

1. Let $f(x) = (-1)^n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be the characteristic polynomial of A .
 By Cayley-Hamilton Thm, $f(A) = (-1)^n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I_n = 0$, for some $a_i \in \mathbb{F}$, $\forall i=0, 1, 2, \dots, n$
 $\Rightarrow A^n = (-1)^n (a_{n-1} A^{n-1} + a_{n-2} A^{n-2} + \dots + a_1 A + a_0 I_n) \in \text{span}\{I_n, A, A^2, \dots, A^{n-1}\}$
 Suppose $A^k \in \text{span}\{I_n, A, A^2, \dots, A^{n-1}\}$, for some $k \in \mathbb{N}$ and $k \geq n$
 $\Rightarrow A^k = b_0 I_n + b_1 A + b_2 A^2 + \dots + b_{n-1} A^{n-1}$, for some $b_i \in \mathbb{F}$, $\forall i=0, 1, 2, \dots, n-1$
 Then, $A^{k+1} = A(A^k) = A(b_0 I_n + b_1 A + b_2 A^2 + \dots + b_{n-1} A^{n-1}) = b_0 A + b_1 A^2 + b_2 A^3 + \dots + b_{n-1} A^n$
 $= b_0 A + b_1 A^2 + \dots + b_{n-2} A^{n-1} + b_{n-1} [(-1)^n (a_{n-1} I_n + a_{n-2} A + \dots + a_1 A^{n-1})]$
 $= c_0 I_n + c_1 A + c_2 A^2 + \dots + c_{n-1} A^{n-1} \in \text{span}\{I_n, A, A^2, \dots, A^{n-1}\}$, for some $c_i \in \mathbb{F}$, $\forall i=0, 1, 2, \dots, n-1$

By the principle of mathematical induction,

$$A^k \in \text{span}\{I_n, A, A^2, \dots, A^{n-1}\}, \quad \forall k \geq n$$

$$\text{hence } \text{span}\{I_n, A, A^2, \dots\} \subseteq \text{span}\{I_n, A, A^2, \dots, A^{n-1}\}$$

$$\Rightarrow \dim(\text{span}\{I_n, A, A^2, \dots\}) \leq \dim(\text{span}\{I_n, A, A^2, \dots, A^{n-1}\}) \leq n.$$

2. Since A is similar to B , $\exists P \in \mathbb{F}^{n \times n}$, invertible s.t. $PAP^{-1} = B$

$$\begin{aligned} \text{Then, } f_B(t) &= \det(B - tI_n) = \det(PAP^{-1} - tPP^{-1}) = \det[P(A - tI_n)P^{-1}] \\ &= \det(P) \cdot \det(A - tI_n) \cdot \det(P^{-1}) = \det(P) \cdot \det(A - tI_n) \cdot \frac{1}{\det(P)} = \det(A - tI_n) = f_A(t). \end{aligned}$$

$$\begin{aligned} t(A) &= \sum_{i=1}^n (A)_{ii} = \sum_{i=1}^n (PAP^{-1})_{ii} = \sum_{i=1}^n \sum_{j=1}^n (PA)_{ij} (P^{-1})_{ji} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (P)_{ik} (A)_{kj} (P^{-1})_{ji} \\ &= \sum_{k=1}^n \sum_{j=1}^n \sum_{i=1}^n (A)_{kj} (P^{-1})_{ji} (P)_{ik} = \sum_{k=1}^n \sum_{j=1}^n (A)_{kj} (P^{-1}P)_{jk} = \sum_{k=1}^n \sum_{j=1}^n (A)_{kj} (I)_{jk} \\ &= \sum_{k=1}^n (A)_{kk} = \text{tr}(A). \end{aligned}$$

$$3. \lim_{m \rightarrow \infty} A_m = L \Leftrightarrow \forall 1 \leq i \leq n, 1 \leq j \leq p, \lim_{m \rightarrow \infty} (A_m)_{ij} = (L)_{ij}$$

$$\lim_{m \rightarrow \infty} B_m = M \Leftrightarrow \forall 1 \leq j \leq p, 1 \leq k \leq s, \lim_{m \rightarrow \infty} (B_m)_{jk} = (M)_{jk}$$

$\therefore \lim_{m \rightarrow \infty} (A_m)_{ij}$ and $\lim_{m \rightarrow \infty} (B_m)_{jk}$ exist, $\forall 1 \leq i \leq n, 1 \leq j \leq p, 1 \leq k \leq s$

$$\Rightarrow \lim_{m \rightarrow \infty} (A_m)_{ij} (B_m)_{jk} = (L)_{ij} (M)_{jk}, \forall 1 \leq i \leq n, 1 \leq j \leq p, 1 \leq k \leq s$$

$$\Rightarrow \sum_{j=1}^p \lim_{m \rightarrow \infty} (A_m)_{ij} (B_m)_{jk} = \sum_{j=1}^p (L)_{ij} (M)_{jk}, \forall 1 \leq i \leq n, 1 \leq k \leq s$$

$$\lim_{m \rightarrow \infty} (A_m B_m)_{ik} = \lim_{m \rightarrow \infty} \sum_{j=1}^p (A_m)_{ij} (B_m)_{jk} = \sum_{j=1}^p \lim_{m \rightarrow \infty} (A_m)_{ij} (B_m)_{jk} = \sum_{j=1}^p (L)_{ij} (M)_{jk} = (LM)_{ik}, \forall 1 \leq i \leq n, 1 \leq k \leq s$$

By definition of the limit of matrix, $\lim_{m \rightarrow \infty} A_m B_m = LM$.

T. Let's let $\{1, x, x^2\}$ be a basis for V.

$$T(1) = 1+x+x^2, T(x) = x+x^2, T(x^2) = x^2+x^2$$

$$[T]_p = [[T(1)]_p \ [T(x)]_p \ [T(x^2)]_p] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

the characteristic polynomial of T, $f_T(x) = (1-x)^3 - (1-x) = (1-x)[(1-x)^2 - 1] = -x(1-x)(2-x)$

the eigenvalues of T are 0, 1, 2 and all distinct,

Thus T is diagonalizable.

(b) the eigenspace of 0, $V(0) = N(T) = \text{span} \left\{ \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$

$$1, V(1) = N(T-I) = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$2, V(2) = N(T-2I) = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

take $\beta = \{x-x^2, 1+x+x^2, x+x^2\}$, then $[T]_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a diagonal matrix.

5. (a) Let \mathbb{U} be any T-invariant subspace containing v

$$\Rightarrow v \in \mathbb{U}, T(v) \in \mathbb{U}$$

$$\text{Since } T(v) \in \mathbb{U}, T(T(v)) = T^2(v) \in \mathbb{U}$$

Suppose that $T^k(v) \in \mathbb{U}$, for some $k \in \mathbb{N}$

$\because \mathbb{U}$ is a T-invariant subspace and $T^k(v) \in \mathbb{U}$

$$\Rightarrow T^{k+1}(v) = T(T^k(v)) \in \mathbb{U}$$

By the principle of mathematical induction,

$$T^k(v) \in \mathbb{U}, \forall k \in \mathbb{N} \cup \{0\}$$

$$\Rightarrow \{v, T(v), T^2(v), \dots\} \subseteq \mathbb{U} \Rightarrow W = \text{span} \{v, T(v), T^2(v), \dots\} \subseteq \mathbb{U}_*$$

(b) v is an eigenvector of T corresponding to $\lambda \in \mathbb{F} \Rightarrow T(v) = \lambda v$

$$T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$$

Suppose that $T^k(v) = \lambda^k v$, for some $k \in \mathbb{N} \cup \{0\}$

$$\text{then, } T^{k+1}(v) = T(T^k(v)) = T(\lambda^k v) = \lambda^k T(v) = \lambda^k \cdot \lambda v = \lambda^{k+1} v$$

By the principle of mathematical induction,

$$T^n(v) = \lambda^n v, \forall n \in \mathbb{N} \cup \{0\}$$

Let $g(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n, \forall a_i \in \mathbb{F}, i=1, 2, \dots, n, \forall n \in \mathbb{N} \cup \{0\}$

$$g(T)v = (a_0 I + a_1 T + a_2 T^2 + \dots + a_n T^n)v = a_0 v + a_1 T v + a_2 T^2 v + \dots + a_n T^n v$$

$$= a_0 v + a_1 \lambda v + a_2 \lambda^2 v + \dots + a_n \lambda^n v = g(\lambda) v$$

b. (a) the characteristic polynomial of A ,

$$f_A(x) = \det(A - xI) = \det \begin{bmatrix} \frac{5}{2}-x & \frac{3}{2} \\ 3 & -2-x \end{bmatrix} = x^2 - \frac{1}{2}x - 5 + \frac{9}{2} = x^2 - \frac{1}{2}x - \frac{1}{2} = (x + \frac{1}{2})(x - 1)$$

\Rightarrow the eigenvalues of A are $1, -\frac{1}{2}$

$$\Rightarrow \text{the eigenspace of } 1, V(1) = N(A - I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$
$$\Rightarrow \text{the eigenspace of } -\frac{1}{2}, V(-\frac{1}{2}) = N(A + \frac{1}{2}I) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Let $Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, then $Q^{-1}AQ = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} = D$ is a diagonal matrix

$$(b) Q^{-1}AQ = D \Rightarrow A = QDQ^{-1}$$

$$\Rightarrow A^m = QD^mQ^{-1}$$

$$\lim_{m \rightarrow \infty} A^m = Q \left(\lim_{m \rightarrow \infty} D^m \right) Q^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$$

7. Let $A \in M_{n \times n}(\mathbb{C})$, then every eigenvalue of A is contained in a Gershgorin's Disk, where the i th Gershgorin's disk

$$C_i = \{z \in \mathbb{C} \mid |z - A_{ii}| < r_i\}$$