If the sequence has no limit, it diverges.

Recall the property of the Euclidean metric on \( \mathbb{C} \simeq \mathbb{R}^2 \): for a complex number \( z = x + iy \), we know that

\[
|x|, |y| \leq |z|, \quad |z| \leq |x| + |y|.
\]

It is then clear that a sequence \( (z_n) \) converges to \( Z \) if and only if \( (\text{Re } z_n) \) converges to \( \text{Re } Z \) and \( (\text{Im } z_n) \) to \( \text{Im } Z \).

**Definition 17.2.** An infinite series of complex numbers is a formal ordered sum

\[
\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_n + \cdots,
\]

where \( (z_n) \) is an infinite sequence of complex numbers. We say that the series converges to the sum \( S \) if the sequence of partial sums converges to \( S \); we then write

\[
\sum_{n=1}^{\infty} z_n = S.
\]

Otherwise we say the series diverges.

Again a series converges if and only both its real part and imaginary part converge. Also it is convenient to define the remainder \( \rho_N \) after \( N \) terms of a series \( \sum_n z_n \):

\[
\rho_N := \sum_{n=N+1}^{\infty} z_n.
\]

It is clear that the series \( \sum_n z_n \) converges if and only the sequence of the remainders \( (\rho_n) \) converges to zero.

**Proposition 17.3.** If a series \( \sum_{n=1}^{\infty} z_n \) converges, then \( \lim_{n \to \infty} z_n = 0 \).

**Proof.** Let \( (S_n) \) be the sequence of partial sums of the series \( \sum_n z_n \), and the series converges to the sum \( S \). By Definition 17.2 we have

\[
\lim_{n \to \infty} S_n = S.
\]

Therefore for every \( \varepsilon > 0 \), there is a positive integer \( N \) such that

\[
|S_n - S| < \frac{\varepsilon}{2} \quad \text{whenever } n > N - 1.
\]
By the identity $z_n = S_n - S_{n-1}$, we see that for $n > N$,

$$|z_n| = |S_n - S_{n-1}| = |(S_n - S) - (S_{n-1} - S)| \leq |S_n - S| + |S_{n-1} - S| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$  

Hence $\lim_{n \to \infty} z_n = 0.$ \hfill \Box

**Remark.** Note that the condition is necessary but not sufficient. An example is the divergent series $\sum_{n=1}^{\infty} (1/n)$.

**Definition 17.4.** We say that a series $\sum_{n=1}^{\infty} z_n$ converges **absolutely** if the series $\sum_{n=1}^{\infty} |z_n|$ of nonnegative real numbers converges.

By the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$, we see that if a series converges absolutely, then the series converges. (This is not a repetition!)

# 18 Taylor series

We turn now to Taylor’s theorem, which says every holomorphic function is analytic. (Or, the true meaning of analytic functions.)

**Theorem 18.1 (Taylor theorem).** Suppose that a function $f$ is analytic throughout an open disk $B(z_0, R_0)$, centered at $z_0$ and with radius $R_0 > 0$. Then, at each point $z$ in that disk, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (|z - z_0| < R_0),$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0, 1, 2, \ldots).$$

That is, the power series here converges to $f(z)$ when $|z - z_0| < R_0$.

Equation (18.1) is the expansion of $f(z)$ into a Taylor series about the point $z_0$. The meaning is that regarding the right-hand side as a function of $z$, the series from the right-hand side converges for each $z \in B(z_0, R_0)$, and its sum is exactly $f(z)$.

**Proof.** To simplify the notation in the proof, we assume that $z_0 = 0$. In general, one can apply a translation $z \mapsto z + z_0$ to obtain the general formula.

Let $C_0$ denote any positively oriented circle $\partial B(0, r_0)$ that is contained in the disk $B(0, R_0)$, but is large enough so that the point $z$ is interior to it. The Cauchy integral formula then applies:

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s) \, ds}{s - z}. \quad (18.3)$$

Now the factor $1/(s - z)$ in the integrand can be put in the form

$$\frac{1}{s - z} = \frac{1}{s} \cdot \frac{1}{1 - \frac{z}{s}} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s - z) s^N}.$$  

Multiplying through this equation by $f(s)$ and then integrating each side with respect to $s$ around $C_0$, we find that

$$\int_{C_0} \frac{f(s) \, ds}{s - z} = \sum_{n=0}^{N-1} \left( \int_{C_0} \frac{f(s) \, ds}{s^{n+1}} \right) z^n + z^N \int_{C_0} \frac{f(s) \, ds}{(s - z) s^N}.$$
The Cauchy integral formula for the derivatives tells us that
\[ \frac{1}{2\pi i} \int_{C_0} \frac{f(s) \, ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!}, \quad (n = 0, 1, 2, \ldots) \]
this reduces, after we multiply through by \(1/(2\pi i)\), to
\[ f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \rho_N(z), \quad (18.4) \]
where
\[ \rho_N(z) = \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s) \, ds}{(s-z)s^N}. \quad (18.5) \]

The Taylor series is then proved once we show that
\[ \lim_{N \to \infty} \rho_N(z) = 0. \]
To do that, suppose that \(|z| = r\) (by our assumption \(r_0 > r\)). Then, if \(s\) is a point on \(C_0\),
\[ |s - z| \geq ||s| - |z|| \geq r_0 - r. \]
Hence, if \(M\) denotes the maximum value of the continuous function \(|f(s)|\) on the compact set \(C_0\), then
\[ |ho_N(z)| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_0 - r)r_0} \cdot 2\pi r_0 = \frac{Mr_0}{r_0 - r} \left( \frac{r}{r_0} \right)^N. \]
But the ratio \(r/r_0 < 1\) since \(z\) is interior to \(C_0\), therefore \(\lim_{N \to \infty} \rho_N(z) = 0\) for every \(z \in B(0, r)\). Because \(r\) is an arbitrary number between 0 and \(R_0\), our theorem is proved.

Remark.

(1) When \(z_0 = 0\), the power series expansion (18.1) is also called the Maclaurin series.

(2) Followed from the Taylor theorem, the series expansion of an analytic function near a point is unique.

Example. Since the function \(f(z) = e^z\) is entire, it has a Maclaurin series representation which is valid for all \(z \in \mathbb{C}\). Here \(f^{(n)}(0) = e^0 = 1\), so
\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots \quad (|z| < \infty). \]

The entire function \(z^2 e^{3z}\) also has a Maclaurin series expansion. By uniqueness, we may compute that series as follows:
\[ z^2 e^{3z} = z^2 \cdot \sum_{n=0}^{\infty} \frac{(3z)^n}{n!} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n \quad (|z| < \infty). \]

Example. We write down a few formulae here for further reference:
\[ \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad (|z| < \infty). \]
\[ \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad (|z| < \infty). \]
\[ \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1). \]


Lecture note on Complex Variables

19 Laurent series

If a function $f$ fails to be analytic at a point $z_0$, we cannot apply Taylor’s theorem at that point. It is often possible, however, to find a series representation for $f(z)$ involving both positive and negative powers of $z - z_0$. We now present the theory of such representations.

**Theorem 19.1 (Laurent theorem).** Suppose that a function $f$ is analytic throughout an annular domain $A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, and let $C$ denote any positively oriented simple closed contour around $z_0$ and lying in that domain. Then, at each point $z$ in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \quad (R_1 < |z - z_0| < R_2), \quad (19.1)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n \in \mathbb{Z}). \quad (19.2)$$

Equation (19.1) is called the **Laurent series** of $f$ in the annulus $A$ around $z_0$.

**Proof.** Again without loss of generality, we will prove the theorem assuming that $z_0 = 0$. The general case can be obtained simply by translation.

Given a point $z$ in the annulus $A$, we first pick two positively oriented circles $|\zeta| = r_1$ and $|\zeta| = r_2$, which are denoted by $C_1$ and $C_2$ respectively, such that $R_1 < r_1 < |z| < r_2 < R_2$. Observe that $f$ is analytic on $C_1$ and $C_2$, as well as in the annular domain between them.

Next, we construct a positively oriented circle $\gamma$ centered at $z$ and small enough to be completely contained in the interior of the annular region $r_1 < |\zeta| < r_2$. It then follows from the extension of the Cauchy-Goursat theorem (Theorem 13.5) that

$$\int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$ 

According to the Cauchy integral formula, the value of the third integral above is $2\pi i f(z)$. Hence

$$f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{z - \zeta} d\zeta. \quad (19.3)$$

For the first integral, $|\zeta| = r_2 > |z|$ for $\zeta \in C_2$. We have faced this situation when we proved the Taylor theorem. Write

$$\frac{1}{\zeta - z} = \sum_{n=0}^{N-1} \frac{z^n}{\zeta^{n+1}} + \frac{z^N}{(\zeta - z)\zeta^N}$$

For the second integral we have the reversed situation: $|\zeta| = r_1 < |z|$ for $\zeta \in C_1$. Therefore we write

$$\frac{1}{z - \zeta} = \sum_{m=0}^{M-1} \frac{\zeta^m}{z^{m+1}} + \frac{\zeta^M}{(z - \zeta)z^M}.$$ 

Multiplying the two series above by $f(\zeta)/(2\pi i)$ and then integrating each side with respect to $\zeta$ around $C_2$ and $C_1$, respectively, we find from Equation (19.3) that

$$f(z) = \sum_{n=-M}^{N-1} a_n z^n + \rho_N(z) + \sigma_M(z), \quad (19.4)$$

32
where the numbers \( a_n, n = -M, -M + 1, \ldots, -1, 0, 1, \ldots, N \) are given by the equations

\[
a_n = \begin{cases} 
\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta & (n \geq 0) \\
\frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta & (n \leq -1)
\end{cases}
\]  

(19.5)

and where the remainders are

\[
\rho_N(z) = \frac{z^N}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z)\zeta^N} d\zeta, \quad \sigma_M(z) = \frac{1}{2\pi i z^M} \int_{C_1} \frac{\zeta^M f(\zeta)}{z - \zeta} d\zeta.
\]

We are now ready to show that

\[
\lim_{N \to \infty} \rho_N(z) = 0 \quad \text{and} \quad \lim_{M \to \infty} \sigma_M(z) = 0.
\]  

(19.6)

Let \( K \) be the maximum of \( |f(\zeta)| \) for \( \zeta \in C_1 \cup C_2 \). By estimating the integrals we find that

\[
|\rho_N(z)| \leq \frac{K}{r_2 - |z|} \cdot \left( \frac{|z|}{r_2} \right)^N, \quad |\sigma_M(z)| \leq \frac{K}{|z| - r_1} \cdot \left( \frac{r_1}{|z|} \right)^M.
\]

Since \( r_1 < |z| < r_2 \), both limits in Equations (19.6) are clear.

Finally we note that the contour integrals in Equation (19.5) can be replaced by those over the contour \( C \), by the principle of deformation of contours. Therefore our theorem is proved.

Example. Replacing \( z \) by \( 1/z \) in the Maclaurin series expansion

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots,
\]

we obtain

\[
e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots + \frac{1}{n!} z^n + \cdots,
\]

which is valid for all \( z \neq 0 \). Note that no positive powers of \( z \) appear here, the coefficients of the positive powers being zero; also that the coefficient of \( 1/z \) is 1, therefore by Theorem 19.1, for a positively oriented simple closed contour around 0,

\[
\int_{C} e^{1/z} dz = 2\pi i a_{-1} = 2\pi i.
\]

Example. Let us consider Laurent series expansions around \( z = 0 \) of the function

\[
f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2},
\]

which has two singular points \( z = 1 \) and \( z = 2 \) and is analytic in the annular domains

\[
|z| < 1, \quad 1 < |z| < 2, \quad 2 < |z|.
\]

Let us denote these domains by \( D_1, D_2, \) and \( D_3 \), respectively.

The series expansion of \( f \) around \( z = 0 \) in \( D_1 \) is in fact a Maclaurin series. To find it, we write

\[
f(z) = \frac{-1}{1 - z} + \frac{1}{2} \cdot \frac{1}{1 - (z/2)}.
\]

Since both \( |z| < 1 \) and \( |z/2| < 1 \) in \( D_1 \), we have

\[
f(z) = -\sum_{n=0}^{\infty} z^n + \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n = \sum_{n=0}^{\infty} \left( -1 + \frac{1}{2^{n+1}} \right) z^n \quad (|z| < 1.) \]

(19.7)
As for the representation in $D_2$, we write

$$f(z) = \frac{1}{z} \cdot \frac{1}{1 - (1/z)} + \frac{1}{2} \cdot \frac{1}{1 - (z/2)}.$$  

Since $|1/z| < 1$ and $|z/2| < 1$ in $D_2$, we have

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad (1 < |z| < 2)$$  

(19.8)

Finally for the representation in $D_3$, we write

$$f(z) = \frac{1}{z} \cdot \frac{1}{1 - (1/z)} + \frac{1}{z} \cdot \frac{1}{1 - (2/z)},$$

and observe that $|1/z| < 1$ and $|2/z| < 1$ for $z \in D_3$. Therefore

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{n=-\infty}^{-1} \left(1 - 2^{-n-1}\right) z^n \quad (2 < |z| < \infty.$$  

(19.9)

One should be aware that the Laurent series of the same function around the same point in different annular domain might be different.

## 20 Absolute and uniform convergence of power series

The following sections are devoted mainly to various properties of power series.

**Theorem 20.1.** If a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$  

(20.1)

converges when $z = z_1 (\neq z_0)$, then it is absolutely convergent at each point $z$ in the open disk $B(z_0, R)$, where $R = |z_1 - z_0| > 0$. 

**Proof.** Without loss of generality, we may assume that $z_0 = 0$, and that

$$\sum_{n=0}^{\infty} a_n z_1^n \quad (z_1 \neq 0)$$

converges. Proposition 17.3 implies that the terms $a_n z_1^n$ are bounded; that is,

$$|a_n z_1^n| \leq M \quad (n = 0, 1, 2, \ldots)$$

for some positive constant $M$. If $|z| < |z_1|$ and we let $\rho$ denote the modulus $|z/z_1|$, we can see that

$$|a_n z^n| = |a_n z_1^n| \cdot \left|\frac{z}{z_1}\right|^n \leq M \rho^n \quad (n = 0, 1, 2, \ldots)$$

where $\rho < 1$. Now the series whose terms are the real numbers $M \rho^n \ (n \in \mathbb{Z}_+)$ is a geometric series, which converges when $\rho < 1$. Hence, by the comparison test for series of real numbers, the series

$$\sum_{n=0}^{\infty} |a_n z^n|$$

converges in the open disk $B(0, |z_1|)$; and the theorem is proved.

**Remark.** The theorem tells us that the set of all points inside some circle centered at $z_0$ is a region of convergence for the power series (20.1), provided that it converges at some point other than $z_0$. The greatest circle centered at $z_0$ such that the series (20.1) converges at each point inside is called the circle of convergence of the series (20.1), and its radius is called the radius of convergence. The series cannot converge at any point outside that circle, for that would violate the maximality of the circle, according to Theorem 20.1.
Our next theorem involves terminology that we must define. Suppose that the power series \( \sum_{n=0}^{\infty} a_n (z - z_0)^n \) has circle of convergence \( |z - z_0| = R \), and let \( S(z) \) and \( S_N(z) \) represent the sum and partial sums, respectively, of that series:

\[
S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad S_N(z) = \sum_{n=0}^{N-1} a_n (z - z_0)^n, \quad (|z - z_0| < R).
\]

Then write the remainder function

\[
\rho_N(z) = S(z) - S_N(z) \quad (|z - z_0| < R).
\]

Since the power series converges for any fixed value \( z \) when \( |z - z_0| < R \), we know that the remainder \( \rho_N(z) \) approaches zero for any such \( z \) as \( N \) tends to infinity. That is, for every positive number \( \varepsilon \), there is a positive \( N_\varepsilon \) such that

\[
|\rho_N(z)| < \varepsilon \quad \text{whenever} \quad N > N_\varepsilon.
\]

Note that the choice of \( N_\varepsilon \) depends on \( \varepsilon \) as well as on \( z \). If there is a uniform choice \( N_\varepsilon \) dependent only of \( \varepsilon \) but independent of \( z \) taken in a specific region, the convergence is said to be uniform in that region.

**Theorem 20.2.** If \( z_1 \) is a point inside the circle of convergence \( |z - z_0| = R \) of a power series

\[
\sum_{n=0}^{\infty} a_n (z - z_0)^n,
\]

then that series converges uniformly in the closed disk \( |z - z_0| \leq R_1 \), where \( R_1 = |z_1 - z_0| \).

**Proof.** Again without loss of generality, we may assume that \( z_0 = 0 \). Since \( z_1 \) lies in the circle of convergence of the power series, we know by Theorem 20.1 that the series actually converges absolutely there, that is, the series

\[
\sum_{n=0}^{\infty} |a_n z_1^n|
\]

converges as well. Note that the terms of the last series are all nonnegative.

For every \( \zeta \in \bar{B}(0, R_1) \), we have \( |\zeta/z_1| \leq 1 \) and

\[
|a_n \zeta^n| \leq |a_n z_1^n| \cdot \left| \frac{\zeta}{z_1^n} \right| \leq |a_n z_1^n|.
\]

Therefore the \( N_\varepsilon \) for any positive \( \varepsilon \) at the point \( z_1 \) works at every point \( \zeta \) in the closed disk \( \bar{B}(0, R_1) \). Using the same \( N_\varepsilon \) and the triangle inequality shows that the convergence of the power series (20.3) is uniform in that disk.

**Corollary 20.3.** A power series (20.3) represents a continuous function \( S(z) \) at each point inside its circle of convergence \( |z - z_0| = R \).

**Proof.** This follows from the standard \( \varepsilon/3 \)-argument that the limit of a sequence of continuous that converges uniformly is still continuous.

**Remark.** The same can be said about Laurent series expansions, provided that we are looking at the exterior of some circle when we talk about the power series with negative powers of \( z - z_0 \).
21 Integration and differentiation of power series

In this section we will establish the analytic properties of a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

(21.1)

within its circle of convergence $|z - z_0| = R$.

**Theorem 21.1.** Let $C$ denote any contour interior to the circle of convergence $|z - z_0| = R$ of the power series (21.1), and let $g(z)$ be any function that is continuous on $C$. The series formed by multiplying each term of the power series by $g(z)$ can be integrated term by term over $C$, that is,

$$\int_C g(z) S(z) \, dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n \, dz.$$  

(21.2)

**Proof.** Since the contour $C$ lies in some closed disk $B(z_0, \rho)$ with $\rho < R$, we know that the series (21.1) converges uniformly on $C$. Any continuous function $g(z)$ on the compact set $C$ must be bounded. Combining these results, we immediately see that Equation (21.2) holds.

If $g(z) = 1$ for each value of $z$ in the open disk bounded by the circle of convergence of the power series (21.1), then

$$\int_C g(z) (z - z_0)^n \, dz = \int_C (z - z_0)^n \, dz = 0 \quad (n = 0, 1, 2, \ldots)$$

given every closed contour $C$ lying in that domain. According to Theorem 21.1, we have

$$\int_C S(z) \, dz = 0.$$  

Therefore by the Morera’s theorem, the function $S(z)$ is analytic throughout the domain. We state this result as a corollary.

**Corollary 21.2.** The sum $S(z)$ of the power series (21.1) is analytic at each point $z$ interior to the circle of convergence of that series.

A nice application of Theorem 21.1 is about the differentiation of a convergent Taylor series.

**Theorem 21.3.** The power series (21.1) can be differentiated term by term. That is, at each point $\zeta$ interior to the circle of convergence of that series,

$$S' (\zeta) = \sum_{n=1}^{\infty} n a_n (\zeta - z_0)^{n-1}.$$  

(21.3)

**Proof.** Let $\zeta$ be an arbitrary point interior to the circle of convergence $|z - z_0| = R$ of the power series (21.1), and $C$ be some positively oriented simple close contour surrounding $\zeta$ and interior to that circle. Define the function

$$g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z - \zeta)^2}$$

at each point $z$ on $C$. Since $g(z)$ is continuous on $C$, Theorem 21.1 tells us that

$$\int_C g(z) S(z) \, dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n \, dz.$$  

(21.4)

Now $S(z)$ is analytic inside and on $C$, and we apply the Cauchy formula for derivatives to write

$$\int_C g(z) S(z) \, dz = \frac{1}{2\pi i} \int_C \frac{S(z)}{(z - \zeta)^2} \, dz = S' (\zeta).$$

On the other hand, when we look into every term on the right hand side of Equation (21.4), we see that

$$\int_C g(z) (z - z_0)^n \, dz = \frac{1}{2\pi i} \int_C \frac{(z - z_0)^n}{(z - \zeta)^2} \, dz = n(\zeta - z_0)^{n-1} \quad (n = 0, 1, 2, \ldots)$$

Therefore the theorem is proved by identifying both sides of Equation (21.3). 

\Box
22 Uniqueness of series representation

The uniqueness of Taylor and Laurent series representations follows readily from Theorem 21.1. We will present the proof of the uniqueness of Taylor series representations here, and that of Laurent series representations is left to interested readers.

Theorem 22.1. If a series
\[ \sum_{n=0}^{\infty} a_n(z - z_0)^n \] (22.1)
converges to \( f(z) \) at all points interior to some circle \( |z - z_0| = R \), then it is the Taylor series expansion of \( f \) in powers of \( z - z_0 \) interior to that circle.

Proof. We write
\[ f(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m, \quad (|z - z_0| < R). \]

Then by Theorem 21.1, we may write
\[ \int_C g_n(z) f(z) \, dz = \sum_{m=0}^{\infty} a_m \int_C g_n(z) (z - z_0)^m \, dz, \]
where \( g_n(z) \) is defined as
\[ g_n(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z - z_0)^{n+1}}, \quad (n = 0, 1, 2, \ldots) \]
and \( C \) is some circle centered at \( z_0 \) and with radius less than \( R \).

By Cauchy integral formulae for the derivative, we see that
\[ \int_C g_n(z) f(z) \, dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz = \frac{f^{(n)}(z_0)}{n!}; \]
on the other hand,
\[ \int_C g_n(z) (z - z_0)^m \, dz = \frac{1}{2\pi i} \int_C (z - z_0)^{m-n-1} \, dz = \begin{cases} 0 & \text{when } m \neq n, \\ 1 & \text{when } m = n. \end{cases} \]
It is then clear that
\[ a_n = \frac{f^{(n)}(z_0)}{n!}, \]
which means that the power series (22.1) is the Taylor series expansion of \( f(z) \) around \( z_0 \).

The same argument can be applied to prove the uniqueness of Laurent series representations. We will state the theorem without proof.

Theorem 22.2. If a series
\[ \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n \]
converges to \( f(z) \) at all points in some annular domain about \( z_0 \), then it is the Laurent series expansion for \( f \) in powers of \( z - z_0 \) for that domain.
The power of these two theorems is that whenever we reach a power series expansion for some unknown function, it is the power series representation of that function as long as we assure that it converges in some disk or annulus.

**Example. (Multiplication of two power series.)** Suppose that each of the power series
\[ f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n \]
converges within some circle \(|z - z_0| = R\). Then the product \( f(z) \, g(z) \) in the open disk \( B(z_0, R) \) has the power series representation
\[ f(z) \, g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \]
where \( c_n = \sum_{k=0}^{n} a_k \, b_{n-k}, \quad n = 0, 1, 2, \ldots \)

**Example.** The Maclaurin series for \( \frac{e^z}{1+z} \) is valid in the disk \(|z| < 1\). The first four terms are easily found by writing
\[
\frac{e^z}{1+z} = (1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \cdots) \cdot (1 + z)^{-1}
\]
\[
= (1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \cdots) \cdot (1 - z^2 - z^3 - \cdots)
\]
\[
= 1 + \frac{1}{2} z^2 - \frac{1}{3} z^3 + \cdots \quad (|z| < 1)
\]

The uniqueness of power series representations can also be used to prove the following important property enjoyed by the class of analytic functions. One should readily come up with a counterexample in the class of smooth functions.

**Theorem 22.3 (Identity principle).** Let \( f, g \) be two analytic functions in some domain \( \Omega \) of \( \mathbb{C} \). If there is a subset \( A \) of \( \Omega \) with an accumulation point in \( \Omega \) such that \( f(z) = g(z) \) for every \( z \in A \), then \( f(z) = g(z) \) for every \( z \in \Omega \).

**Proof.** Without loss of generality, we may assume that \( g(z) = 0 \) for every \( z \in \Omega \) (replacing \( f, g \) by \( f - g, 0 \), respectively.) Define a subset \( E \) of \( \Omega \) by
\[
E := \{ z \in \Omega \mid f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \ldots \}.
\]

\( E \) is certainly closed because every \( f^{(n)} \) is continuous. Suppose \( z_0 \in E \), then the Taylor series expansion of \( f \) about \( z_0 \) is identically zero in a neighborhood of \( z_0 \). This shows that \( E \) is an open subset of \( \Omega \).

Now we have to show that \( E \) is nonempty. Let \( \zeta \) be an accumulation point of \( A \), that is, there is a sequence \((z_n)_{n \in \mathbb{N}}\) in \( A \) such that \( z_n \to \zeta \) as \( n \to \infty \). Let \( \sum_{k=0}^{\infty} a_k (z - \zeta)^k \) be the Taylor series representation of \( f \) about \( \zeta \). If this series is not identically zero, there is a smallest nonnegative integer \( m \) such that \( a_0 = \cdots = a_{m-1} = 0 \) but \( a_m \neq 0 \). Therefore \( f(z) \) can be written as
\[
f(z) = a_m (z - \zeta)^m \cdot \left( 1 + \sum_{k=m+1}^{\infty} \frac{a_k}{a_m} (z - \zeta)^{k-m} \right).
\]
The latter factor of \( f \) represents another analytic function in some neighborhood of \( \zeta \) in \( \Omega \). There is an \( \varepsilon > 0 \) such that
\[
B(\zeta, \varepsilon) \subset \Omega \quad \text{and} \quad \left| 1 + \sum_{k=m+1}^{\infty} \frac{a_k}{a_m} (z - \zeta)^{k-m} \right| > \frac{1}{2} \quad \forall z \in B(\zeta, \varepsilon).
\]
This shows that \( f \) does not have any zeroes in the deleted neighborhood \( B'(\zeta, \varepsilon) \), which contradicts to the choice of \( \zeta \). Therefore \( \zeta \in E \). Now \( E \) is a nonempty, closed and open subset of the connect set \( \Omega \), hence \( E \) must be \( \Omega \) and \( f(z) = 0 \) for every \( z \in \Omega \). \( \square \)
23 Infinite products

An infinite product of complex numbers

\[ p_1 p_2 \cdots p_n \cdots = \prod_{n=1}^{\infty} p_n \quad (23.1) \]

is evaluated by taking the limit of the partial products \( P_n = p_1 p_2 \cdots p_n \). It is said to converge to the value \( P = \lim_{n \to \infty} P_n \) if this limit exists and is different from zero. There are good reasons for excluding the value zero. For one thing, if the value \( P = 0 \) were permitted, any infinite product with one factor 0 would converge, and the convergence would not depend on the whole sequence of factors. On the other hand, in certain connections this convention is too radical. In fact, we wish to express a function as an infinite product, and this must be possible even if the function has zeroes. For this reason we make the following agreement: The infinite product (23.1) is said to converge if and only if at most a finite number of the factors are zeroes, and if the partial products formed by the nonvanishing factors tend to a finite limit which is different from zero.

In a convergent product the general factor \( p_n \) tends to 1; this is clear by writing \( p_n = \frac{P_n}{P_{n-1}} \), the zero factors being omitted. In view of this fact it is preferable to write all infinite products in the form

\[ \prod_{n=1}^{\infty} (1 + a_n) \quad (23.2) \]

so that \( a_n \to 0 \) is a necessary condition for convergence.

If no factor is zero, it is natural to compare the product (23.2) with the infinite series

\[ \sum_{n=1}^{\infty} \log(1 + a_n) \quad (23.3) \]

Since the \( a_n \)'s are complex we must agree on a definite branch of the logarithms, and we decide to choose the principal branch in each term. Denote the partial sums of (23.3) by \( S_n \). Then \( P_n = \exp(S_n) \), and if \( S_n \to S \) it follows that \( P_n \) tends to the limit \( P = e^S \) which is never zero. In other words, the convergence of (23.3) is a sufficient condition for the convergence of (23.2).

In order to prove that the condition is also necessary, suppose that \( P_n \to P \neq 0 \). It is not true, in general, that the series (23.3), formed with the principal values, converges to the principal value of \( \log P \); what we wish to show is that it converges to some value of \( \log P \). For greater clarity we shall temporarily adopt the usage of denoting the principal value of the logarithm by \( \text{Log} \) and its imaginary part by \( \text{Arg} \).

Because \( P_n/P \to 1 \) it is clear that \( \text{Log}(P_n/P) \to 0 \) as \( n \to \infty \). There exists an integer \( h_n \) such that

\[ \text{Log}(P_n/P) = S_n - \text{Log} P + h_n \cdot 2\pi i \]

We pass to the differences to obtain

\[ (h_{n+1} - h_n) \cdot 2\pi i = \text{Log}(P_{n+1}/P) - \text{Log}(P_n/P) - \text{Log}(1 + a_{n+1}) \]

and hence

\[ (h_{n+1} - h_n) \cdot 2\pi = \text{Arg}(P_{n+1}/P) - \text{Arg}(P_n/P) - \text{Arg}(1 + a_{n+1}) \]

By definition, \( |\text{Arg}(1 + a_{n+1})| \leq \pi \), and we know that

\[ \text{Arg}(P_{n+1}/P) - \text{Arg}(P_n/P) \to 0. \]

For large \( n \) this is incompatible with the previous equation unless \( h_{n+1} = h_n \). Hence \( h_n \) is ultimately equal to a fixed integer, and it follows from \( \text{Log}(P_n/P) = S_n - \text{Log} P + h \cdot 2\pi i \) that \( S_n \to \text{Log} P - h \cdot 2\pi i \). We have proved:
Theorem 23.1. The infinite product $\prod_{n=1}^{\infty} (1+a_n)$ with $a_n \neq -1$ converges simultaneously with the series $\sum_{n=1}^{\infty} \log(1+a_n)$ whose terms represent the values of the principal branch of the logarithm.

The question of convergence of a product can thus be reduced to the more familiar question concerning the convergence of a series. It can be further reduced by observing that the series (23.3) converges absolutely at the same time as the simpler series $\sum |a_n|$. This is an immediate consequence of the fact that

$$\lim_{z \to 0} \frac{\log(1+z)}{z} = 1.$$ 

If either the series (23.3) of $\sum |a_n|$ converges, we have $a_n \to 0$, and for a given $\varepsilon > 0$ the double inequality

$$(1-\varepsilon)|a_n| < |\log(1+a_n)| < (1+\varepsilon)|a_n|$$

will hold for sufficiently large $n$. It follows immediately that the two series are in fact simultaneously absolutely convergent.

An infinite product is said to be absolutely convergent if and only if the corresponding series (23.3) converges absolutely. With this terminology we can state our result in the following terms:

Theorem 23.2. A necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty} (1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty} |a_n|$.

In the last theorem the emphasis is on absolute convergence. By simple examples it can be shown that the convergence of $\sum a_n$ is neither sufficient nor necessary for the convergence of the product $\prod(1+a_n)$. 
Lecture note on Complex Variables

24 Residue theorem

Definition 24.1. A point $z_0$ is called a singular point of a function $f$ if $f$ fails to be analytic at $z_0$ but is analytic at some point in every neighborhood of $z_0$. A singular point $z_0$ is said to be isolated if, in addition, there is a deleted neighborhood $B'(z_0, \varepsilon)$ of $z_0$ throughout which $f$ is analytic.

When $z_0$ is an isolated singularity of a function $f$, there is a positive number $R$ such that $f$ is analytic at each point $z$ in the deleted neighborhood $B'(z_0, R)$. Consequently, in that neighborhood $f(z)$ is represented by a Laurent series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - z_0)^m}$, $\ (z \in B'(z_0, R))$, (24.1)

which converges uniformly on every compact subset of $B'(z_0, R)$. As demonstrated before, the coefficients $a_n, b_m$ have certain integral representations. In particular,

$$b_m = \frac{1}{2\pi i} \int_C f(z) \frac{dz}{(z - z_0)^{m-1}} \quad (m = 1, 2, \ldots)$$

where $C$ is any positively oriented simple closed contour around $z_0$ and lying in the punctured disk $B'(z_0, R)$. When $m = 1$, this expression for $b_m$ can be written

$$b_1 = \frac{1}{2\pi i} \int_C f(z) \, dz. \quad (24.2)$$

Definition 24.2. The complex number $b_1$, which is the coefficient of $1/(z - z_0)$ in the expansion (24.1), is called the residue of $f$ at the isolated singularity $z_0$. We often use the notation

$$\text{Res}_{z=z_0} f(z),$$

to denote the residue $b_1$.

Example. Consider the integral

$$\int_C \frac{dz}{z(z - 2)^4},$$

where $C$ is the positively oriented circle $|z - 2| = 1$. While we may evaluate this integral by the Cauchy integral formula for derivatives (exercise?), or here we use the notion of residue. The integrand is analytic in the deleted neighborhood $B'(2, 2)$, in which it has the following Laurent series expansion as:

$$\frac{1}{z(z - 2)^4} = \frac{1}{(z - 2)^4} \cdot \frac{1}{z} = \frac{1}{(z - 2)^4} \cdot \frac{2 + (z - 2)}{z} = \frac{1}{(z - 2)^4} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (z - 2)^k}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (z - 2)^{k-4}}{2^{k+1}}.$$

Now we are going to read off the coefficient of the term $(z - 2)^{-1}$, which is the case when $k = 3$. Therefore the residue of the function $1/(z(z - 2)^4)$ at $z = 2$ is $(-1)^3/2^4 = -1/16$, and the contour we are asked to evaluate is

$$\int_C \frac{dz}{z(z - 2)^4} = 2\pi i \cdot \frac{-1}{16} = \frac{-\pi i}{8}.$$
If a function $f$ has only a finite number of singular points interior to a given simple closed contour $C$, then they must be isolated. The following theorem, which is known as Cauchy’s residue theorem, is a precise statement of the fact that if $f$ is also analytic on $C$ and $C$ is described in the positive sense, then the value of the integral of $f$ around $C$ is $2\pi i$ times the sum of the residues of $f$ at those singular points.

**Theorem 24.3 (Residue theorem).** Let $C$ be a positively oriented simple closed contour. If a function $f$ is analytic inside and on $C$ except for a finite number of singular points $z_k$ ($k = 1, 2, \ldots, n$) inside $C$, then

$$\int_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res } f(z).$$

(24.3)

**Proof.** Let the points $z_k$, $k = 1, 2, \ldots, n$, be centers of positively oriented circles $C_k$ which are interior to $C$ and are so small that no two of them have points in common. The circles $C_k$, together with the simple closed contour $C$, form the boundary of a closed region throughout which $f$ is analytic and whose interior is a multiply connected domain. Hence by Theorem 13.5 (deformation of contours,) we have

$$\int_C f(z) \, dz - n \sum_{k=1}^{n} \int_{C_k} f(z) \, dz = 0.$$

This in turn implies Equation (24.3) because

$$\int_{C_k} f(z) \, dz = 2\pi i \text{Res } f(z),$$

(k = 1, 2, \ldots, n),

and the proof is complete. □

**Example.** Let us use the residue theorem to evaluate the integral

$$\int_C \frac{5z - 2}{z(z - 1)} \, dz,$$

where $C$ is the circle $|z|=2$, described counterclockwise. The integrand has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to $C$. We can find both residues at $z = 0$ and $z = 1$ by looking at the perspective Laurent series. Firstly when $0 < |z| < 1$,

$$\frac{5z - 2}{z(z - 1)} = \frac{5z - 2}{z} \cdot \frac{-1}{1 - z} = \left(-\frac{2}{z} + 5\right)(-1 - z - z^2 - \cdots),$$

therefore

$$\text{Res }_{z=0} \frac{5z - 2}{z(z - 1)} = (-2)(-1) = 2.$$

Similarly when $0 < |z - 1| < 1$,

$$\frac{5z - 2}{z(z - 1)} = \frac{5(z - 1) + 3}{z - 1} \cdot \frac{1}{1 + (z - 1)} = \left(\frac{3}{z - 1} + 5\right)[1 - (z - 1) + (z - 1)^2 - \cdots];$$

hence

$$\text{Res }_{z=1} \frac{5z - 2}{z(z - 1)} = (3)(1) = 3.$$

Thus by the residue theorem, we see that

$$\int_C \frac{5z - 2}{z(z - 1)} \, dz = 2\pi i \left(\text{Res }_{z=0} \frac{5z - 2}{z(z - 1)} + \text{Res }_{z=1} \frac{5z - 2}{z(z - 1)}\right) = 2\pi i(2 + 3) = 10\pi i.$$

In this example, it is actually simpler to write the integrand as the sum of its partial fractions:

$$\frac{5z - 2}{z(z - 1)} = \frac{2}{z} + \frac{3}{z - 1}.$$
Since \(2/z\) is already a Laurent series in \(B'(0,1)\) and \(3/(z - 1)\) is analytic in that deleted neighborhood, we see that \(\text{Res}_{z=0} \frac{5z - 2}{z(z-1)} = 2\); likewise for the residue at \(z = 1\). So we are able to retain the result we just concluded in the last paragraph.

If the function \(f\) in Theorem 24.3 is, in addition, analytic at each point in the finite plane exterior to \(C\), it is sometimes more efficient to evaluate the integrand of \(f\) around \(C\) by a single residue of a certain related function. We state the method as a theorem.

**Theorem 24.4.** If a function \(f\) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour \(C\), then

\[
\int_C f(z) \, dz = 2\pi i \text{Res}_{z=0} \left[ \frac{1}{z^2} f \left( \frac{1}{z} \right) \right].
\]

**Proof.** Let \(C_R\) be the positively oriented circle \(|z| = R\) with sufficiently large \(R > 0\) so that the contour \(C\) is interior to it. Then if \(C_0\) denotes a positively oriented circle \(|z| = R_0\), where \(R_0 > R\), we know from the Laurent’s theorem (cf. Theorem 19.1) that

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R < |z| < \infty),
\]

where

\[
c_n = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z^{n+1}} \, dz \quad (n \in \mathbb{Z}).
\]

By writing \(n = -1\), we find that

\[
\int_{C_0} f(z) \, dz = 2\pi i c_{-1}.
\]

One should be alerted that \(c_{-1}\) is not the residue of \(f\) at \(z = 0\), for the validity of the Laurent series expansion lies in a different domain. But, if we replace \(z\) by \(1/z\), we see that

\[
\frac{1}{z^2} f \left( \frac{1}{z} \right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} \quad (0 < |z| < \frac{1}{R})
\]

and hence that

\[
c_{-1} = \text{Res}_{z=0} \left[ \frac{1}{z^2} f \left( \frac{1}{z} \right) \right].
\]

Therefore we conclude that

\[
\int_{C_0} f(z) \, dz = \text{Res}_{z=0} \left[ \frac{1}{z^2} f \left( \frac{1}{z} \right) \right].
\]

Lastly, we observe that the function \(f\) is analytic on both \(C\) and \(C_0\) as well as the region between those two contours, \(\int_C f(z) \, dz = \int_{C_0} f(z) \, dz\) and our formula is proved.

**Example.** Now we compute the integral in the previous example by Theorem 24.4. Note that in this example the hypotheses in Theorem 24.4 are satisfied. So we need to find the Laurent series expansion of the stated function in an appropriate region: (denote \(f(z) = (5z - 2)/|z(z-1)|\))

\[
\frac{1}{z^2} f \left( \frac{1}{z} \right) = \frac{5 - 2z}{z(1-z)} = \frac{5 - 2z}{z} \cdot \frac{1}{1-z}
\]

\[
= \left( \frac{5}{z} - 2 \right) (1 + z + z^2 + \cdots)
\]

\[
= \frac{5}{z} + 3z + 3z + \cdots \quad (0 < |z| < 1),
\]

we see the residue to be used is 5. Hence

\[
\int_C \frac{5z - 2}{z(z - 1)} \, dz = 2\pi i \cdot 5 = 10\pi i,
\]

as already shown in the previous example.
25 Classification of isolated singular points

We saw in the previous section that the theory of residues is based on the fact if \( f \) has an isolated singular point \( z_0 \), then \( f(z) \) can be represented by a Laurent series

\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} + \cdots
\]

in some deleted neighborhood \( B'(z_0, \varepsilon) \). The portion

\[
\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_m}{(z - z_0)^m} + \cdots
\]

of the series, involving the negative powers of \( z - z_0 \), is called the principal part of \( f \) at \( z_0 \). There are three types of isolated singular points for a function \( f \), as we state the classification theorem as follows.

**Theorem 25.1.** Let \( z_0 \) be an isolated singular point of a function \( f \). One of the following situations will happen:

1. \( f \) is bounded around \( z_0 \); in this case the principal part of \( f \) at \( z_0 \) vanishes, and \( z_0 \) is called a **removable singularity** of \( f \).

2. \( \lim_{z \to z_0} |f(z)| = \infty \). In this case, \( z_0 \) is called a **pole** of \( f \), and the principal part of \( f \) at \( z_0 \) consists of finite number of terms.

3. \( \limsup_{z \to z_0} |f(z)| = \infty \) but \( \lim_{z \to z_0} |f(z)| \) does not exist. In fact there is a positive \( \varepsilon > 0 \) such that for every \( 0 < \varepsilon < z_0 \), the function \( f \) is analytic in the deleted neighborhood \( B'(z_0, \varepsilon) \), and the image \( f(B'(z_0, \varepsilon)) \) is dense in \( \mathbb{C} \); here \( z_0 \) is called an **essential singularity** of \( f \), and the principal part of \( f \) at \( z_0 \) consists of infinite number of terms.

**Proof.** We shall show first that if \( f \) is bounded around \( z_0 \), then \( \lim_{z \to z_0} f(z) \) exists in \( \mathbb{C} \). Indeed, the function

\[
g(z) = \begin{cases} 
(z - z_0)f(z) & \text{if } z \neq z_0 \\
0 & \text{if } z = z_0 
\end{cases}
\]

is continuous in some neighborhood of \( z_0 \) and analytic near \( z_0 \). By Morera’s theorem \( g \) is analytic in \( B(z_0, \delta) \) for some \( \delta > 0 \), and \( z_0 \) is a zero of \( g \). Using the Taylor series expansion of \( g \) at \( z_0 \) one sees that there is another analytic function \( h \) near \( z_0 \) with \( g(z) = (z - z_0)h(z) \). Since \( f(z) = h(z) \) in \( B'(z_0, \delta) \), we see that

\[
\lim_{z \to z_0} h(z) = \lim_{z \to z_0} f(z) = h(z_0).
\]

And we can use the Taylor series expansion of \( h \) at \( z_0 \) to see that the principal part of \( f \) at \( z_0 \) is zero.

If \( f \) is not bounded around \( z_0 \), \( \lim_{z \to z_0} |f(z)| = +\infty \) in \( \mathbb{R} \). This can be divided into two subcases: either

\[
\lim_{z \to z_0} |f(z)| = +\infty \text{ or } \lim_{z \to z_0} |f(z)| \text{ does not exist in } \mathbb{R}.
\]

When \( \lim_{z \to z_0} |f(z)| = +\infty \), we consider \( F(z) = \frac{1}{f(z)} \), which has an isolated singular point at \( z_0 \). Furthermore, \( F \) is bounded around \( z_0 \). Therefore by (1) \( z_0 \) is a removable singularity of \( F \), and we may define

\[
F(z_0) = \lim_{z \to z_0} F(z) = 0
\]

so that the extension is analytic in some open disk \( B(z_0, \delta) \). Looking at the Taylor series expansion of \( F \) about \( z_0 \), there is a positive integer \( k \) and an analytic function \( G(z) \) around \( z_0 \) with \( G(z_0) \neq 0 \) such that

\[
F(z) = (z - z_0)^k G(z).
\]
Writing $1/G(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in $B(z_0, \delta)$, we see that the original function $f(z)$ has the following Laurent series expansion in the deleted neighborhood $B'(z_0, \delta)$:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^{n-k} = \frac{a_0}{z - z_0} + \cdots + \frac{a_{k-1}}{z - z_0} + a_k + a_{k+1}(z - z_0) + \cdots$$

This proves the statement in (2). Also it is clear that if the principal part of $f$ at $z_0$ consists of only finite number of terms and is not zero, $|f(z)|$ will tend to infinity as $z$ tends to $z_0$.

The last case is $\limsup_{z \to z_0} |f(z)| = +\infty$ but $\lim_{z \to z_0} |f(z)|$ does not exist in $\mathbb{R}$, and we need to prove the statement about density in (3). Assume otherwise. That means there are $\rho > 0$, $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| > \delta$ in $B'(z_0, \rho)$. Then the function

$$H(z) = \frac{1}{f(z) - w}$$

is analytic and bounded in $B'(z_0, \rho)$. Its singularity at $z_0$ is then removable by (1), and denote again by $H$ its extension across $z_0$. It is impossible that $H(z_0) \neq 0$, otherwise $f$ will be bounded in a neighborhood of $z_0$. On the other hand if $H(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{1}{|H(z)|} = \lim_{z \to z_0} |f(z) - w| = \infty,$$

contradicting to the assumption. It is also obvious that the principal part of $f$ at $z_0$ must have infinite number of terms.
26 Residues at poles

When a function $f$ has an isolated singularity at a point $z_0$, the basic method for identifying $z_0$ as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z-z_0)$. By the classification theorem (Theorem 25.1) we know that the principal part of $f$ at $z_0$ has only finite number of terms. Hence we make the following definition:

**Definition 26.1.** An isolated singularity $z_0$ of a function $f$ is a pole of order $m$ if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

(26.1)

where $\phi(z)$ is analytic and nonzero at $z_0$.

This definition is explained as follows. When we factor out the lowest power of $z-z_0$ from the Laurent series expansion of $f$ around $z_0$, we obtain

$$f(z) = \frac{a_{-m}}{z-z_0} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$= \frac{1}{(z-z_0)^m}(a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_0(z-z_0)^m + \cdots)$$

with $m \in \mathbb{N}$ and $a_{-m} \neq 0$. If we call

$$\phi(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots + a_0(z-z_0)^m + \cdots,$$

then $\phi(z)$ is analytic in some neighborhood of $z_0$ and $\phi(z_0) = a_{-m} \neq 0$. This positive integer $m$ is then called the order of pole of $f$ at $z_0$. Once we identify the isolated singularity $z_0$ of $f$ as a pole, we have a convenient way of finding the residue of $f$ at $z_0$ as stated in the following theorem.

**Theorem 26.2.** Let $z_0$ be a pole of order $m$ of a function $f$, and write $f$ near $z_0$ as in Equation (26.1). Then

$$\text{Res}_{z=z_0} f(z) = \begin{cases} \phi(z_0) & \text{if } m = 1; \\ \frac{\phi^{(m-1)}(z_0)}{(m-1)!} & \text{if } m > 1. \end{cases}$$

**Proof.** The analytic function $\phi(z)$ has a Taylor series expansion

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

in some neighborhood $B(z_0, \varepsilon)$. And from Equation (26.1) it follows that

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0} + \sum_{n=m}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

(26.2)

when $0 < |z-z_0| < \varepsilon$. Hence the residue of $f$ at $z_0$ can be read off from the Laurent series expansion (Equation (26.2)) of $f$ near $z_0$, as stated in the theorem.

**Example.** Let $f(z) = \frac{z^3 + 2z}{(z-i)^3}$. The point $z = i$ is a pole of order 3 of $f$, and

$$f(z) = \frac{\phi(z)}{(z-i)^3}, \quad \text{where} \quad \phi(z) = z^3 + 2z.$$
The function $\phi(z)$ is entire and $\phi(i) \neq 0$. Hence the residue of $f$ at $i$ is

$$\text{Res}_{z=i} f(z) = \frac{\phi''(i)}{2!} = \frac{6i}{2} = 3i.$$ 

While the above theorem can be extremely useful, the identification of an isolated singularity as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

**Example.** Let $f(z) = \frac{\sinh z}{z^4}$. If we write $f(z) = \phi(z)/z^4$ with $\phi(z) = \sinh z$, it is incorrect that 0 is a pole of order 4 of $f$, because $\phi(0) = 0$ and Definition 26.1 cannot be applied. Instead we write down the Laurent series of $f$ near 0 directly as

$$f(z) = \frac{1}{z^4} \left( z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots \right) = \frac{1}{z^4} + \frac{1}{6z} + \frac{z}{5!} + \cdots + \frac{z^{2n-3}}{(2n+1)!} + \cdots.$$

Therefore $\text{Res}_{z=0} f(z) = 1/6$.

**Definition 26.3.** Let $f$ be a non-constant analytic function in a neighborhood of $z_0$ and $f(z_0) = 0$. There is a unique positive integer $m$ such that

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \quad \text{but} \quad f^{(m)}(z_0) \neq 0.$$

We say that $z_0$ is a zero of order $m$ of the function $f$.

The reason that this integer $m$ exists and is unique is as follows. We write down the Taylor series expansion of $f$ around $z_0$:

$$f(z) = a_0 + a_1(z-z_0) + \cdots + a_m(z-z_0)^m + \cdots.$$

Clearly $a_0 = 0$ because $f(z_0) = 0$. Since $f(z)$ is not identically zero, there is at least one non-zero coefficient $a_n$'s. Among them, there is a smallest $m \in \mathbb{N}$ with $a_m \neq 0$. Because

$$a_n = \frac{f^{(n)}(z_0)}{n!} \quad \forall n \in \mathbb{N} \cup \{0\},$$

we see the claim in the definition is valid.

**Proposition 26.4.** Let two functions $p$ and $q$ be analytic in a neighborhood of a point $z_0$, and suppose that $p(z_0) \neq 0$. If $q$ has a zero of order $m$ at $z_0$, $m \in \mathbb{N}$, then the quotient $q(z)/p(z)$ has a pole of order $m$ there.

**Proof.** By Definition 26.3, we may write $q(z)$ around $z_0$ as

$$q(z) = (z-z_0)^m g(z)$$

with $g(z)$ analytic around $z_0$ and $g(z_0) \neq 0$. This enables us to write

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{(z-z_0)^m}.$$ 

Since the function $p(z)/g(z)$ is analytic around $z_0$ and non-zero there, by Definition 26.1 we conclude that the function $p(z)/q(z)$ has a pole of order $m$ at $z_0$.

The above proposition leads us to the following useful method for identifying simple poles and finding the corresponding residues.

**Corollary 26.5.** Let two functions $p$ and $q$ be analytic around a point $z_0$. If

$$p(z_0) \neq 0, \quad q(z_0) = 0, \quad \text{and} \quad q'(z_0) \neq 0,$$

then $z_0$ is a simple pole of the quotient $p(z)/q(z)$; furthermore,

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}.$$
Proof. Since \( z_0 \) is a simple zero of \( q(z) \), it follows immediately from Proposition 26.4 that \( z_0 \) is a simple pole of the quotient \( p(z)/q(z) \). To calculate the residue, we write
\[
q(z) = (z - z_0) g(z),
\]
where \( g \) is analytic around \( z_0 \) and \( g(z_0) \neq 0 \). Hence
\[
\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}.
\]
Now applying Theorem 26.2 with \( \phi(z) = p(z)/g(z) \) and \( m = 1 \), we see that
\[
\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)}.
\]
But \( g(z_0) = q'(z_0) \), as is seen by differentiating both sides of Equation (26.3) and setting \( z = z_0 \), the proof of the corollary is complete.

Example. Let us find the residue of the function
\[
f(z) = \frac{z}{z^4 + 4}
\]
at the isolated singular point
\[
z_0 = \sqrt{2} e^{i\pi/4} = 1 + i.
\]
Writing \( p(z) = z \) and \( q(z) = z^4 + 4 \), we check that \( p(z_0) \neq 0 \) and \( z_0 \) is a simple zero of \( q(z) \). Hence by Corollary 26.5,
\[
\text{Res}_{z=1+i} f(z) = \frac{p(z_0)}{q'(z_0)} = \frac{1 + i}{4(1 + i)^3} = \frac{-i}{8}.
\]
27 Evaluation of definite and improper integrals

The calculus of residues provides a very efficient tool for the evaluation of definite and improper integrals. It is particularly important when it is impossible to find the indefinite integral explicitly, but even if the ordinary methods of calculus can be applied the use of residues is frequently a labor-saving device. The fact that the calculus of residues yields complex rather than real integrals is no disadvantage, for clearly the evaluation of a complex integral is equivalent to the evaluation of two definite integrals.

There are, however, some serious limitations, and the method is far from infallible. In the first place, the integrand must be closely connected with some analytic function. This is not very serious, for usually we are only required to integrate elementary functions, and they can all be extended to the complex domain. It is much more serious that the real integral is always extended over an interval. A special device must be used in order to reduce the problem to one which concerns integration over a closed curve. There are a number of ways in which this can be accomplished, but they all apply under rather special circumstances. The technique can be learned at the hand of typical examples, but even complete mastery does not guarantee success.

1. Let \( R(u, v) \) be a rational function of two variables \( u \) and \( v \) which is continuous on \( u^2 + v^2 = 1 \). Consider the integral

\[
I = \int_0^{2\pi} R(\cos \theta, \sin \theta) \, d\theta.
\]

Introducing the variable \( z = e^{i\theta} \), we can write

\[
\cos \theta = \frac{1}{2} (z + \frac{1}{z}), \quad \sin \theta = \frac{1}{2i} (z - \frac{1}{z}),
\]

and the integral becomes

\[
I = \int_C R \left( \frac{1}{2} (z + \frac{1}{z}) , \frac{1}{2i} (z - \frac{1}{z}) \right) \frac{dz}{iz},
\]

where \( C \) is the unit circle \( t \mapsto e^{2\pi it} \). If

\[
f(z) = \frac{1}{iz} \cdot R \left( \frac{1}{2} (z + \frac{1}{z}) , \frac{1}{2i} (z - \frac{1}{z}) \right),
\]

then

\[
I = 2\pi i \sum_{|z| < 1} \text{Res} f(z).
\]

As an example, let us compute

\[
\int_0^\pi \frac{d\theta}{a + \cos \theta}, \quad a > 1.
\] (27.1)

This integral is not extended over \( [0, 2\pi] \), but since \( \cos \theta \) takes the same values in the intervals \( [0, \pi] \) and \( [\pi, 2\pi] \) it is clear that the integral from 0 to \( \pi \) is one-half of the integral from 0 to \( 2\pi \). Taking this into account we find that the integral equals

\[
\int_{|z|=1} \frac{1}{z^2 + 2az + 1} \cdot \frac{dz}{iz}.
\]

The denominator can be factor into \((z - \alpha)(z - \beta)\) with

\[
\alpha = -a + \sqrt{a^2 - 1}, \quad \beta = -a - \sqrt{a^2 - 1}.
\]

Evidently \(|\alpha| < 1\), \(|\beta| > 1\), and the residue at \( \alpha \) is \((\alpha - \beta)^{-1}\). The value of the integral (27.1) is found to be

\[
\int_0^\pi \frac{d\theta}{a + \cos \theta}, \quad a > 1.
\]
2. An integral of the form
\[ \int_{-\infty}^{\infty} R(x) \, dx \]
converges if and only if in the rational function \( R(x) \) the degree of the denominator is at least two units higher than that of the numerator (use the comparison test) and if no pole lies on the real axis. The standard procedure is to integrate the complex function \( R(z) \) over a closed curve consisting of a line segment \([-\rho, \rho]\) and the semicircle from \( \rho \) to \(-\rho\) in the upper half plane. If \( \rho \) is large enough this curve encloses all poles in the upper half plane, and the corresponding integral is equal to \( 2\pi i \) times the sum of the residues in the upper half plane. We perform the following estimate on the integral on the semicircle (\( C_\rho \) denotes the semicircular path):
\[
\left| \int_{C_\rho} R(z) \, dz \right| \leq \int_{C_\rho} |R(z)||dz| \leq \frac{C}{\rho^2} \cdot 2\pi \rho \to 0
\]
as \( \rho \to \infty \) (\( C \) is some constant for sufficient large \( \rho \)). Therefore we obtain\(^1\)
\[
\int_{-\infty}^{\infty} R(x) \, dx = 2\pi i \sum_{\text{Im } a > 0} \text{Res } z = a R(z).
\]

One more note: we actually computed the principal value of the improper integral, which is defined as follows:
\[
P.V. \int_{-\infty}^{\infty} R(x) \, dx := \lim_{\rho \to \infty} \int_{-\rho}^{\rho} R(x) \, dx.
\]
Since our rational function has the aforementioned hypothesis on degrees, the improper integral converges absolutely and can be obtained from its principal value.

As an application, one can show that
\[
\int_{0}^{\infty} \frac{x^2 \, dx}{(x^2 + 9)(x^2 + 4)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 \, dx}{(x^2 + 9)(x^2 + 4)} = \frac{\pi}{200}.
\]

3. The same method can be applied to an integral of the form
\[ \int_{-\infty}^{\infty} R(x) \, e^{ix} \, dx \]
whose real and imaginary parts determine the important integrals
\[ \int_{-\infty}^{\infty} R(x) \cos x \, dx, \quad \int_{-\infty}^{\infty} R(x) \sin x \, dx. \]
Since \( |e^{iz}| = e^{-\text{Im} z} \) is bounded in the upper half plane, we can again conclude that the integral over the semicircle is zero, provided that the rational function \( R(z) \) has the same constraints on degrees and poles as in Case 2. We then obtain
\[
\int_{-\infty}^{\infty} R(x) \, e^{ix} \, dx = 2\pi i \sum_{\text{Im } a > 0} \text{Res } z = a R(z) \, e^{iz}.
\]
As an application, one shows that
\[
\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2 + 1)^2} \, dx = \frac{2\pi}{e^3},
\]
\(^1\)All these computations will be done explicitly in the classroom.
Lecture note on Complex Variables

3. (Continued) On the other hand, if the rational function $R(x)$ is expressed as the quotient of two polynomials $P(x)/Q(x)$, with $\deg P \leq \deg Q - 1$, we need the following result to estimate the contour integral on the semicircular path.

We need the following result. Let $f$ be a continuous function in the upper half-plane $\text{Im } z > 0$, tending to zero when $|z| \to \infty$. Let $C_r$ be the semicircular path from $r$ to $-r$ ($r > 0$). It is easily checked that the following inequality holds in elementary calculus:

$$\frac{2\theta}{\pi} \leq \sin \theta, \quad \forall \theta \in [0, \frac{\pi}{2}].$$

Let us write $z = re^{i\theta}$, $0 \leq \theta \leq \pi$. The integral becomes

$$I_r := \int_{C_r} f(z) e^{iaz} \, dz = \int_{0}^{\pi} f(re^{i\theta}) e^{iaz} \, d\theta.$$

Let $M_r$ be the maximum value of $|f(z)|$ on $C_r$. Then

$$|I_r| \leq r \int_{0}^{\pi} |f(re^{i\theta})| e^{-\alpha r \sin \theta} \, d\theta \leq r M_r \int_{0}^{\pi} e^{-\alpha r \sin \theta} \, d\theta \leq 2r M_r \int_{0}^{\pi/2} e^{-2\alpha r \theta/\pi} \, d\theta = 2r M_r \frac{1 - e^{-\alpha r}}{2\alpha r/\pi} \leq \frac{\pi}{\alpha} M_r.$$

Therefore

$$\lim_{r \to \infty} \int_{C_r} f(z) e^{iaz} \, dz = 0.$$

For example,

$$\int_{0}^{\infty} \frac{x \sin x \, dx}{x^2 + 2x + 2} = \frac{\pi}{2e} \left(\sin 1 + \cos 1\right).$$

4. We are still interested in finding the principal values of

$$\int_{-\infty}^{\infty} R(x) e^{iax} \, dx,$$

where $\alpha > 0$ is fixed and $R(x) = P(x)/Q(x)$ is a rational function with $\deg P \leq \deg Q - 1$, but now we allow $Q(x)$ has a finite number of simple zeroes on the real axis. We now illustrate how we are going to deal with those simple poles of $R(x)$.

Again we state a lemma here: Let $\psi(z)$ be a meromorphic function with a simple pole at $z = 0$. Let $\varepsilon$ be a small positive number and $C_{\varepsilon}$ be the semicircular path from $-\varepsilon$ to $\varepsilon$ in the upper half-plane, (which is clockwise.) Then

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon}} \psi(z) \, dz = -i\pi \text{Res } \psi(z).$$

The proof goes as follows. Write $\psi(z) = \frac{b_{-1}}{z} + \varphi(z)$ with $\varphi$ an analytic function in some neighborhood of 0. Clearly

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon}} \varphi(z) \, dz = 0$$

because $\varphi$ is bounded around 0. On the other hand,

$$\lim_{\varepsilon \to 0} \int_{C_{\varepsilon}} \frac{1}{z} \, dz = -i\pi,$$
by using the parameterization $z = \varepsilon e^{i\theta}, \theta : \pi \to 0$ of the path $C_\varepsilon$. Hence the lemma is proved.

Combing the estimates in 1–3., we see that, for $\alpha > 0$,

$$\text{P.V.} \int_{-\infty}^{\infty} R(x) e^{i\alpha x} \, dx = \pi i \left[ \sum_{a \in \mathbb{R}} \text{Res} R(z) e^{i\alpha z} + 2 \sum_{\Im b > 0} \text{Res} R(z) e^{i\alpha z} \right].$$

For instance,

$$\int_0^\infty \frac{\sin x}{x} \, dx = \frac{1}{2i} \text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx = \frac{\pi}{2}.$$

5. Again we are interested in

$$\int_{-\infty}^{\infty} R(x) \, dx$$

where $R(x) = P(x)/Q(x)$ is a rational function, $\deg P \leq \deg Q - 2$, and $Q(x)$ does not have poles on the positive real axis. This time we consider the auxiliary function $f(z) := R(z) \log z$ integrated along the contour $\gamma_{r,\varepsilon,h}$. (Figures are drawn on the blackboard.)

For $\varepsilon, h > 0$, small, and $r > 0$, large, we have

$$\int_{\gamma_{r,\varepsilon,h}} f(z) \, dz = 2\pi i \sum_{Q(a) = 0} \text{Res} R(z) \log(z).$$

For $\varepsilon, r$ fixed, we have the limits, uniform on $x \in [\varepsilon, r]$,

$$\lim_{h \to 0^+} \lim_{\Im z \to 0^+} R(z) \log z = R(x) \ln x$$

$$\lim_{h \to 0^-} \lim_{\Im z \to 0^-} R(z) \log z = R(x)(\ln x + 2\pi i).$$

Therefore, if $\gamma_\rho$ denotes the circle of center 0 and radius $\rho$, positively oriented, we have

$$\int_{\gamma_r - \gamma_\varepsilon} R(z) \log z \, dz - 2\pi i \int_{\varepsilon}^r R(x) \, dx = 2\pi i \sum_{Q(a) = 0} \text{Res} R(z) \log z.$$

As $r \to \infty$ and $\varepsilon \to 0^+$, the first integral tends to zero, hence we conclude that

$$\int_0^{\infty} R(x) \, dx = - \sum_{Q(a) = 0} \text{Res} R(z) \log z.$$ 

As an example, we compute

$$\int_0^{\infty} \frac{dx}{(x^2 + 4)^2} = \frac{\pi}{32}.$$

Using the same contour $\gamma_{r,\varepsilon,h}$, we can compute

$$\int_0^{\infty} \frac{\ln x}{(x^2 + 4)^2} = \frac{\pi}{32} (\ln 2 - 1).$$
28 Argument principle

A function $f$ is said to be meromorphic in a domain $D$ if it is analytic throughout $D$ except possibly for poles. Suppose now that $f$ is meromorphic in the domain interior to a positively oriented simple closed contour $C$ and that it is analytic and nonzero on $C$. The image $\Gamma$ of $C$ under the transformation $w = f(z)$ is again a closed contour, not necessarily simple, in the $w$-plane. As a point $z$ traverses $C$ in the positive direction, its image $w$ traverses $\Gamma$ in a particular direction that determines the orientation of $\Gamma$. Note that, since $f$ has no zeroes on $C$, the contour $\Gamma$ does not pass through the origin in the $w$-plane.

Let $w$ and $w_0$ be points on $\Gamma$, where $w_0$ is fixed and $\phi_0$ is a value of $\text{arg } w_0$. Then let $\text{arg } w$ vary continuously, starting with the value $\phi_0$, as the point $w$ begins at the point $w_0$ and traverses $\Gamma$ once in the direction of orientation assigned to it by the mapping $w = f(z)$. When $w$ returns to the starting point $w_0$, arg $w$ assumes a particular value of arg $w_0$, which we denote by $\phi_1$. Thus the change in arg $w$ as $w$ describes $\Gamma$ once in its direction of orientation is $\phi_1 - \phi_0$. This change is, of course, independent of the point $w_0$ chosen to determine it. Since $w = f(z)$, the number $\phi_1 - \phi_0$ is, in fact, the change in $\text{arg } f(z)$ as $z$ describes $C$ once in the positive direction, starting with a point $z_0$; and we write

$$\Delta_C \arg f(z) = \phi_1 - \phi_0.$$ 

The value of $\Delta_C \arg f(z)$ is evidently an integral multiple of $2\pi$, and the integer

$$\frac{1}{2\pi} \Delta_C \arg f(z)$$

represents the number of times the point $w$ winds around the origin in the $w$-plane. For that reason, this integer is sometimes called the winding number of $\Gamma$ with respect to the origin $w = 0$. It is positive if $\Gamma$ winds around the origin in the counterclockwise direction and negative if it winds clockwise around that point. The winding number is always zero when $\Gamma$ does not enclose the origin.

The winding number can be determined from the number of zeroes and poles of $f$ interior to $C$. The numbers of zeroes and poles are necessarily finite if $f$ is not identically zero. Suppose now that $f$ has $Z$ zeroes and $P$ poles in the domain interior to $C$, counting with multiplicity, that is, we agree that $f$ has $m$ zeroes at a point $z$ if it has a zero of order $m$ there, likewise for poles. Under this assumption, we are now ready to state the argument principle.

**Theorem 28.1 (Argument principle).** Let a function $f$ be meromorphic in the domain interior to a positively oriented simple closed contour $C$, and suppose that $f$ is analytic and nonzero on $C$. If, counting multiplicities, $Z$ is the number of zeroes and $P$ the number of poles inside $C$, then,

$$\frac{1}{2\pi} \Delta_C \arg f(z) = Z - P. \quad (28.1)$$

**Proof.** We will evaluate the integral of $f'(z)/f(z)$ around the contour $C$ in two different ways. First, we let $z = z(t) (a \leq t \leq b)$ be a parametric representation for $C$, so that

$$\int_C \frac{f'(z)}{f(z)} \, dz = \int_a^b \frac{f'[z(t)] z'(t)}{f[z(t)]} \, dt. \quad (28.2)$$

Since, under the transformation $w = f(z)$, the image $\Gamma$ of $C$ never passes through the origin in the $w$-plane, the image of any point $z = z(t)$ on $C$ can be expressed in exponential form as $w = \rho(t) e^{i\phi(t)}$. Thus

$$f[z(t)] = \rho(t) e^{i\phi(t)} \quad (a \leq t \leq b). \quad (28.3)$$

The numerator of the integrand at the right-hand side of (28.2) becomes

$$f'[z(t)] z'(t) = \frac{d}{dt} f[z(t)] = \frac{d}{dt} [\rho(t) e^{i\phi(t)}] = \rho'(t) e^{i\phi(t)} + i \rho(t) e^{i\phi(t)} \phi'(t). \quad (28.4)$$
Inasmuch as \( \rho'(t) \) and \( \phi'(t) \) are piecewise continuous on the interval \( a \leq t \leq b \), we can now use Equations (28.3) and (28.4) to write integral (28.2) as follows:

\[
\int_C \frac{f'(z)}{f(z)} \, dz = \int_a^b \frac{\rho'(t)}{\rho(t)} \, dt + i \int_a^b \phi'(t) \, dt = (\ln \rho(t) + i \phi(t))_a^b.
\]

But \( \rho(b) = \rho(a) \) and \( \phi(b) - \phi(a) = \Delta_C \arg f(z) \), hence

\[
\int_C \frac{f'(z)}{f(z)} \, dz = i \Delta_C \arg f(z).
\]

Another way to evaluate integral (28.2) is to use Cauchy’s residue theorem. To be specific, we observe that the integrand \( f'(z)/f(z) \) is analytic inside and on \( C \) except at the points inside \( C \) at which the zeroes and the poles of \( f \) occur. If \( f \) has a zero of order \( m_0 \) at \( z_0 \), then

\[
f(z) = (z - z_0)^{m_0} g(z),
\]

where \( g(z) \) is analytic and nonzero at \( z_0 \). Hence

\[
f'(z) = m_0(z - z_0)^{m_0-1}g(z) + (z - z_0)^{m_0}g'(z),
\]

or

\[
\frac{f'(z)}{f(z)} = \frac{m_0}{z - z_0} + \frac{g'(z)}{g(z)}.
\]

Since \( g'(z)/g(z) \) is analytic at \( z_0 \), we see that

\[
\text{Res}_{z=z_0} \frac{f'(z)}{f(z)} = m_0.
\]

Similarly, when \( f \) has a pole of order \( m_0 \) at \( z_0 \), then

\[
\text{Res}_{z=z_0} \frac{f'(z)}{f(z)} = -m_0.
\]

Now we apply the residue theorem, and find that

\[
\int_C \frac{f'(z)}{f(z)} \, dz = 2\pi i (Z - P).
\]

Therefore Equation (28.1) is proved.

**Example.** The function \( f(z) = 1/z^2 \) has only a pole of order 2 at the origin and no zeroes. If we parameterize the unit circle \( C \) by \( z = e^{i\theta}, 0 \leq \theta \leq 2\pi \), then we see that

\[
-2 = Z - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi} \Delta_C \arg f(z).
\]

The next theorem, which is known as Rouché’s theorem, is an immediate consequence of Theorem 28.1 and is important in locating regions of the complex plane in which there may be zeroes of a given analytic function.

**Theorem 28.2 (Rouché’s theorem).** Let two functions \( f(z) \) and \( g(z) \) be analytic inside and on a simple closed contour \( C \), and suppose that \( |f(z)| > |g(z)| \) at each point of \( C \). Then \( f(z) \) and \( f(z) + g(z) \) have the same number of zeroes, counting with multiplicities, inside \( C \).

**Proof.** The orientation of \( C \) is immaterial here, so we may assume that the orientation is positive. We begin with an observation that neither \( f(z) \) nor \( f(z) + g(z) \) can assume zero on \( C \), since

\[
|f(z)| > |g(z)| \geq 0, \quad |f(z) + g(z)| \geq |f(z)| - |g(z)| > 0
\]

if \( z \in C \).
If \( Z_f \) and \( Z_{f+g} \) denote the number of zeroes, counting with multiplicities, of \( f(z) \) and \( f(z) + g(z) \), respectively, inside \( C \), we know from Theorem 28.1 that

\[
Z_f = \frac{1}{2\pi} \Delta_C \arg f(z) \quad \text{and} \quad Z_{f+g} = \frac{1}{2\pi} \Delta_C \arg [f(z) + g(z)].
\]

Consequently, since

\[
\Delta_C \arg [f(z) + g(z)] = \Delta_C \arg \left\{ f(z) \left[1 + \frac{g(z)}{f(z)}\right]\right\} = \Delta_C \arg f(z) + \Delta_C \arg \left[1 + \frac{g(z)}{f(z)}\right],
\]

it is clear that

\[
Z_{f+g} = Z_f + \frac{1}{2\pi} \Delta_C \arg F(z), \quad \text{where} \quad F(z) = 1 + \frac{g(z)}{f(z)}. \tag{28.5}
\]

However, for any point \( z \in C \),

\[
|F(z) - 1| = \left| \frac{g(z)}{f(z)} \right| < 1;
\]

and this means that, under the transformation \( w = F(z) \), the image of \( C \) lies in the open disk \(|w - 1| < 1\). That image does not enclose the origin \( w = 0 \). Hence \( \Delta_C \arg F(z) = 0 \), and Equation (28.5) reduces to \( Z_{f+g} = Z_f \), the theorem is then proved.

**Example.** In order to determine the number of roots of the equation

\[
z^7 - 4z^3 + z - 1 = 0
\]

inside the circle \(|z| = 1\), we write

\[
f(z) = -4z^3 \quad \text{and} \quad g(z) = z^7 + z - 1.
\]

Then observe that \(|f(z)| = 4 > 3 = |z^7| + |z| + 1 \geq |g(z)| \) when \(|z| = 1\). The hypotheses for Rouché’s theorem are thus satisfied. Consequently, since \( f(z) \) has 3 zeroes, counting with multiplicities, inside the circle \(|z| = 1\), so does \( f(z) + g(z) \). That is, the equation \( z^7 - 4z^3 + z - 1 = 0 \) has three solutions there.
Lecture note on Complex Variables

29 The Gamma function

The Gamma function $\Gamma$ has been intensively studied in complex analysis. In this section we are going to investigate some of its properties.

**Definition 29.1.** The Gamma function is the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

for $\text{Re} \, z > 0$.

When $\text{Re} \, z > 0$, this improper integral converges. Let us write $\Gamma(x)$ for positive real $x$ for a moment. An integration by parts shows that

$$\Gamma(x + 1) = \int_0^\infty e^{-t} t^x \, dt = x \int_0^\infty e^{-t} t^{x-1} \, dt = x \Gamma(x).$$

Furthermore $\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1$. By induction we immediately see that

$$\Gamma(n) = (n-1)! = (n-1)(n-2) \cdots 2 \cdot 1, \quad \forall n \in \mathbb{N}.$$

Now we will give the analytic continuation of the Gamma function as a meromorphic function on $\mathbb{C}$. We write the integral in the form

$$\int_0^\infty = \int_0^1 + \int_1^\infty.$$

Observe that the integral from 1 to $\infty$ defines an entire function of $z$, say

$$H(z) = \int_1^\infty e^{-t} t^{z-1} \, dt$$

because the convergence problem for $t$ near $0^+$ has disappeared. For the other integral

$$P(z) = \int_0^1 e^{-t} t^{z-1} \, dt,$$

we can write down the MacLaurin series for $e^{-t}$:

$$e^{-t} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n.$$

Note that this series converges uniformly over any compact subset of $\mathbb{C}$, therefore we can interchange the sum and the integral to obtain

$$P(z) = \int_0^1 \left( \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^{z+n} \right) \, dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z+n-1} \, dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n}.$$

Then

$$\Gamma(z) = P(z) + H(z)$$

gives the so-called Mittag-Leffler decomposition of the Gamma function in terms of its principal parts, and gives one form for its meromorphic continuation to $\mathbb{C}$.

Another approach is to write down the Gamma function in terms of convergent infinite product. Let us start with the following lemma.
Lemma 29.2.

\[ 0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n} \]

for every \( n \in \mathbb{N} \), and \( t \in [0, n] \).

**Proof.** The function \( \chi_n(t) = e^t (1 - \frac{t}{n})^n \) has the property that \( \chi_n(0) = 1 \), \( \chi_n(n) = 0 \), and

\[ \chi_n'(t) = e^t \left[ (1 - \frac{t}{n})^n + n \left( 1 - \frac{t}{n} \right)^{n-1} \cdot \left( -\frac{1}{n} \right) \right] = e^t \left( 1 - \frac{t}{n} \right)^{n-1} \left( -\frac{1}{n} \right) \leq 0. \]

For the other inequality, we consider \( \theta_n(t) = 1 - e^t (1 - \frac{t}{n})^n - \frac{t^2}{n} \). Clearly \( \theta_n(0) = 0 \) and

\[ \theta_n'(t) = \frac{t}{n} e^t \left( 1 - \frac{t}{n} \right)^{n-1} - \frac{2t}{n} = \frac{t}{n} \left[ e^t \left( 1 - \frac{t}{n} \right)^{n-1} - 2 \right]. \]

Let \( g_n(t) = e^t (1 - \frac{t}{n})^{n-1} - 2 \). Then

\[ g_n'(t) = e^t \left[ (1 - \frac{t}{n})^{n-1} - \frac{n-1}{n} \left( 1 - \frac{t}{n} \right)^{n-2} \right] = e^t \left( 1 - \frac{t}{n} \right)^{n-2} \left( 1 - \frac{t}{n} - 1 + \frac{1}{n} \right) = e^t \left( 1 - \frac{t}{n} \right)^{n-2} \left( 1 - \frac{t}{n} \right) \frac{1}{n}. \]

\( g_n'(t) = 0 \) only when \( t = 1 \), and \( g_n(1) \) is a maximum for \( g_n(t) \). One checks that for \( n \geq 2 \), \( (1 - \frac{1}{n})^{n-1} \leq 1/2 \), hence \( g_n(1) \leq e/2 - 2 < 0 \). This implies that \( \theta_n(t) \) is always decreasing when \( n \geq 2 \). For \( n = 1 \), we know that \( g_1(t) \leq 0 \) when \( t \in [0, \log 2] \), \( g_1(t) \geq 0 \) when \( t \in [\log 2, 1] \). Hence \( \theta_1(t) \) is decreasing on \([0, \log 2]\), increasing on \([\log 2, 1]\). Nevertheless \( \theta_1(1) = 0 \), this concludes the proof of the lemma.

Let us define the auxiliary functions \( \varphi_n \) for \( n \in \mathbb{N} \) by

\[ \varphi_n(z) := \int_0^n \left( 1 - \frac{t}{n} \right)^{n-1} dt. \]

**Lemma 29.3.** The functions \( \varphi_n \) converge to \( \Gamma \) locally uniformly in the half-plane \( \{ z \mid \Re{z} > 0 \} \).

**Proof.** Let \( x = \Re{z} \). If \( 0 < x \leq 1 \),

\[ \left| \Gamma(z) - \int_0^n e^{-t} t^{z-1} dt \right| \leq \int_n^\infty e^{-t} t^{z-1} dt \leq n^{z-1} \int_n^\infty e^{-t} dt \leq \frac{e^{-n}}{n^{1-z}}. \]

If \( x > 1 \) we can integrate by parts and obtain

\[ \int_n^\infty e^{-t} t^{z-1} dt = e^{-n} n^{z-1} + (x-1) \int_n^\infty e^{-t} t^{x-2} dt. \]

It is clear that for any \( 0 < A < B < \infty \) there is a constant \( C \) such that whenever \( A \leq \Re{z} \leq B \) we have

\[ \left| \Gamma(z) - \int_0^n e^{-t} t^{z-1} dt \right| \leq C n^{B-1} e^{-n} \to 0 \quad \text{as} \quad n \to \infty. \]

On the other hand, by Lemma 29.2

\[ \left| \varphi_n(z) - \int_0^n e^{-t} t^{z-1} dt \right| \leq \int_0^n t^{z-1} \left( e^{-t} - \left( 1 - \frac{t}{n} \right)^n \right) dt \leq \int_0^n t^{z-1} \frac{t^2 e^{-t}}{n} dt \]

\[ \leq \frac{1}{n} \int_0^n e^{-t} t^{x+1} dt \leq \frac{\Gamma(x+2)}{n} \to 0 \quad \text{as} \quad n \to \infty. \]

Hence the lemma is proved.

Let us make the change of variable \( t = n \tau \) in \( \varphi_n \):

\[ \varphi_n(z) = \int_0^n \left( 1 - \frac{t}{n} \right)^{n-1} dt = n z \int_0^1 (1 - \tau)^{n-1} \tau^{z-1} d\tau = n z \int_0^1 (1 - \tau)^{n-1} \tau^z d\tau = \cdots \]

\[ = n z \cdot \frac{n!}{z(z+1) \cdots (z+n-1)} \int_0^1 \tau^{z+n-1} d\tau = \frac{n^z n!}{z(z+1) \cdots (z+n)} = \frac{n^z}{z^1(1+\frac{z}{2})(\frac{z}{n}+1)} \cdots (\frac{z}{n}+1) \]

57
Define the Euler-Mascheroni constant \( \gamma := \lim_{n \to \infty} \left( \sum_{j=1}^{\infty} \frac{1}{j} - \log n \right) \) (\( \approx 0.57721 \ldots \)) and let \( \gamma_n = \sum_{j=1}^{n} \frac{1}{j} - \log n - \gamma \).

Rewriting \( n^z \) as \( n^z = \exp \left( \sum_{j=1}^{n} \frac{1}{j} - \gamma - \gamma_n \right) z \).

Plug it in to the last expression of \( \varphi_n(z) \) and let \( n \to \infty \), we see that
\[
\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{j=1}^{\infty} \left( 1 + \frac{z}{j} \right) e^{-\frac{z}{j}}
\quad \text{(29.1)}
\]
when \( \text{Re} \, z > 0 \).

**Lemma 29.4.** The infinite product
\[
\prod_{j=1}^{\infty} \left( 1 + \frac{z}{j} \right) e^{-\frac{z}{j}}
\]
converges uniformly on any compact subset of \( \mathbb{C} \).

**Proof.** By Theorem 23.1 it is equivalent to show that the sum of the logarithm
\[
\sum_{j=1}^{\infty} \left[ \log \left( 1 + \frac{z}{j} \right) - \frac{z}{j} \right]
\quad \text{(29.2)}
\]
converges uniformly on any finite closed disk \( \overline{B}(0, R) \), \( R \in \mathbb{N} \). We use the MacLaurin series expansion of the branch of \( \log(1 - \zeta) \) with \( \log 1 = 0 \):
\[
\log(1 - \zeta) = -\sum_{k=1}^{\infty} \frac{\zeta^k}{k}.
\]
Hence with \( \zeta_j = -\frac{z}{j} \), we have when \( j > 2R \) and \( |z| \leq R \),
\[
\left| \log \left( 1 + \frac{z}{j} \right) - \frac{z}{j} \right| \leq \sum_{k=2}^{\infty} \frac{|\zeta_j|^k}{k} \leq \frac{1}{2} \frac{|\zeta_j|^2}{1 - |\zeta_j|} < |\zeta_j|^2.
\]
Because the sequence \( \sum_{n=1}^{\infty} n^{-2} \) converges, we can choose a positive integer \( N > 2R \) for any given \( \varepsilon > 0 \) so that the tail of the series (29.2) is bounded by
\[
\sum_{j=N}^{\infty} |\zeta_j|^2 = \sum_{j=N}^{\infty} \left| \frac{z}{j} \right|^2 \leq \left( \sum_{j=N}^{\infty} \frac{1}{j^2} \right) \cdot R^2 < \varepsilon.
\]

This lemma shows that Equation (29.1) is a *Weierstrass product*. Because the infinite product converges uniformly on any compact subset of \( \mathbb{C} \), it defines an entire function with simple zeroes only at all non-positive integers. This will prove that the Gamma function \( \Gamma \) never vanishes as a meromorphic function on \( \mathbb{C} \), and it has simple poles only at all non-positive integers.
30 The Stirling’s formula

In this section we are going to prove the asymptotic formula for $n!$, which is known as the Stirling’s formula. Let us start with some basic formulae in elementary calculus.

Lemma 30.1. Let $n \in \mathbb{N}$. Then

$$\int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{(n-1)(n-3)\cdots3\cdot1}{n(n-2)\cdots4\cdot2} \cdot \frac{\pi}{2} & \text{when } n \text{ is even;} \\ \frac{(n-1)(n-3)\cdots4}{n(n-2)\cdots5\cdot3} & \text{when } n \text{ is odd.} \end{cases} \quad (30.1)$$

Proof. Define $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Then integration by parts shows

$$I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx = -\sin^{n-1} x \cos x \bigg|_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x \, dx = (n-1)(I_{n-2} - I_n).$$

Hence we obtain a recursive relation:

$$I_n = \frac{n-1}{n} \cdot I_{n-2} \quad \forall n \in \mathbb{N}, n \geq 2.$$

Use this recursion formula and $I_0 = \frac{\pi}{2}$, $I_1 = 1$, the Equation (30.1) drops out.

Lemma 30.2 (Wallis formula).

$$\lim_{n \to \infty} \frac{1}{n} \left( \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right)^2 = \pi.$$  

Or equivalently,

$$\lim_{n \to \infty} \frac{2^{2n} \,(n!)^2}{(2n)! \sqrt{n}} = \sqrt{\pi}. \quad (30.2)$$

Proof. We observe in the previous lemma,

$$0 < I_{2n+1} < I_{2n} < I_{2n-1} \quad \forall n \in \mathbb{N}.$$

Moving things around, we find

$$\pi < \frac{1}{n} \left( \frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} \right)^2 < \frac{2n+1}{2n} \pi.$$

Now let $n \to \infty$ and apply the squeeze theorem, we reach the conclusion.

For the second identity, we observe that

$$\frac{2 \cdot 4 \cdots (2n)}{1 \cdot 3 \cdots (2n-1)} = \frac{2^{2n} \,(n!)^2}{(2n)!}.$$

Therefore a straightforward computation proves the second limit.

Theorem 30.3 (Stirling’s formula).

$$\lim_{n \to \infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1.$$
Proof. Define a sequence of positive numbers

\[ a_n := \frac{n!}{n^n e^{-n} \sqrt{n}} \quad \forall n \in \mathbb{N}. \]

Our goal is to show that the sequence \((a_n)\) converges to a positive number \(\alpha\). Write

\[ \frac{a_n}{a_{n+1}} = \frac{n!}{n^n e^{-n} \sqrt{n}} \frac{(n+1)^{n+1} e^{-(n+1)} \sqrt{n+1}}{(n+1)!} = \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} \cdot \frac{1}{e}. \]

For sufficiently large \(n\),

\[ \log \frac{a_n}{a_{n+1}} = \left(n + \frac{1}{2}\right) \log \left(1 + \frac{1}{n}\right) - 1 = \left(n + \frac{1}{2}\right) \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + O(n^{-4})\right) - 1 \]
\[ = (1 - 1) + \left(\frac{1}{2} - \frac{1}{2}\right) \frac{1}{n} + \left(\frac{1}{3} - \frac{1}{4}\right) \frac{1}{n^2} + O(n^{-3}) \]
\[ = \frac{1}{12n^2} + O(n^{-3}) \]

Hence there exists an integer \(N \in \mathbb{N}\) such that for every \(n \geq N\),

\[ 0 < \log \frac{a_n}{a_{n+1}} \leq \frac{1}{6n^2}. \]

Now for \(M \geq N\),

\[ \log a_N - \log a_M = \sum_{n=N}^{M-1} \log \frac{a_n}{a_{n+1}} \leq \sum_{n=N}^{M-1} \frac{1}{6n^2} \leq \frac{\pi^2}{36}, \]

that is,

\[ a_M \geq \exp \left(\log a_N - \frac{\pi^2}{36}\right) =: C > 0. \]

Therefore the sequence \((a_n)\), whose tail is decreasing and bounded below by a positive constant \(C\), converges to some \(\alpha > 0\). Define

\[ \varepsilon_n := a_n - \alpha. \]

Then \(\varepsilon \to 0\) as \(n \to \infty\). And also

\[ \frac{(n!)^2}{n^{2n} e^{-2n} (\sqrt{n})^2} = (\alpha + \varepsilon_n)^2 \]
\[ \frac{(2n)!}{(2n)^n e^{-2n} \sqrt{2n}} = \alpha + \varepsilon_{2n}. \]

By taking the quotient of these equations and applying the Wallis formula (30.2), we see that

\[ \alpha = \lim_{n \to \infty} \frac{(\alpha + \varepsilon_n)^2}{\alpha + \varepsilon_{2n}} = \sqrt{2\pi}, \]

which proves the Stirling’s formula. \(\square\)
31 Conformal equivalence

Let us start looking at the mapping properties of analytic functions. In this section all curves \( z(t) \) are assumed to be such that \( z'(t) \neq 0 \).

**Definition 31.1.** Suppose two smooth curves \( C_1 \) and \( C_2 \) intersect at a point \( z_0 \). Then the angle from \( C_1 \) to \( C_2 \), denoted by \( \angle(C_1, C_2) \), is defined as the angle measured counterclockwise from the tangent line of \( C_1 \) at \( z_0 \) to that of \( C_2 \) at \( z_0 \).

**Definition 31.2.** Suppose \( f \) is defined in a neighborhood of \( z_0 \). \( f \) is said to be conformal at \( z_0 \) if \( f \) preserves the angle there. That is, for each pair of smooth curves \( C_1 \) and \( C_2 \) intersecting at \( z_0 \), \( \angle(C_1, C_2) = \angle(\Gamma_1, \Gamma_2) \), where \( \Gamma_1 = f(C_1) \) and \( \Gamma_2 = f(C_2) \). Similarly, we say \( f \) is conformal in a region \( D \) if \( f \) is conformal at all points \( z \in D \).

Note that the function \( f(z) = z^2 \) is not conformal at \( z = 0 \). For example, the positive real axis and the positive imaginary axis are mapped onto the positive real axis and the negative real axis, respectively. However, as we shall see below, it is conformal at all other points of the complex plane.

**Definition 31.3.** A function \( f \) is locally injective at \( z_0 \) if for some \( \delta > 0 \) and any distinct \( z_1, z_2 \in B(z_0, \delta) \), \( f(z_1) \neq f(z_2) \). \( f \) is locally injective in a region \( D \) if \( f \) is locally injective at every \( z \in D \).

**Theorem 31.4.** Suppose \( f \) is analytic at \( z_0 \) and \( f'(z_0) \neq 0 \). Then \( f \) is conformal and locally injective at \( z_0 \).

**Proof.** Let \( C : z(t) = x(t) + iy(t) \) be a smooth curve with \( z(t_0) = z_0 \). Then the tangent line at \( z_0 \) has the direction \( z'(t_0) \) so that its angle of inclination with the positive real axis is \( \arg z'(t_0) \). If we set \( \Gamma = f(C) \), then \( \Gamma \) can be parameterized by \( \omega(t) = f(z(t)) \) and the angle of inclination of its tangent line at \( f(z_0) \) is equal to

\[
\arg \omega'(t_0) = \arg(f'(z_0) z'(t_0)) = \arg f'(z_0) + \arg z'(t_0)
\]

when \( z'(t_0) \neq 0 \). Hence the function \( f \) maps all curves at \( z_0 \) in such a way that the angles of inclination are increased by the constant \( \arg f'(z_0) \). Thus, if \( C_1 \) and \( C_2 \) meet at \( z_0 \) and \( \Gamma_1, \Gamma_2 \) are their respective images under \( f \), it follows that \( \angle(\Gamma_1, \Gamma_2) = \angle(C_1, C_2) \).

To show \( f \) is injective in a neighborhood at \( z_0 \), let \( f(z_0) = \alpha \) and take \( \delta' > 0 \) small enough so that \( f(z_0) - \alpha \) has no other zeroes in \( B(z_0, \delta') \). Such a \( \delta' \) can always be found for otherwise we would have \( f'(z_0) = 0 \) by Weierstrass' preparation theorem.

If we let \( C = C(z_0, \delta') \) and \( \Gamma = f(C) \), it follows by the argument principle (cf. Theorem 28.1) that

\[
2\pi i = \int_C \frac{f'(z)}{f(z) - \alpha} \, dz = \int_{\Gamma} \frac{d\omega}{\omega - \alpha} = \int_{\Gamma} \frac{d\omega}{\omega - \beta}
\]

for all \( \beta \) in some sufficiently small disc \( B(\alpha, \varepsilon) \), since the winding number is locally constant. It we then take \( \delta \leq \delta' \) so that \( B(z_0, \delta) \subseteq f^{-1}(B(\alpha, \varepsilon)) \) it follows that for any \( z_1, z_2 \in B(z_0, \delta) \),

\[
2\pi i = \int_{\Gamma} \frac{d\omega}{\omega - f(z_j)} = \int_C \frac{f'(z) \, dz}{f(z) - f(z_j)}, \quad j = 1, 2;
\]

that is, the values \( f(z_1) \) and \( f(z_2) \) are both assumed once inside \( C \) so that \( f(z_1) \neq f(z_2) \) if \( z_1 \neq z_2 \). \( \square \)

**Example.** \( f(z) = e^z \) has a nonzero derivative at all points, hence it is everywhere conformal and locally injective. By the conformality of \( f \), the images of the orthogonal lines \( x = \text{constant} \) and \( y = \text{constant} \) under the mapping \( f \) are themselves orthogonal. It is easy to show that \( f \) maps the vertical lines \( x = \text{constant} \) onto circles centered at the origin and maps the horizontal lines \( y = \text{constant} \) onto rays from the origin.

**Example.** Let \( f(z) = z^2 \). Since \( f'(z) = 2z \neq 0 \) except at \( z = 0 \), \( f \) is conformal throughout \( z \neq 0 \). Thus, if we set \( f = u + iv \) it follows that the preimages of the curves \( u = c_1 \), \( v = c_2 \) for \( c_1, c_2 \neq 0 \) must be orthogonal. Indeed,
since \( u(z) = x^2 - y^2, v(z) = 2xy \) these preimages are the orthogonal systems of hyperbolas given by \( x^2 - y^2 = c_1, 2xy = c_2 \).

To analyze the mapping properties of an analytic function \( f \) at a point \( z \) where \( f'(z) = 0 \), we first consider the following special case.

**Definition 31.5.** Let \( k \) be a positive integer. \( f \) is a \( k \)-to-1 mapping from \( D_1 \) onto \( D_2 \) if for every \( \alpha \in D_2 \), the equation \( f(z) = \alpha \) has \( k \) roots (counting multiplicity) in \( D_1 \).

**Lemma 31.6.** Let \( f(z) = z^k \), \( k \) a positive integer. Then \( f \) magnifies angles at 0 by a factor of \( k \) and maps the disc \( B(0, \delta) \), \( \delta > 0 \) onto the disc \( B(0, \delta^k) \) in a \( k \)-to-1 manner.

**Proof.** Since \( f(re^{i\theta}) = r^k e^{ik\theta} \), \( f \) maps the ray from 0 with argument \( \theta \) onto the ray from 0 with argument \( k\theta \). Hence the angle at 0 between any two rays is magnified by a factor of \( k \). To see that \( f(z) = \alpha, \alpha \in B(0, \delta^k) \) has \( k \) roots in \( B(0, \delta) \), recall that if \( \alpha \neq 0 \) there are \( k \) distinct roots all lying on the circle \( |z| = |\alpha|^{1/k} \). If \( \alpha = 0 \), the equation \( z^k = 0 \) has a \( k \)-fold root at the origin.

We can now consider an extension of Theorem 31.4.

**Theorem 31.7.** Suppose \( f \) is analytic at \( z_0 \) with \( f'(z_0) = 0 \). Then, unless \( f \) is constant, in some sufficiently small open set containing \( z_0 \), \( f \) is a \( k \)-to-1 mapping and \( f \) magnifies angles at \( z_0 \) by a factor of \( k \), where \( k \) is the least positive integer for which \( f^{(k)}(z_0) \neq 0 \).

**Proof.** By translation we may assume that \( f(z_0) = 0 \). Then, by hypothesis, there is an analytic function \( g(z) \) in a neighborhood of \( z_0 \) with \( g(z_0) \neq 0 \) and

\[
 f(z) = (z - z_0)^k g(z).
\]

Since \( g(z_0) \neq 0 \), \( g \) has an analytic \( k \)th root in some disc \( B(z_0, \delta) \) (e.g. \( \tilde{g} = \log g \) and take \( \exp(\frac{1}{k} \tilde{g}(z)) \)). Thus, in that disc,

\[
 f(z) = |h(z)|^k
\]

where \( h \) is an analytic function defined by

\[
 h(z) = (z - z_0)^{1/k}(z)
\]

with \( h(z_0) = 0 \) and \( h'(z_0) = g^{1/k}(z_0) \neq 0 \). Hence, in a sufficiently small neighborhood \( D \) of \( z_0 \), \( f \) is the composition of the locally injective and conformal mapping \( h \) followed by the mapping \( z^k \). Since \( z^k \) magnifies angles at 0 by a factor of \( k \), it follows that \( f \) magnifies angles at \( z_0 \) by \( k \). Also, since \( z^k \) is \( k \)-to-1 on discs about 0, it follows that if \( B(0, \varepsilon) \subseteq h(D) \) and \( \eta = h^{-1}(B(0, \varepsilon)) \), then \( f \) is \( k \)-to-1 on \( \eta \).

The previous results combine to yield the following properties of injective analytic functions.

**Theorem 31.8.** Suppose \( f \) is an injective analytic function in a region \( D \). Then

1. \( f^{-1} \) exists and is analytic in \( f(D) \).
2. \( f \) and \( f^{-1} \) are conformal in \( D \) and \( f(D) \), respectively.

**Proof.** Since \( f \) is injective, \( f' \neq 0 \). Hence \( f^{-1} \) is also analytic. Furthermore, \((f^{-1})' = 1/f' \) so that \( f^{-1} \) has a nonzero derivative. Thus \( f \) and \( f^{-1} \) are both conformal.

The above theorem motivates the following definitions.

**Definition 31.9.**

1. An injective analytic mapping is called a conformal mapping.
2. Two regions \( D_1 \) and \( D_2 \) are said to be conformally equivalent if there exists a conformal mapping of \( D_1 \) onto \( D_2 \).

We leave it as an exercise to verify that “conformal equivalence” satisfies the usual axioms of an equivalence relation. In particular, we note that the transitive property follows from the fact that the composition of two conformal mappings is also a conformal mapping, and we will use this fact in the next sections.
Lecture note on Complex Variables

32 Special mappings

In this section we will look at some special mappings. They are usually regarded as building blocks of more complicated mappings, that is, we can take the composite of these special mappings to produce the desired result.

I Elementary transformations

(i) $w = az + b$, the affine transformations ($a \neq 0$).

The linear map $w = az + b$ is an injective map of the entire plane onto itself. The effect of the mapping on a given domain can be seen by viewing it as a composition $w = f_3 \circ f_2 \circ f_1$ of the mappings

1. $f_1(z) = kx$, $k = |a| > 0$.
2. $f_2(z) = e^{i\theta}z$, $\theta = \arg a$.
3. $f_3(z) = z + b$.

A mapping of the form $w = kx$, $k > 0$, is called a magnification. It sends each point onto another point along the same ray from the origin, multiplying its magnitude by a factor of $k$. The mapping $w = e^{i\theta}z$ is a counterclockwise rotation through an angle $\theta$. Finally, $w = z + b$ is called a translation since it translates each point by the complex number $b$.

(ii) $w = z^\alpha$, $\alpha$ real.

The function $w = z^\alpha = \exp(\alpha \log z)$ is analytic in every simply connected domain that does not contain 0. If we take the branch of log $z$ which is positive on the positive real axis, then $z^\alpha$ will map the positive axis onto itself. The point $z = re^{i\theta}$ is mapped onto $r^\alpha e^{i\alpha \theta}$ and hence $w = z^\alpha$ maps the wedge $S = \{z \mid \theta_1 < \arg z < \theta_2\}$ onto the wedge $T = \{w \mid \alpha \theta_1 < \arg w < \alpha \theta_2\}$. If, moreover, $\alpha \theta_2 - \alpha \theta_1 \leq 2\pi$, i.e., $\theta_2 - \theta_1 \leq 2\pi/\alpha$, the mapping is a conformal mapping from $S$ onto $T$.

(iii) $w = \exp z$.

Since $e^z = e^x e^{iy}$, the function $w = e^z$ maps the strip $y_1 < y < y_2$ onto the wedge $y_1 < \arg w < y_2$. If $y_2 - y_1 \leq 2\pi$, the mapping is injective and conformal. For example, the strip $0 < y < \pi$ is mapped conformally onto the upper half plane.

II The bilinear transformation $w = \frac{az + b}{cz + d}$.

The mapping

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0$$

(32.1)

is called a bilinear transformation or a Möbius transformation. The condition $ad - bc \neq 0$ insures that $f$ is identically constant. Since

$$f'(z) = \frac{ad - bc}{(cz + d)^2} \neq 0,$$

$f$ is locally injective and conformal. In fact, it is easily checked that $f$ is globally injective.

The bilinear transformation (32.1) maps the full plane, minus the point $z = -d/c$, onto the full plane minus the point $w = a/c$, since the equation

$$w = \frac{az + b}{cz + d}$$

has the explicit solution

$$z = \frac{dw - b}{-cw + a} \quad \forall w \neq \frac{a}{c}.$$

In fact, if we consider the limiting values $f(\infty) = a/c$ and $f(-d/c) = \infty$ we can say that $f$ is an automorphism of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. 

63
The set of all bilinear transformations forms a group under composition. Indeed, the mapping \( \varphi : \text{GL}(2, \mathbb{C}) \to \mathcal{M} \) \((\text{GL}(2, \mathbb{C}) \) is the group of all invertible \(2 \times 2\) matrices over \(\mathbb{C}\) and \(\mathcal{M}\) is the set of all bilinear transformations) defined by
\[
\varphi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = f, \quad f(z) = \frac{az + b}{cz + d}
\]
is an epimorphism. One verifies that the kernel of \( \varphi \) is a multiple of the identity matrix \( I_2 \), hence
\[
\mathcal{M} \cong \frac{\text{GL}(2, \mathbb{C})}{\{ \lambda I_2 \mid \lambda \neq 0 \}}
\]
by the 1st isomorphism theorem. A set of generators of \( \frac{\text{GL}(2, \mathbb{C})}{\{ \lambda I_2 \mid \lambda \neq 0 \}} \) are of the form
\[
\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
and the first two correspond to affine transformations that map circles onto circles and lines onto lines. Hence we will study the map
\[
f(z) = \frac{1}{z}
\]
and deduce the properties of bilinear transformations.

**Proposition 32.1.** The function \( w = f(z) = 1/z \) maps circles and lines onto circles and lines.

**Proof.** Using the stereographic projection, we view lines in the complex plane as circles on the Riemann surface. Therefore we will treat lines and circles equally, and call circles passing through the point of infinity as lines. Thus, an equation for circles is
\[
|z - \alpha| = r, \quad \alpha \in S^2, r > 0.
\]
Apply the transformation and get
\[
\left| \frac{1}{w} - \alpha \right| = r \implies \left| w - \frac{1}{\alpha} \right| = \frac{r}{|\alpha|} |w|,
\]
which is an equation for an Apollonius circle. 

**Theorem 32.2.** A bilinear transformation
\[
f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,
\]
maps circles and lines onto circles and lines.
Lecture note on Complex Variables

33 Bilinear transformations

A set of generators of \( GL(2, \mathbb{C}) \) are of the form

\[
\begin{pmatrix}
\lambda & 0 \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

and the first two correspond to affine transformations that map circles onto circles and lines onto lines. Hence we will study the map

\[ f(z) = \frac{1}{z} \]

and deduce the properties of bilinear transformations.

**Proposition 33.1.** The function \( w = f(z) = 1/z \) maps circles and lines onto circles and lines.

*Proof.* Using the stereographic projection, we view lines in the complex plane as circles on the Riemann surface. Therefore we will treat lines and circles equally, and call circles passing through the point of infinity as lines. Thus, an equation for circles is

\[ |z - \alpha| = r, \quad \alpha \in S^2, r > 0. \]

Apply the transformation and get

\[ \left| \frac{1}{w} - \alpha \right| = r \Rightarrow \left| w - \frac{1}{\alpha} \right| = \frac{r}{|\alpha|}|w|, \]

which is an equation for an Apollonius circle. \( \square \)

**Theorem 33.2.** A bilinear transformation

\[ f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \]

maps circles and lines onto circles and lines. \( \square \)

**Example.** We will find a conformal mapping \( f \) of the semi-disc \( S = \{ z \mid |z| < 1, \text{Im} z > 0 \} \) onto the upper half-plane.

Note that because \( g(z) = 1/(z + 1) \) has a pole at \(-1\), it maps the line segment \([-1, 1]\) and the upper semi-circle onto infinite rays. Furthermore, the two rays must intersect at \( g(1) = \frac{1}{2} \), and by the conformality of \( g \), they intersect orthogonally. By considering several points, it can then be seen that \( g \) maps the segment \([-1, 1]\) onto the ray \([\frac{1}{2}, \infty)\) and the semi-circle onto the ray \([\frac{1}{2}, \frac{1}{2} - i\infty)\). The semi-disc is then mapped onto the right-bottom region bounded by those two rays.

Thus, the desired mapping \( f \) is given by

\[ f(z) = \left[ i \left( g(z) - \frac{1}{2} \right) \right]^2 = -\frac{(z - 1)^2}{4(z + 1)^2}. \]

**Definition 33.3.** A conformal mapping of a region onto itself is called an automorphism of that region.

**Lemma 33.4.** Suppose \( f : D_1 \to D_2 \) is a conformal mapping. Then

1. Any other conformal mapping \( h : D_1 \to D_2 \) is of the form \( g \circ f \),
2. Any automorphism \( h \) of \( D_1 \) is of the form \( f^{-1} \circ g \circ f \),
where $g$ is an automorphism of $D_2$.

Proof. In (1), the mapping $g := h \circ f^{-1}$ is an automorphism of $D_2$, hence $h = g \circ f$.

In (2), if $h$ is an automorphism of $D_1$, then $g := f \circ h \circ f^{-1}$ is an automorphism of $D_2$. Hence $h = f^{-1} \circ g \circ f$. □

We now consider the problem of determining all the automorphisms of the unit disc. First we need

Lemma 33.5 (Schwarz’ lemma). Suppose that $f$ is analytic in the unit disc and $|f(z)| \leq 1$ whenever $|z| < 1$. If $f(0) = 0$, then

$$|f(z)| \leq |z| \quad \text{and} \quad |f'(0)| \leq 1.$$ 

Furthermore, either equality holds (except the that at the presumed point $z = 0$) if and only if $f(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$.

Proof. Define the analytic function

$$g(z) = \begin{cases} \frac{f(z)}{z} & 0 < |z| < 1, \\ f'(0) & z = 0. \end{cases}$$

(Why is $g$ analytic?) Since $|g(z)| \leq \frac{1}{r}$ on the circle $|z| = r < 1$, by letting $r \to 1^-$ and applying the maximum principle, we find that $|g(z)| \leq 1$ throughout the unit disc. Hence the two inequalities are proven. Furthermore, if $|g(z_0)| = 1$ for some $z_0 \in B(0, 1)$, then again by the maximum principle $g$ would be a constant of modulus 1, and $f(z) = e^{i\theta}z$. □

Lemma 33.6. The only automorphisms of the unit disc with $f(0) = 0$ are given by $f(z) = e^{i\theta}z$, $\theta \in \mathbb{R}$.

Proof. Since both $f$ and $f^{-1}$ are analytic on the unit disc $B(0, 1)$, by Schwarz’ lemma we have (denoting $w = f(z), z = f^{-1}(w)$)

$$|w| = |f(z)| \leq |z|, \quad |z| = |f^{-1}(w)| \leq |w|$$

for any point $|z| < 1$. Hence $|z| = |f(z)|$ and the second statement of the Schwarz’ lemma shows that $f$ must be a rotation.

Now assume that $f$ is an arbitrary automorphism of the unit disc $B(0, 1)$ with $f(\alpha) = 0$ for some $\alpha \in B(0, 1)$. Define

$$g(z) = \frac{z - \alpha}{1 - \alpha z}.$$ 

When $|z| = 1$,

$$|g(z)| = \frac{|z - \alpha|}{|1 - \alpha z|} = \frac{|\bar{z}| \cdot |z - \alpha|}{|1 - \alpha z|} = \frac{|1 - \alpha \bar{z}|}{|1 - \alpha z|} = 1.$$ 

Hence $g$ maps the unit circle $|z| = 1$ onto itself. Since $g(\alpha) = 0$ and $|\alpha| < 1$, $g$ is indeed an automorphism of the unit disc. Then $h = f \circ g^{-1}$ is an automorphism of the disc with $h(0) = 0$. By Lemma 33.6, $h$ must be of the form $e^{i\theta}$.

Hence

$$f(z) = h(g(z)) = e^{i\theta} \frac{z - \alpha}{1 - \alpha z}.$$ 

So we have proved the following theorem.

Theorem 33.7. An automorphism of the unit disc is of the form

$$f(z) = e^{i\theta} \frac{z - \alpha}{1 - \alpha z}, \quad |\alpha| < 1, \theta \in \mathbb{R}.$$ 

□

66
34 Poisson integral formula

Let \( C_0 = \partial B(0; r_0) \), \( r_0 > 0 \), denote the positively oriented circle centered at the origin with radius \( r_0 \), and suppose that a function \( f \) is holomorphic within and on \( C_0 \). By the Cauchy integral formula,

\[
f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]  

(34.1)

expresses the value of \( f \) at any point \( z \) interior to \( C_0 \) in terms of those on the boundary \( C_0 \). We would like to do the same thing for harmonic functions.

Let \( z_1 \) be the point on \( \mathbb{C} \) which is the reflection point of \( z \) with respect to the circle \( C_0 \). We write \( z = re^{i\alpha} \), where \( 0 < r < r_0 \), and express \( z_1 \) as follows:

\[
z_1 = \frac{r^2}{r} e^{i\alpha} = \frac{r^2}{z} = \frac{\zeta \bar{z}}{\bar{z}}
\]

if \( \zeta \) is a point on \( C_0 \).

By the Cauchy integral formula again,

\[
0 = \frac{1}{2\pi i} \int_{C_0} \frac{f(\zeta)}{\zeta - z_1} \, d\zeta
\]  

(34.2)

since \( z_1 \) is exterior to \( C_0 \). We use the parametrization \( \zeta = r_0 e^{i\theta} \), \( 0 \leq \theta \leq 2\pi \) and substract (34.2) from (34.1) to get

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{\zeta - z} - \frac{1}{\zeta - z_1} \right] f(\zeta) r_0 e^{i\theta} \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{r^2}{z}} \right] f(\zeta) \zeta \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left[ \frac{\zeta}{\zeta - z} + \frac{\bar{z}}{\zeta - \bar{z}} \right] f(\zeta) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{\zeta(\bar{\zeta} - \bar{z}) + \bar{z}(\zeta - z)}{|\zeta - z|^2} f(\zeta) \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2r_0r \cos(\theta - \alpha) + r^2} f(r_0 e^{i\theta}) \, d\theta
\]

(34.3)

Note that the last line of the formula above also works when \( r = 0 \); in that case, the formula reduces to

\[
f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(r_0 e^{i\theta}) \, d\theta.
\]

The kernel

\[
P(r_0, r, \theta - \alpha) = \frac{r_0^2 - r^2}{r_0^2 - 2r_0r \cos(\theta - \alpha) + r^2}
\]  

(34.4)

is clearly a real-valued function. We have shown that the real part of a holomorphic function is harmonic. Hence if \( u \) is the real part of the holomorphic function \( f \), it follows that

\[
u(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} P(r_0, r, \theta - \alpha) u(r_0, \theta) \, d\theta.
\]

(34.5)
This is the Poisson integral formula for the harmonic function $u$ in the open disk bounded by the circle $r = r_0$, and $P(r_0, r, \theta - \alpha)$ is called the Poisson kernel.

Along the computation, we note that the complex numbers $\bar{z}/\bar{\zeta} - \bar{z}$ and $z/(\zeta - z)$ have the same real parts, hence

$$P(r_0, r, \theta - \alpha) = \text{Re} \left( \frac{\zeta}{\zeta - z} + \frac{z}{\zeta - z} \right) = \text{Re} \left( \frac{\zeta + z}{\zeta - z} \right). \quad (34.6)$$

Thus $P(r_0, r, \theta - \alpha)$ is a harmonic function of $r$ and $\theta$ interior to the circle $C_0$ for each fixed $\zeta$ on $C_0$. Also when $f(z) = 1$ on $C_0$, we put it into the Poisson integral formula and immediately see that

$$\frac{1}{2\pi} \int_0^{2\pi} P(r_0, r, \theta - \alpha) \, d\theta = 1.$$

Let us work the other way around. Suppose that $u = u(\theta)$ is a piecewise continuous function on $C_0$, and we define a new function $U(r, \alpha)$ according to (34.5) in the interior of $C_0$, namely,

$$U(r, \alpha) := \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - r_0^2}{r^2 - 2r_0r \cos(\theta - \alpha) + r^2} u(\theta) \, d\theta, \quad (r < r_0) \quad (34.7)$$

What property will the function $U$ have?

**Theorem 34.1.** (a) $U$ is harmonic in the interior $B(0; r_0)$.

(b) The limit

$$\lim_{r \rightarrow r_0} U(r, \alpha) = u(\alpha), \quad (34.8)$$

holds at the angle $\alpha$ where $u$ is continuous.

**Proof.** (a) By (34.6), we see that $P$, being the real part of a holomorphic function of $z$, is harmonic. Since $u$ is uniformly continuous on the compact set $C_0$, we can safely apply the Laplace operator $\Delta$ to (34.7) and conclude that

$$\Delta U(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} \Delta P(r_0, r, \theta - \alpha) u(\theta) \, d\theta = 0,$$

hence $U$ is itself harmonic too.

(b) We cannot take the limit $r \rightarrow r_0$ directly because the kernel $P$ will blow up! Instead, we will analyze the integral more carefully. For $\varepsilon > 0$, there is a $\varphi$ such that $|u(\theta) - u(\alpha)| < \varepsilon/2$ if $|\theta - \alpha| \leq \varphi$ by continuity. Thus

$$U(r, \theta) - u(\theta) = I_1(r) + I_2(r),$$

where

$$I_1(r) = \frac{1}{2\pi} \int_{\alpha - \varphi}^{\alpha + \varphi} P(r, r_0, \theta - \alpha)(u(\theta) - u(\alpha)) \, d\theta$$

$$I_2(r) = \frac{1}{2\pi} \int_{\alpha + \varphi}^{\alpha + 2\pi} P(r, r_0, \theta - \alpha)(u(\theta) - u(\alpha)) \, d\theta.$$

The fact that $P$ is a positive function implies that

$$|I_1(r)| \leq \frac{1}{2\pi} \int_{\alpha - \varphi}^{\alpha + \varphi} P(r, r_0, \theta - \alpha)|u(\theta) - u(\alpha)| \, d\theta \leq \frac{\varepsilon}{4\pi} \int_0^{2\pi} P(r, r_0, \theta - \alpha) \, d\theta = \frac{\varepsilon}{2}.$$  

On the other hand, if $\alpha + \varphi \leq \theta \leq \alpha - \varphi + 2\pi$, the denominator $|\zeta - z|^2$ of $P$ is bounded below by a positive number $m$. Therefore if $M$ is an upper bound of the continuous function $|u(\theta) - u(\alpha)|$ on the interval $0 \leq \theta \leq 2\pi$, it follows that

$$|I_2(r)| \leq \frac{(r_0^2 - r^2)M \cdot 2\pi}{2\pi m} < \frac{2r_0 M (r_0 - r)}{m} < \frac{2Mr_0 \delta}{m} = \frac{\varepsilon}{2},$$

68
whenever $0 < r_0 - r < \delta$, where $\delta = \frac{m \varepsilon}{4M r_0}$. Finally, combining both estimates we conclude that

$$|U(r, \theta) - u(\theta)| \leq |I_1(r)| + |I_2(r)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $0 < r_0 - r < \delta$. Hence the continuity is proved. \hfill \square

So given a piecewise function $u$ on $C_0$, the function $U$ defined by the integral (34.7) is called a solution of the Dirichlet problem for the disk $r < r_0$.

The formula (34.3) has a nice interpretation: the value of a harmonic function at the center of the circle $r = r_0$ is the average of the boundary value on the circle. This is the Gauss’s mean value theorem as we saw before, which readily implies the maximum and minimum principle for harmonic functions: a harmonic function cannot have its maximum or minimum at any interior point unless it is a constant. Also an easy corollary is that a continuous function on $B(0; r_0)$ which is harmonic in the interior is completely determined by its values at the boundary.