第一次期中考試

范洪源 (2011.03.29)

1. Let A be an $n \times n$ matrix. Prove that
$$\dim(\operatorname{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$
(10 pts)

2. For any two similar matices $A, B \in \mathbb{F}^{n\times n}$, prove that $f_A(t) = f_B(t)$ and $tr(A) = tr(B)$. (10 pts)

3. Let $A_m \in \mathbb{C}^{n\times p}$ and $B_m \in \mathbb{C}^{p\times s}$ for $m \geq 1$. Prove that if $\lim_{m\to\infty} A_m = L$ and $\lim_{m\to\infty} B_m = M$, then $\lim_{m\to\infty} A_m B_m = LM$. (10 pts)

4. Let $V = P_2(\mathbb{R})$ and define a linear operator T on V by
$$T(f(x)) = f(0) + f(1)(x + x^2), \quad f(x) \in V.$$
   (a) Is T diagonalizable? Give your reasons. (10 pts)
   (b) If T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix. (10 pts)

5. Let T be a linear operator on a vector space V over the scalar field $\mathbb{F}$, and let v be a nonzero vector in V. Prove that
   (a) If W is a T-cyclic subspace of V generated by v, then it is contained in any T-invariant subspace containing v. (10 pts)
   (b) If v is an eigenvector of T corresponding to the eigenvalue $\lambda \in \mathbb{F}$, then $g(T)(v) = g(\lambda)v$ for any polynomial $g(t)$. (10 pts)

6. Let $A = \begin{bmatrix} 5/2 & -3/2 \\ 3 & -2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$.
   (a) Find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. (10 pts)
   (b) Find $\lim_{m\to\infty} A^m$ if the limit exists. (10 pts)

7. State woithout proof the *Gerschgorin's Disk Theorem.* (10 pts)