

$$\begin{aligned}
 & \det \begin{bmatrix} -2-t & -1 & 2 \\ -2 & 1-t & 2 \\ -2 & -3 & 4-t \end{bmatrix} = \det \begin{bmatrix} -4 & -2 & 0 & 6-t \\ -4 & -2 & -t & 6-t \\ -2 & -3 & 1 & 4-t \end{bmatrix} \\
 & = -t \det \begin{bmatrix} 2-t & 0 & -2t \\ -4 & -2t & 6-t \\ -2 & -3 & 4-t \end{bmatrix} - \det \begin{bmatrix} 2-t & 0 & -2t \\ -4 & -2t & 6-t \\ -4 & -2 & 6-t \end{bmatrix} \\
 & = -t(2-t) \det \begin{bmatrix} 1 & 0 & 0 \\ -4 & -2t & 2-t \\ -2 & -3 & 2-t \end{bmatrix} - (2-t) \det \begin{bmatrix} 1 & 0 & 0 \\ -4 & -2t & 2-t \\ -4 & -2 & 2-t \end{bmatrix} \\
 & = (2-t)^2 [(-t)(-2-t+3) - (-2-t+2)] = (2-t)^2 t^2
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad V(2) = \mathcal{N}(A-2I) &= \mathcal{N} \left(\begin{bmatrix} -2 & -3 & 1 & 2 \\ -2 & -1 & -1 & 2 \\ -2 & 1 & -3 & 2 \\ -2 & -3 & 1 & 2 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} -2 & -3 & 1 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
 &= \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
 \end{aligned}$$

$$\begin{aligned}
 V(0) = \mathcal{N}(A) &= \mathcal{N} \left(\begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 2 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & -2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} 0 & -3 & 1 & 2 \\ 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right) \\
 &= \mathcal{N} \left(\begin{bmatrix} 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}
 \end{aligned}$$

$$\text{gm}(0) = \dim(V(0)) = 1 \neq 2 = \text{am}(0)$$

$$\mathcal{N}(A^2) = \mathcal{N} \left(\begin{bmatrix} 0 & -8 & 4 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & -8 & 4 & 8 \end{bmatrix} \right) = \mathcal{N} \left(\begin{bmatrix} 0 & 2 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

the dot diagram for $\lambda=0$

$$\text{hence, take } Q = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \text{ then } J = Q^{-1} A Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a Jordan form}$$

$$(c) \text{ since } A^2(2I-A) = 0 \text{ but } A(2I-A) = \begin{bmatrix} 0 & 5 & -3 & -2 \\ 2 & 1 & -1 & -2 \\ 2 & 1 & -1 & -2 \\ 2 & 5 & -3 & -4 \end{bmatrix} \neq 0$$

$$m_A(t) = t^2(t-2)$$

2. Let $\dim(V) = n < \infty$, since V is T -cyclic, the minimal polynomial of T is $m_T(t) = (t-1)^n f_T(t)$, where $f_T(t)$ is the characteristic polynomial of T .

Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the distinct eigenvalues of T .

then $f_T(t) = (\lambda_1 - t)^{\alpha_1} (\lambda_2 - t)^{\alpha_2} \dots (\lambda_r - t)^{\alpha_r}$, for some $\alpha_1, \dots, \alpha_r \in \mathbb{N}$ and $\sum_{j=1}^r \alpha_j = n$.

$$\Rightarrow \dim(K(\lambda_i)) = \alpha_i, \quad \forall i=1, 2, \dots, r$$

Since the number of the 1st column of dot diagram for any eigenvalue λ_i equals to the degree for λ_i of the minimal polynomial $m_T(t)$

$$\text{and } m_T(t) = (t-\lambda_1)^{\alpha_1} (t-\lambda_2)^{\alpha_2} \dots (t-\lambda_r)^{\alpha_r},$$

the number of the 1st column of dot diagram for any eigenvalue $\lambda_i = \alpha_i = \dim(K(\lambda_i))$

\Rightarrow the dot diagrams for all λ_i are all one column. $\rightarrow \text{D}$

On the other hand, T is diagonalizable,

\Rightarrow the dot diagrams for all λ_i have only one dot in each column

\Rightarrow the dot diagrams for all λ_i are all one row. $\rightarrow \text{E}$

By D, E, the dot diagrams for all λ_i have only one dot

i.e. $\dim(K(\lambda_i)) = 1 \Rightarrow$ each of the eigenspaces of T is one-dimensional.

(\Leftarrow) Since T is diagonalizable, $\dim(K(\lambda_i)) = \dim(V(\lambda_i))$

the eigenspaces of T are all one-dimensional.

$$\Rightarrow \dim(K(\lambda_i)) = \dim(V(\lambda_i)) = 1$$

$\Rightarrow f_T(t) = (\lambda_1 - t)(\lambda_2 - t) \dots (\lambda_n - t)$, where $\lambda_1, \dots, \lambda_n$ are n distinct eigenvalues for T .

Let v_i be the eigenvector corresponding to λ_i , $\forall i=1, 2, \dots, n$

Let $v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$, for c_1, c_2, \dots, c_n not all zero.

Claim: $\{v, T(v), \dots, T^{n-1}(v)\}$ forms a basis for V

Suppose $a_0 v + a_1 T(v) + \dots + a_{n-1} T^{n-1}(v) = 0$,

since $T(v) = T(c_1 v_1) + T(c_2 v_2) + \dots + T(c_n v_n) = \lambda_1 c_1 v_1 + \lambda_2 c_2 v_2 + \dots + \lambda_n c_n v_n$

$$T^k(v) = \lambda_1^k c_1 v_1 + \lambda_2^k c_2 v_2 + \dots + \lambda_n^k c_n v_n, \quad \forall k=1, 2, \dots, n$$

$$0 = a_0 \sum_{i=1}^n c_i v_i + a_1 \sum_{i=1}^n \lambda_i c_i v_i + \dots + a_{n-1} \sum_{i=1}^n \lambda_i^{n-1} c_i v_i = \sum_{j=0}^{n-1} a_j \sum_{i=1}^n \lambda_i^j c_i v_i = \sum_{j=0}^{n-1} \sum_{i=1}^n a_j \lambda_i^j c_i v_i$$

$$= [c_1 v_1 \dots c_n v_n] \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Since v_1, v_2, \dots, v_n are linear independent, and a_1, a_2, \dots, a_n not all zero are

$$\Rightarrow \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = 0$$

Since $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct, then

$$\det \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} = \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j) \neq 0$$

$$\Rightarrow A = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \text{ is invertible} \Rightarrow A \text{ is nonsingular}$$

$$\Rightarrow A \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = 0 \Rightarrow a_0 = a_1 = \dots = a_{n-1} = 0 \Rightarrow \{v, T(v), \dots, T^{n-1}(v)\} \text{ is linear independent} \quad \text{--- (c)}$$

since $\dim(V) = n = \dim(\text{span}\{v, T(v), \dots, T^{n-1}(v)\})$, and $\text{span}\{v, T(v), \dots, T^{n-1}(v)\} \subseteq V$,
 $V = \text{span}\{v, T(v), \dots, T^{n-1}(v)\} \quad \text{--- (d)}$

By (d), $\{v, T(v), \dots, T^{n-1}(v)\}$ forms a basis for V and hence V is T -cyclic.

3 (a) Let $\dim(V) = n$.

$$\text{Since } \text{rank}(U^m) = \text{rank}(U^{m+1}) \Rightarrow \text{nullity}(U^m) = \text{nullity}(U^{m+1})$$

$$\therefore \ker(U^m) \subseteq \ker(U^{m+1}), \therefore \ker(U^m) = \ker(U^{m+1}), \text{ i.e. } \mathcal{N}(U^m) = \mathcal{N}(U^{m+1})$$

$$\forall x \in \mathcal{N}(U^{m+2}), U^{m+2}(x) = 0 \Rightarrow U^{m+1}(U(x)) = 0$$

$$\text{Since } \mathcal{N}(U^m) = \mathcal{N}(U^{m+1}), U^m(U(x)) = 0 \Rightarrow U^{m+1}(x) = 0$$

$$\Rightarrow x \in \mathcal{N}(U^{m+1}) \Rightarrow \mathcal{N}(U^{m+2}) \subseteq \mathcal{N}(U^{m+1})$$

$$\text{On the other hand, } \forall y \in \mathcal{N}(U^{m+1}), U^{m+1}(y) = 0 \Rightarrow U(U^{m+1}(y)) = 0 \Rightarrow U^{m+2}(y) = 0$$

$$\Rightarrow y \in \mathcal{N}(U^{m+2}) \Rightarrow \mathcal{N}(U^{m+1}) \subseteq \mathcal{N}(U^{m+2})$$

$$\Rightarrow \mathcal{N}(U^{m+1}) = \mathcal{N}(U^{m+2}) \Rightarrow \text{nullity}(U^{m+1}) = \text{nullity}(U^{m+2}) \Rightarrow \text{rank}(U^{m+1}) = \text{rank}(U^{m+2})$$

SUPPOSE $N(U^P) = N(U^{P+1})$, $P > m$

then, $\forall z \in N(U^{P+2})$, $U^{P+2}(z) = 0 \Rightarrow U^{P+1}(U(z)) = 0 \Rightarrow U^P(U(z)) = 0 \Rightarrow U^{P+1}(z) = 0$
 $\Rightarrow z \in N(U^{P+1}) \Rightarrow N(U^{P+2}) \subseteq N(U^{P+1})$

$\forall w \in N(U^{P+1})$, $U^{P+1}(w) = 0 \Rightarrow U(U^{P+1}(w)) = U(0) = 0 \Rightarrow U^{P+2}(w) = 0 \Rightarrow w \in N(U^{P+2})$
 $\Rightarrow N(U^{P+1}) \subseteq N(U^{P+2})$

$\Rightarrow N(U^{P+1}) = N(U^{P+2}) \Rightarrow \text{nullity}(U^{P+1}) = \text{nullity}(U^{P+2}) \Rightarrow \text{rank}(U^{P+1}) = \text{rank}(U^{P+2})$

$\Rightarrow N(U^m) = N(U^{m+1}) = \dots = N(U^P) = N(U^{P+1}) = \dots$, $\forall P > m$,

i.e. $N(U^m) = N(U^k)$, $\forall k \geq m$

and hence $\text{nullity}(U^m) = \text{nullity}(U^k) \Rightarrow \text{rank}(U^m) = n - \text{nullity}(U^m) = n - \text{nullity}(U^k) = \text{rank}(U^k)$

(b) By (a), we know $\text{rank}((T-\lambda I)^m) = \text{rank}((T-\lambda I)^{m+1}) \Rightarrow N((T-\lambda I)^m) = N((T-\lambda I)^k)$, $\forall k \geq m$
 since $N((T-\lambda I)^k) \subseteq N((T-\lambda I)^{k+1})$, $\forall k \in \mathbb{N}$.

and $K_\lambda = \bigcup_{i=1}^{\infty} N((T-\lambda I)^i)$,

$K_\lambda = \bigcup_{i=m}^{\infty} N((T-\lambda I)^i) = \lim_{k \rightarrow \infty} \bigcup_{i=m}^k N((T-\lambda I)^i) = \lim_{k \rightarrow \infty} N((T-\lambda I)^k)$

$\therefore k > m$, $\therefore K_\lambda = \lim_{k \rightarrow \infty} N((T-\lambda I)^k) = \lim_{k \rightarrow \infty} N((T-\lambda I)^m) = N((T-\lambda I)^m)$

(a)

4. Let $\gamma = \{1, x, x^2, x^3\}$ be a basis for $P_3(\mathbb{R})$

$A = [T]_\gamma = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, and $f_T(t) = f_A(t) = \det(A - tI)$

$f_T(t) = \det \begin{bmatrix} 2-t & 1 & 0 & 0 \\ 0 & 2-t & 2 & 0 \\ 0 & 0 & 2-t & 3 \\ 0 & 0 & 0 & 2-t \end{bmatrix} = (2-t)^4$

(b) Since $\text{rank}(A - zI) = \dim(V(A - zI)) = 4 - 3 = 1$

The dot diagram must be

$$(A - zI) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad (A - zI)^2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad (A - zI)^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A - zI)^4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

Take $\beta = \{6, 6x, 3x^2, x^3\}$, then $[T]_{\beta} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is a Jordan canonical form.

$$(c) (A - zI)^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0, \quad \text{but } (A - zI)^4 = 0$$

hence $m_T(t) = m_{\beta}(t) = (t - z)^4$

I. Suppose that $m_T(t)$, $n_T(t)$ are minimal polynomials of T

• $m_T(t)$ is a minimal polynomial and $n_T(t) = 0$

$$\Rightarrow m_T(t) \mid n_T(t)$$

• $n_T(t)$ is a minimal polynomial and $m_T(t) = 0$

$$\Rightarrow n_T(t) \mid m_T(t)$$

$$\Rightarrow m_T(t) = k n_T(t), \quad \text{for some } k \in \mathbb{R}$$

Since $m_T(t)$ and $n_T(t)$ are minimal polynomials,

$m_T(t)$ and $n_T(t)$ are both monic.

$$\Rightarrow m_T(t) = n_T(t)$$

hence the minimal polynomial of T is unique one.