Chapter 1
Limits and Their Properties
(極限與其性質)

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本章預定授課範圍

1.2 Functions and Their Graphs

1.3 Inverse Functions

1.4 Exponential and Logarithmic Functions

1.5 Finding Limits Graphically and Numerically

1.6 Evaluating Limits Analytically

1.7 Continuity and One-Sided Limits

1.8 Infinite Limits

Useful Notations
(常用的數學符號)

1. A set (集合) is a collection of specified objects, and usually denoted by A, B, $C, \cdots$.

$$\mathbb{R} = \{x \,|\, x \text{ is a real number (實數)}\}$$

$$\mathbb{N} = \{x \,|\, x \text{ is a positive integer (正整數)}\}$$

$$\mathbb{Z} = \{x \,|\, x \text{ is an integer (整數)}\}$$

$$\mathbb{Q} = \{x \,|\, x \text{ is a rational number (有理數)}\}$$

2. $x \in A$: x belongs to (屬於) A, i.e., x is an element of the set A.

$$\pi \in \mathbb{R}, \quad 5 \in \mathbb{N}, \quad -7 \in \mathbb{Z} \quad \text{and} \quad \frac{2}{3} \in \mathbb{Q}.$$

$A \subseteq B$: A is a subset (子集合) of B, i.e., if $x \in A$, then $x \in B$.

3. The union (聯集) and intersection (交集) of two sets A and B are defined by

$$A \cup B = \{x | x \in A \quad \text{or} \quad x \in B\},$$
$$A \cap B = \{x | x \in A \quad \text{and} \quad x \in B\}.$$

4. $\forall$: for all (對於所有).

5. $\exists$: exist (存在).

6. $\nexists$: does not exist (不存在).

7. s.t. or $\ni$: such that (使得).

8. $\Longrightarrow$: imply that (意指).

9. $\Longleftrightarrow$: if and only if (若且唯若).

10. Greek letters: (常用希臘字符)

$$\alpha \text{ (alpha)}, \quad \beta \text{ (beta)}, \quad \gamma \text{ (gamma)}, \quad \delta \text{ (delta)},$$
$$\varepsilon \text{ (epsilon)}, \quad \theta \text{ (theta)}, \quad \lambda \text{ (lambda)}, \quad \mu \text{ (mu)},$$
$$\rho \text{ (rho)}, \quad \tau \text{ (tau)}, \quad \phi \text{ (phi)}, \quad \omega \text{ (omega)}, \cdots$$

11. The intervals (區間) in $\mathbb{R}$ are often denoted by I, e.g.,

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\},$$

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\},$$

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}.$$

12. i.e.: that is (也就是說).

13. e.g.: for example (舉例來說).

14. w.r.t.: with respect to (關於).

15. Def: Definition (定義), Thm: Theorem (定理).

Section 1.2
Functions and Their Graphs
(函數與其圖形)

Def (實值函數的定義)

Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$.

(1) A real-valued function f from X to Y, denoted by $f : X \to Y$, is a correspondence (對應) that assigns to (指派) each $x \in X$ one unique (唯一的) value $y \in Y$.

(2) The set $X = \text{dom}(f)$ is called the domain (定義域) of f.

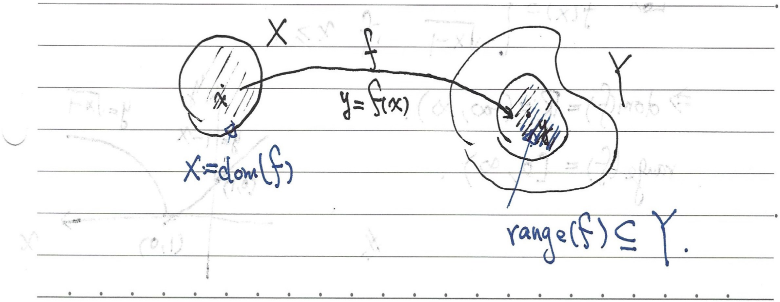
(3) The subset of Y defined by

$$\text{range}(f) = \{y \in Y \mid \exists\, x \in X \text{ s.t. } y = f(x)\}$$

is called the range (值域) of f.

(4) x is the independent variable (自變數) and y is the dependent variable (應變數) of f.

函數映射的示意圖

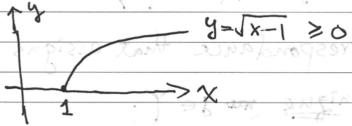


Example 2 : Find dom(f) and range(f).

(a) $f(x)=\sqrt{x-1}$

$\Rightarrow \text{dom}(f)=\{x\in\mathbb{R} \mid x-1\geq 0\}=[1,\infty)$

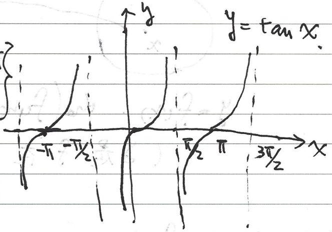
$\text{range}(f)=\{f(x) \mid x\geq 1\}=[0,\infty)$



(b) $f(x)=\tan x$

$\Rightarrow \text{dom}(f)=\{x\in\mathbb{R} \mid x\neq (n+\frac{1}{2})\pi \text{ with } n\in\mathbb{Z}\}$

$\text{range}(f)=\{f(x) \mid x\neq (n+\frac{1}{2})\pi \text{ with } n\in\mathbb{Z}\}$
$=\mathbb{R}=(-\infty,\infty)$.



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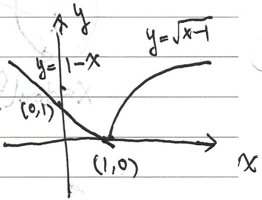
Example 3: (分段函数の定义域与值域)

Let $f(x)=\begin{cases} 1-x & \text{if } x<1 \\ \sqrt{x-1} & \text{if } x\geq 1. \end{cases}$

$\Rightarrow \text{dom}(f)=\mathbb{R}=(-\infty,\infty)$.

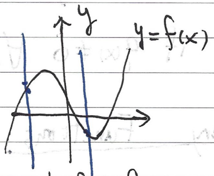
$\text{range}(f)=[0,\infty)$. #



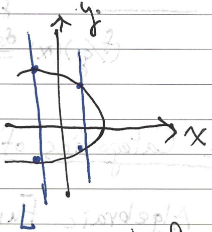
* Graph of a Function :

Thm: (Vertical Line Test)

Any vertical line intersects the graph of $y=f(x)$ at most ~~exactly~~ once.



Graph of $y=f(x)$



Not a function graph!

Def (函數的基本運算; 1/2)

Let f and g be real-valued functions defined on $X \subseteq \mathbb{R}$.

(1) The sum $f+g$ of f and g is defined by

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in X.$$

(2) The difference $f-g$ of f and g is defined by

$$(f-g)(x) = f(x) - g(x) \quad \forall x \in X.$$

(3) For any $k \in \mathbb{R}$, the constant multiple kf of f is defined by

$$(kf)(x) = k \cdot f(x) \quad \forall x \in X.$$

Def (函數的基本運算; 2/2)

(4) The product fg of f and g is defined by

$$(fg)(x) = f(x) \cdot g(x) \quad \forall\, x \in X.$$

(5) The quotient $\frac{f}{g}$ of f and g is defined by

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} \quad \forall\, x \in X,$$

provided that $g(x) \neq 0 \quad \forall\, x \in X$.

Three Categories of Elementary Functions (1/2)

1. Algebraic Functions (代數函數)
   (a) polynomial (function) of nth degree ($n \in \mathbb{N}$)

   $$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad \text{with } a_n \neq 0.$$

   (b) rational function (有理函數)

   $$f(x) = p(x)/q(x),$$

   where $p(x)$ and $q(x) \neq 0$ are polynomials.

   (c) radical function (根式函數)

   $$f(x) = x^{1/n} = \sqrt[n]{x} \quad \text{with } n \in \mathbb{N}.$$

Note (根式函數的定義域與值域)

Let $f(x) = \sqrt[n]{x}$ with $n \in \mathbb{N}$.

- When n is odd (奇數), we know that

$$\text{dom}(f) = \text{range}(f) = (-\infty, \infty) = \mathbb{R}.$$

- When n is even (偶數), we see that

$$\text{dom}(f) = \text{range}(f) = [0, \infty).$$

Three Categories of Elementary Functions (2/2)

2. Trigonometric Functions (三角函數)

$$f(x) = \sin x, \cos x, \tan x, \cot x, \sec x, \csc x.$$

3. Exponential and Logarithmic Functions (指數與對數函數)

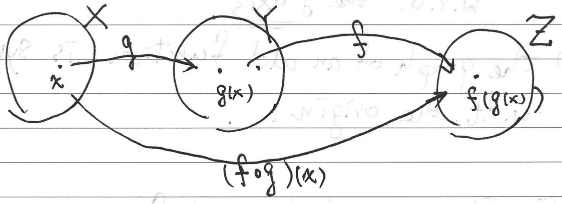
$$f(x) = a^x \quad \text{or} \quad f(x) = \log_a x,$$

where $0 < a \neq 1$. (See Section 1.4 later!)

Def (合成函數的定義)

Let X, Y and Z be subsets of $\mathbb{R}$. The composite function (合成函數) of $f : Y \to Z$ and $g : X \to Y$ is defined by

$$(f \circ g)(x) = f(g(x)) \quad \forall\, x \in X.$$



Example 4: Let $f(x)=2x-3$ and $g(x)=\cos x$.

(a) $(f\circ g)(x)=f(g(x))=2\cos x-3 \quad \forall x\in\mathbb{R}$.

(b) $(g\circ f)(x)=g(f(x))=\cos(2x-3) \quad \forall x\in\mathbb{R}$.

Note: $(f\circ g)(x)\neq(g\circ f)(x)$ in general !

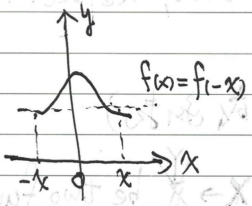
#

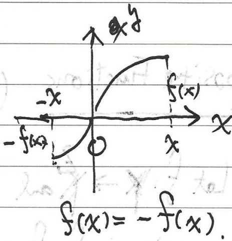
Def (Even and Odd Functions)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$.

(1) f is an even function (偶函數) if $f(-x) = f(x) \quad \forall x \in X$.

(2) f is an odd function (奇函數) if $f(-x) = -f(x) \quad \forall x \in X$.





Notes :

Notes

- The graph of an even function is symmetric w.r.t. the y-axis. (偶函數的圖形對稱於 y-軸)
- The graph of an odd function is symmetric w.r.t. the origin. (奇函數的圖形對稱於原點)

Example 5: Determine whether each function is even, odd or neither.

(a) $f(x) = x^3 - x$ is odd, since $f(-x) = (-x)^3 - (-x)$
$= -x^3 + x = -(x^3 - x) = -f(x)$. $\forall x \in \mathbb{R}$.

(b) $g(x) = 1 + \cos x$ is even because $g(-x) = 1 + \cos(-x)$
$= 1 + \cos x = g(x)$ $\forall x \in \mathbb{R}$.

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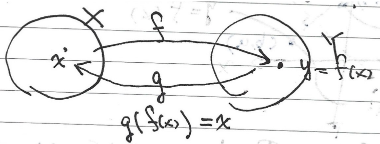
Section 1.3
Inverse Functions
(反函數)

Def (反函數的定義)

Let $f: X \to Y$ be a function, where $X, Y \subseteq \mathbb{R}$. A function $g: Y \to X$ is called the inverse function (反函數) of f if

$$g(f(x)) = x \quad \forall\, x \in X \quad \text{and} \quad f(g(y)) = y \quad \forall\, y \in Y.$$

In this case, we denote $g = f^{-1}$. (讀作 f-inverse)



Remark: If the inverse function g $\exists$, then $g = f^{-1}$ is unique.

... is unique.
#

Example 1: Show that $f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\frac{x+1}{2}}$ are inverse functions of each other.

Sol: $\because$ $f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x$ $\forall x \in \mathbb{R}$

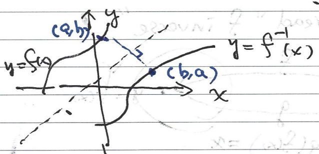
and $g(f(x)) = \sqrt[3]{\frac{(2x^3-1)+1}{2}} = \sqrt[3]{x^3} = x$ $\forall x \in \mathbb{R}$.

$\therefore$ f and g are inverse functions of each other.

#

Remarks

1. If f^{-1} $\exists$, then $(f^{-1})^{-1} = f$.
2. In general, it is true that $f^{-1}(x) \neq \frac{1}{f(x)}$.
3. The graph of f^{-1} is a reflection (反射) of the graph of f in the line $y = x$, i.e., $b = f(a) \Longleftrightarrow f^{-1}(b) = a$.



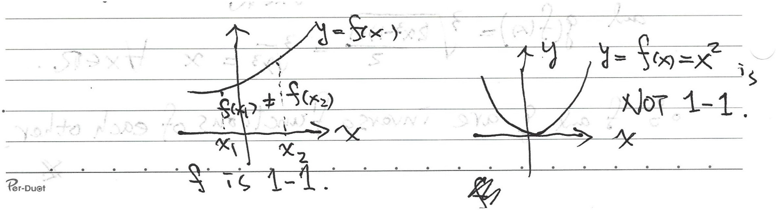
(f 與 f^{-1} 的圖形 反對稱於直線 $y=x$)

Main Questions

- Does f always have an inverse function f^{-1}?
- When does f^{-1} exist for any real-valued function f?

Def (一對一函數的定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$. If $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2 \in X$, then f is an one-to-one function. (一對一函數; 簡寫成 1-1)



Existence of f^{-1}

Thm (反函數 f^{-1} 的存在性)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$. Then

$$f^{-1} \quad \exists \iff f \text{ is one-to-one on } X.$$

Example 2: (判斷 f^{-1} の存在性).

(a) Let $f(x) = x^3 - 1$. Then f is 1-1 on $\mathbb{R}$.

<u>Reason</u>: suppose that $f(x_1) = f(x_2)$ $\forall x_1, x_2 \in \mathbb{R}$.

$\Rightarrow x_1^3 - 1 = x_2^3 - 1 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1-1 on $\mathbb{R}$.

So, $f(x) = x^3 - 1$ has an inverse function. #

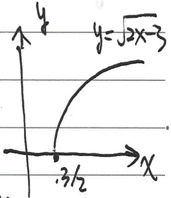
(b) Let $f(x) = x^3 - x + 1$. $\forall x \in \mathbb{R}$. #

$\Rightarrow$ f is NOT 1-1 on $\mathbb{R}$, since

$f(-1) = f(0) = f(1) = 1$.

$\Rightarrow$ $f^{-1} \nexists$ by the above Thm. #

Example 3: Find f^{-1} for $f(x)=\sqrt{2x-3}$ $\forall x\in[\frac{3}{2},\infty)$.



Sol: $\because f:[\frac{3}{2},\infty)\to[0,\infty)$ is 1-1

$\therefore f^{-1}:[0,\infty)\to[\frac{3}{2},\infty)$ $\exists$!

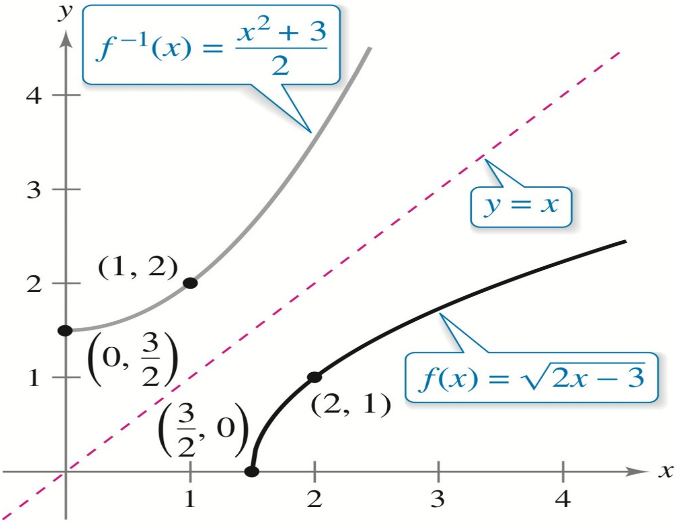
$$y=\sqrt{2x-3}\Leftrightarrow y^2=2x-3\Leftrightarrow x=\frac{y^2+3}{2}=f^{-1}(y).$$

So, the inverse function of f is

$$f^{-1}(x)=\frac{x^2+3}{2}\quad\forall x\in[0,\infty).$$

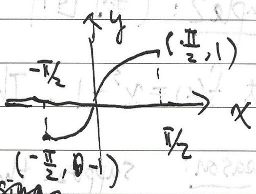
$\#$

示意圖 (承上頁)



$f^{-1}(x) = \frac{x^2+3}{2}$
$y = x$
$f(x) = \sqrt{2x-3}$
(1, 2)
$\left(0, \frac{3}{2}\right)$
$\left(\frac{3}{2}, 0\right)$
(2, 1)
y
x
1
2
3
4

Example 4: $f(x)=\sin x$ is NOT 1-1 on $\mathbb{R}$ because $f(0)=f(\pi)=0$, but it is 1-1 on $[-\frac{\pi}{2},\frac{\pi}{2}]$.



$\Rightarrow f^{-1}=\sin^{-1}$

$=\arcsin : [-1,1] \to [-\frac{\pi}{2},\frac{\pi}{2}]$ ∎

- In order to obtain the inverse trigonometric functions, we need to restrict the domains of six trigonometric functions.
- Conventionally, the following functions

$$\sin : [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1,1], \quad \cos : [0,\pi] \to [-1,1]$$
$$\tan : (\frac{-\pi}{2}, \frac{\pi}{2}) \to (-\infty,\infty), \quad \cot : (0,\pi) \to (-\infty,\infty)$$
$$\sec : [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \to (-\infty,-1] \cup [1,\infty),$$
$$\csc : [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \to (-\infty,-1] \cup [1,\infty)$$

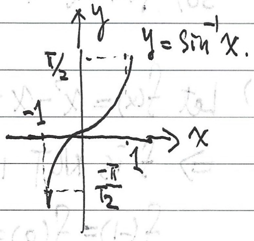
are both 1-1 on the restricted domains.

① $y = \arcsin x = \sin^{-1} x$

$\Leftrightarrow \quad \sin y = x$

Domain: $-1 \le x \le 1$

Range: $-\pi/2 \le y \le \pi/2$

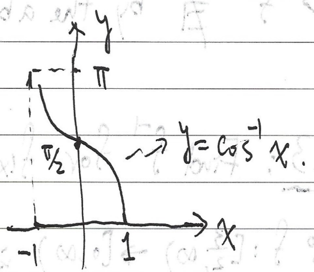


② $y = \arccos x = \cos^{-1} x$

$\Leftrightarrow \quad \cos y = x$

Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$.

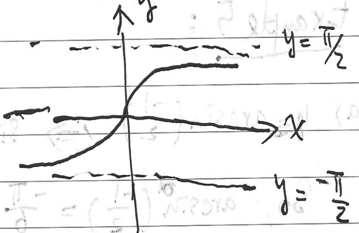


③ $y = \arctan x = \tan^{-1} x$

$\Leftrightarrow \tan y = x$

Domain: $-\infty < x < \infty$

Range: $-\frac{\pi}{2} < y < \pi/2$



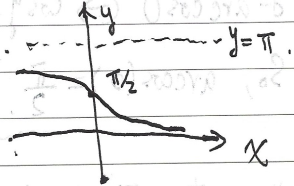
$y = \tan^{-1} x$.

④ $y = \text{arc}\cot x = \cot^{-1} x$.

$\Leftrightarrow \cot y = x$.

Domain: $-\infty < x < \infty$

Range: $0 < y < \pi$.



$y = \cot^{-1} x$.

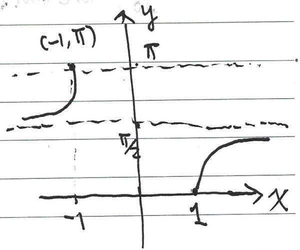
⑤ $y = \text{arcsec}\, x = \sec^{-1} \ldots$

⑤ $y = \operatorname{arcsec} x = \sec^{-1} x$

$\Leftrightarrow \sec y = x$

Domain: $|x| \geq 1$

Range: $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$



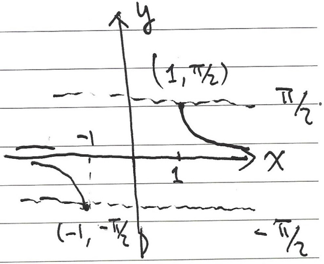
$y = \sec^{-1} x$.

⑥ $y = \operatorname{arccsc} x = \csc^{-1} x$

$\Leftrightarrow \csc y = x$

Domain: $|x| \geq 1$

Range: $-\pi/2 \leq y \leq \pi/2$, $y \neq 0$



Example 5 :

(a) $y = \arcsin\left(\frac{-1}{2}\right) \Leftrightarrow \sin y = \frac{-1}{2} \Leftrightarrow y = \frac{-\pi}{6}$

So, $\arcsin\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$ #

(b) $y = \arccos 0 \Leftrightarrow \cos y = 0 \Leftrightarrow y = \pi/2$.

So, $\arccos(0) = \frac{\pi}{2}$. #

(c) $y = \arctan\sqrt{3} \Leftrightarrow \tan y = \sqrt{3} \Leftrightarrow y = \frac{\pi}{3}$.

So, $\arctan\sqrt{3} = \pi/3$. #

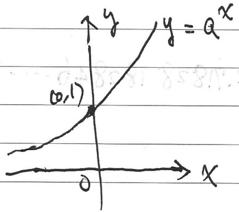
Section 1.4
Exponential and Logarithmic Functions
(指數函數與對數函數)

Def (以 a 為底的指數函數)

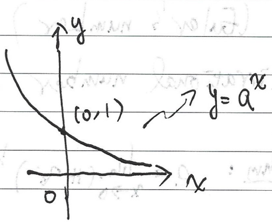
The exponential function with base number a is defined by

$$f(x) = a^x \quad \forall\, x \in \mathbb{R},$$

where $0 < a \neq 1$.



$a > 1$



$0 < a < 1$

Example 2: Sketch the graphs of $y = 2^x$, $y = (\frac{1}{2})^x$ and $y = 3^x$.

Sol:

x	-3	-2	-1	0	1	2	3	4
2^x	$1/8$	$1/4$	$1/2$	1	2	4	8	16
2^{-x}	8	4	2	1	$1/2$	$1/4$	$1/8$	$1/16$
3^x	$1/27$	$1/9$	$1/3$	1	3	9	27	81

Thm (Basic Properties of a^x)

Let $f(x) = a^x$ with $0 < a \neq 1$. Then

1. $\mathrm{dom}(f) = \mathbb{R} = (-\infty, \infty)$.
2. $\mathrm{range}(f) = (0, \infty)$, i.e., $f(x) = a^x > 0 \quad \forall\, x \in \mathbb{R}$.
3. $f(0) = a^0 = 1$, i.e., the y-intercept of f is $(0, 1)$, and $f(1) = a$.
4. f is one-to-one on $\mathbb{R}$, i.e., $f^{-1} : (0, \infty) \to \mathbb{R} \quad \exists$.

Thm (Laws of Exponents; 指數律)

If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^x a^y = a^{x+y}$.
2. $(a^x)^y = a^{xy} = (a^y)^x$.
3. $(ab)^x = a^x b^x$.
4. $\dfrac{a^x}{a^y} = a^{x-y}$.
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.

Def (Euler's number; 歐拉數或尤拉數)

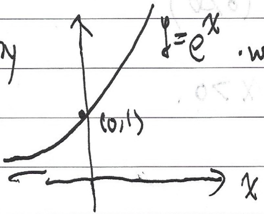
The irrational number $e = \lim_{x \to 0} (1 + x)^{1/x} \approx 2.71828182846 \cdots$.

Note: see Example 3 for observing the behavior of $f(x) = (1 + x)^{1/x}$ as x approaches 0.

Definition of e^x

For the base number $a = e > 1$, $f(x) = a^x = e^x$ is called the natural exponential function (自然指數函數).

Example 4: The graph of $f(x)=e^x$ $\forall x\in\mathbb{R}$ is given by



with $e>1$.

The Inverse Function of e^x

Since the function $f(x) = e^x$ is one-to-one on $\mathbb{R}$, it must have an inverse function $f^{-1} : (0, \infty) \longrightarrow \mathbb{R} = (-\infty, \infty)$!

Definition of $\ln x$

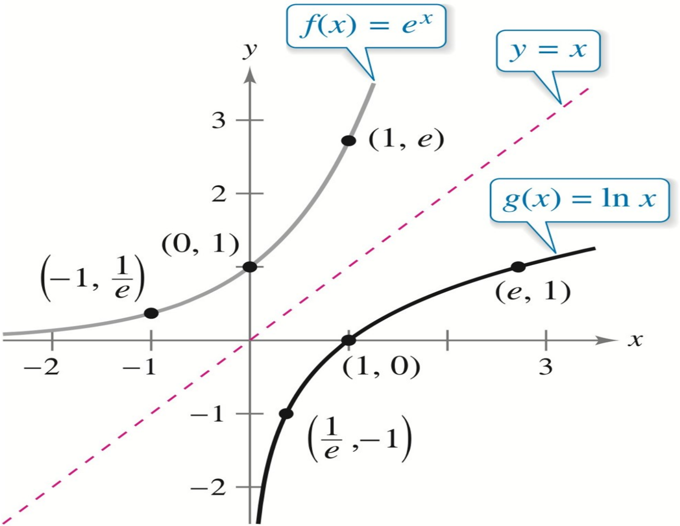
The inverse function of $f(x) = e^x$, denoted by $f^{-1}(x) = \ln x$, is called the natural logarithmic function (自然對數函數). Moreover, we have

$$y = \ln x \quad \forall\, x > 0 \iff e^y = x \quad \forall\, y \in \mathbb{R}.$$

Thm (Basic Properties of $\ln x$)

Let $f(x) = \ln x$. Then

1. $\text{dom}(f) = (0, \infty)$.
2. $\text{range}(f) = \mathbb{R} = (-\infty, \infty)$.
3. $f(1) = ln 1 = 0$, i.e., the x-intercept of f is $(1, 0)$, and $f(e) = \ln e = 1$.
4. f is one-to-one on $(0, \infty)$, i.e., $f^{-1} : \mathbb{R} \to (0, \infty) \quad \exists$.
5. $\ln(e^x) = x$ for $x \in \mathbb{R}$, and $e^{\ln x} = x$ for $x > 0$.


$f(x) = e^x$
$y = x$
$g(x) = \ln x$
y
x
3
2
-1
-2
-2
-1
3
$(1, e)$
$(0, 1)$
$\left(-1, \frac{1}{e}\right)$
$(e, 1)$
$(1, 0)$
$\left(\frac{1}{e}, -1\right)$

Thm (Laws of Logarithms; 對數律)

If $x > 0$, $y > 0$ and $z \in \mathbb{R}$, then

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln(x^z) = z \cdot \ln x$.

Example 5 (續):

(a), (b), (c) 自行閱讀.

(d) $$\ln\frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln\left[(x^2+3)^2\right] - \ln\left[x(x^2+1)^{1/3}\right].$$

$$= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1).$$ #

Example 6: Solve the equations.

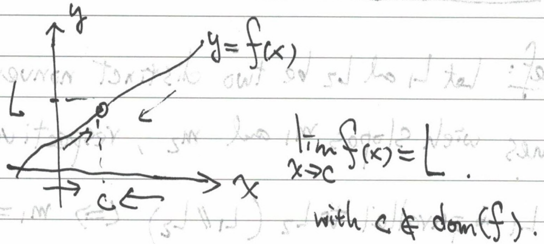
(a). $7 = e^{x+1} \Rightarrow \ln 7 = (x+1)\ln e = x+1 \Rightarrow x = \ln 7 - 1$ #

(b) $\ln(2x-3) = 5 \Rightarrow 2x-3 = e^5 \Rightarrow x = \frac{1}{2}(e^5+3)$. #

Section 1.5
Finding Limits Graphically and Numerically
(從圖形上與數值上求極限)

Def (函數極限值的非正式定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \notin X$. If $f(x)$ becomes arbitrarily close to a unique number $L \in \mathbb{R}$ as x approaches c from either side, then the limit of f is L as x approaches c, denoted by $\lim_{x \to c} f(x) = L$.



Example 1: Find $\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1}$ numerically.

Sol:

Let $f(x) = \frac{x}{\sqrt{x+1}-1}$ for $x \neq 0$.

Note that $\text{dom}(f) = (-\infty, 0) \cup (0, \infty)$ and $c = 0 \notin \text{dom}(f)$.

x	-10^{-2}	-10^{-3}	-10^{-4}	0	10^{-4}	10^{-3}	10^{-2}
$f(x)$			1.99995	?	2.00005		

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{\sqrt{x+1}-1} = 2. \quad \#$$

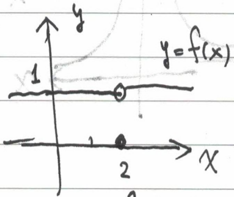
Per-Duet

Example 2:

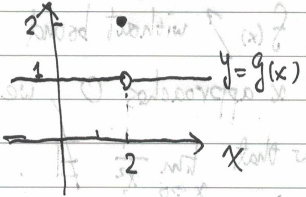
If $f(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$ and $g(x) = \begin{cases} 1, & x \neq 2 \\ 2, & x = 2. \end{cases}$

then $\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = 1$.

$g(2) = 2$



$f(2) = 0$.



Type I (在 c 點兩邊的極限值不相等)

If $f(x) \to L_1$ and $f(x) \to L_2$, where $L_1 \neq L_2$, as x approaches c from either side, then $\lim_{x \to c} f(x) \nexists$.

Example 3: Show that $\lim_{x\to 0} \frac{|x|}{x} \not\exists$.

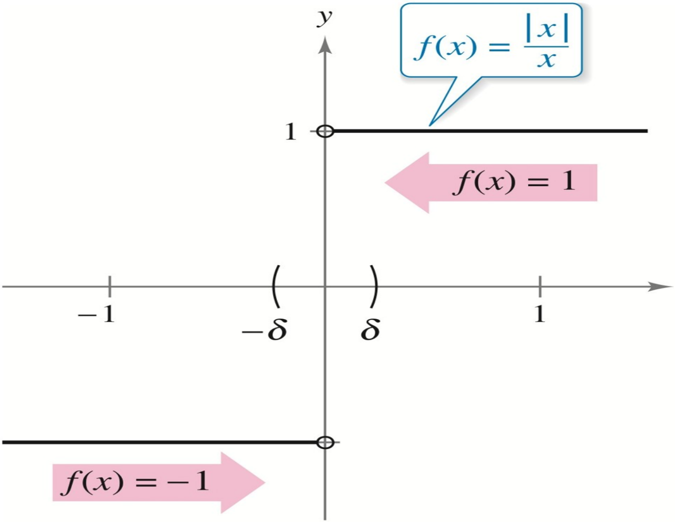
Sol: Let $f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & \text{if } x > 0 \\ \frac{-x}{x} = -1 & \text{if } x < 0. \end{cases}$

$\Rightarrow$ $f(x) \to 1$ as $x \to 0$ from the right side.

$f(x) \to -1$ as $x \to 0$ from the left side.

$\Rightarrow \lim_{x\to 0} \frac{|x|}{x} \not\exists$.

示意圖 (承上例)

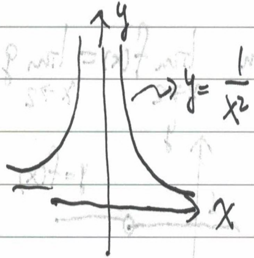


Type II (函數值在 c 點附近無上下界)

If $f(x)$ increases (遞增) or decreases (遞減) without bound as x approaches c from either side, then $\lim_{x\to c} f(x) \quad \nexists$.

Example 4 : ($f(x)$ becomes unbounded)

Consider $f(x) = \frac{1}{x^2}$ for $x \neq 0$.



Since $f(x) \nearrow$ without bound as x approaches 0, we know that $\lim_{x \to 0} \frac{1}{x^2} \nexists$.

Fact: $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Type III (函數值在 c 點附近震盪)

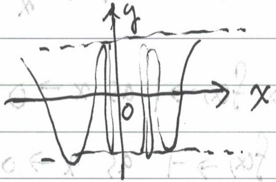
If $f(x)$ oscillates (震盪) between two fixed values as x approaches c from either side, then $\lim_{x \to c} f(x) \quad \nexists$.

Example 5: ($f(x)$ is oscillating)

Find the limit $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$.

Sol: $\because$ $f(x) = \sin\left(\frac{1}{x}\right)$ oscillates between -1 and 1 as x approaches 0.

$\therefore$ $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \nexists$

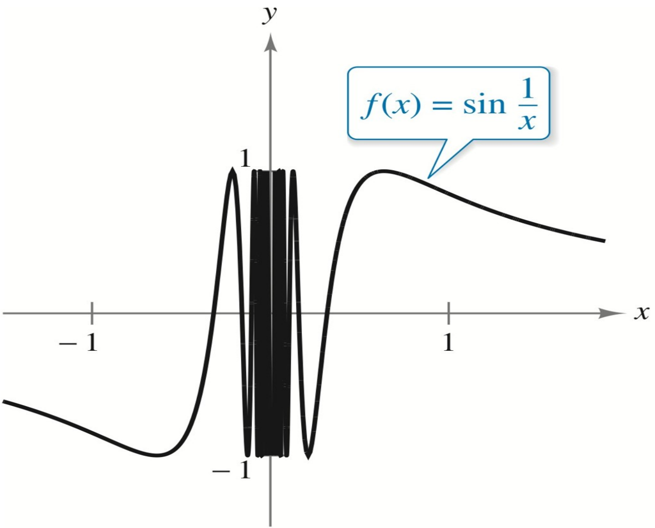


Facts: ① For $x = \frac{2}{(4n+1)\pi}$ $\forall n \in \mathbb{Z}$, $\sin\left(\frac{1}{x}\right) = \sin\frac{(4n+1)\pi}{2} = 1$ $\forall n \in \mathbb{Z}$.

② For $x = \frac{2}{(4n-1)\pi}$ $\forall n \in \mathbb{Z}$, $\sin\left(\frac{1}{x}\right) = -1$ $\forall n \in \mathbb{Z}$.

#

示意圖 (承上例)



Note: 函數值在 -1 和 1 之間來回 [振盪，但沒碰到 y-軸喔!

Def (函數極限值的正式定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \notin X$. Then

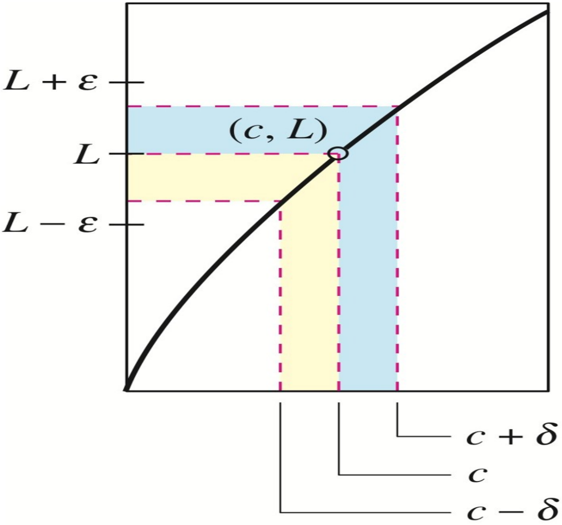
$$\lim_{x \to c} f(x) = L.$$

$$\iff \forall \varepsilon > 0,\ \exists\, \delta > 0 \text{ s.t. if } 0 < |x - c| < \delta \text{ (and } x \in X\text{)},$$

$$\text{then } |f(x) - L| < \varepsilon.$$

Note: this is also called the ε-δ definition of a limit.

極限的正式定義



Example 6: Find a $\delta > 0$ s.t. if $0 < |x-3| < \delta$,

then $|(2x-5)-1| < \varepsilon = 0.01$.

Sol: Want $|(2x-5)-1| = |2x-6| = 2|x-3| < 0.01$

$\Rightarrow |x-3| < \frac{1}{2}(0.01) = 0.005 = \delta$

So, we may choose $\delta = 0.005$.

Remark: Any $\delta \in (0, 0.005]$ also works in this example!

Example 7: Show that $\lim_{x\to 2}(3x-2)=4$.

Pf: Let $f(x)=3x-2$ and $L=4$.

Given $\varepsilon>0$ arbitrarily.

Choose $\delta=\varepsilon/3>0$.

If $0<|x-2|<\delta$, then $|f(x)-L|=|(3x-2)-4|=3|x-2|$

$$<3\delta=3\left(\frac{\varepsilon}{3}\right)=\varepsilon.$$

Thus, $\lim_{x\to 2}(3x-2)=4$ by the ε-δ Def. #

Section 1.6
Evaluating Limits Analytically
(從分析上求極限)

Thm (Basic Limit Laws; 1/2)

Let $b, c \in \mathbb{R}$ and let f and g be real-valued functions with

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = K.$$

(1) $\lim_{x \to c} b = b$ and $\lim_{x \to c} |x| = |c|$.

(2) $\lim_{x \to c} x^n = c^n$ and $\lim_{x \to c} \left[f(x)\right]^n = L^n \quad \forall\, n \in \mathbb{N}$.

(3) $\lim_{x \to c} [b \cdot f(x)] = b \cdot \left[\lim_{x \to c} f(x)\right] = b \cdot L$.

(4) $\lim_{x \to c} \left[f(x) \pm g(x)\right] = \left[\lim_{x \to c} f(x)\right] \pm \left[\lim_{x \to c} g(x)\right] = L \pm K$.

Thm (Basic Limit Laws; 2/2)

(5) $\lim_{x \to c} \left[f(x) \cdot g(x) \right] = \left[\lim_{x \to c} f(x) \right] \cdot \left[\lim_{x \to c} g(x) \right] = L \cdot K.$

(6) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{K}$ if $K \neq 0$.

(7) The Limit of $f \circ g$: (合成函數的極限值)

If $\lim_{x \to c} f(x) = f(K)$, then $\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x) \right) = f(K)$.

Examples 1 and 2:

(a) $\lim_{x\to 2} 3 = 3$

(b) $\lim_{x\to(-4)} x = -4$

(c) $\lim_{x\to 2} x^2 = (2)^2 = 4.$

(d) $\lim_{x\to 2}(4x^2+3) = 4\left(\lim_{x\to 2} x^2\right) + \lim_{x\to 2} 3 = 4(2^2)+3 = 19.$ #

Thm (Limits of Elementary Functions; 1/2)

Let c be a real number in the domain of the given function.

(1) If $p(x)$ is a polynomial, then $\lim_{x \to c} p(x) = p(c)$.

(2) If $r(x) = p(x)/q(x)$ is a rational function with $q(c) \neq 0$, then $\lim_{x \to c} r(x) = r(c) = p(c)/q(c)$.

(3) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c} \quad \forall\, n \in \mathbb{N}$, where $c \geq 0$ when n is even and $c \in \mathbb{R}$ when n is odd.

Thm (Limits of Elementary Functions; 2/2)

(4) Limits of 6 trigonometric functions are given by

$$\lim_{x\to c}\sin x = \sin c,\quad \lim_{x\to c}\cos x = \cos c,\quad \lim_{x\to c}\tan x = \tan c,$$
$$\lim_{x\to c}\cot x = \cot c,\quad \lim_{x\to c}\sec x = \sec c,\quad \lim_{x\to c}\csc x = \csc c.$$

(5) $\lim_{x\to c} a^x = a^c$ for $a > 0$ and $c \in \mathbb{R}$.

(6) $\lim_{x\to c} \ln x = \ln c$ for $c > 0$.

Example 3: Find $\lim\limits_{x\to 1}\dfrac{x^2+x+2}{x+1}$.

Sol: $\lim\limits_{x\to 1}\dfrac{x^2+x+2}{x+1}=\dfrac{1+1+2}{1+1}=\dfrac{4}{2}=2$. #

Example 4: (根號函數の極限)

(a)

$$\lim_{x\to 0}\sqrt{x^2+4}=\sqrt{\lim_{x\to 0}(x^2+4)}=\sqrt{0+4}=2.$$

(b)

$$\lim_{x\to 3}\sqrt[3]{2x^2-10}=\sqrt[3]{18-10}=\sqrt[3]{8}=2. \#$$

Example 5 : (Limits of Transcendental Functions)

(a) $\lim_{x\to 0} \tan x = \tan(0) = 0$.

(b) $\lim_{x\to 0} \sin^2 x = \left(\lim_{x\to 0} \sin x\right)^2 = 0^2 = 0$.

(c) $\lim_{x\to(-1)} xe^x = \left(\lim_{x\to(-1)} x\right)\left(\lim_{x\to(-1)} e^x\right) = -e^{-1}$.

(d) $\lim_{x\to e} \ln x^3 = \left(\lim_{x\to e} 3\ln x\right) = (3)(1) = 3$. #

Thm 1.7 (化簡函數後求極限值)

If $\exists\,\delta > 0$ s.t. $f(x) = g(x) \quad \forall\, x \in (c-\delta, c) \cup (c.c+\delta)$, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x).$$

Note: 將 $f(x)$ 簡化為 $g(x)$ 後，兩者在 c 點附近的極限值相等!

Example 7 : (Dividing Out Technique)

$$\lim_{x\to(-3)} \frac{x^2+x-6}{x+3} = \lim_{x\to(-3)} \frac{(x+3)(x-2)}{x+3} = \lim_{x\to(-3)} (x-2)$$

$$= -3-2 = -5. \quad \#$$

有理化技巧.

Example 8 : (Rationalizing Technique)

有理化技巧.

$$\lim_{x\to0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to0} \left(\frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \right)$$

$$= \lim_{x\to0} \frac{x}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{2}. \quad \#$$

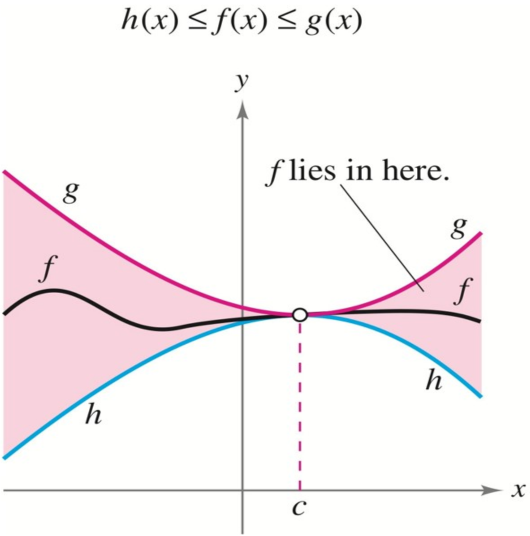
Thm 1.8 (Squeeze (or Sandwich) Thm; 夾擠定理或夾擊定理)

If $\exists\, \delta > 0$ s.t. $h(x) \leq f(x) \leq g(x) \quad \forall\, x \in (c-\delta, c) \cup (c, c+\delta)$, and

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x),$$

then $\lim_{x \to c} f(x) = L$.

Thm 1.8 的示意圖

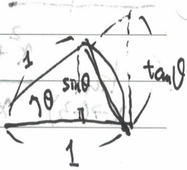


Thm 1.9 (Some Special Limits)

1. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$
2. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0.$
3. $\lim_{x \to 0} (1+x)^{1/x} = e.$

pf: ① Note that for $0<\theta<\pi/2$, we have

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$



$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \quad (同乘以 \tfrac{2}{\sin\theta})$$

$$\Rightarrow \cos\theta < \frac{\sin\theta}{\theta} < 1 \quad \forall\, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$ since

$\cos\theta = \cos(-\theta)$ and $\dfrac{\sin\theta}{\theta} = \dfrac{\sin(-\theta)}{-\theta}$ are even.

From the Squeeze Thm and $\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} 1 = 1 \Rightarrow$

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1}$$

#

(2) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\times\frac{1+\cos\theta}{1+\cos\theta}\right)$

$=\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}=\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)=(1)(0)$

$=0$ ※

Example 9: $\lim_{x\to 0}\frac{\tan x}{x} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right) = (1)(1) = 1$. #

Example 10: Find $\lim_{x\to 0}\frac{\sin 4x}{x}$.

Sol: Let $\theta = 4x$. Then $\theta \to 0$ as $x \to 0$.

So, $\lim_{x\to 0}\frac{\sin 4x}{x} = \lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)\times 4 = 4\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)$

$= (4)(1) = 4$. #

Section 1.7
Continuity and One-Sided Limits
(連續性與單邊極限)

Def (實值函數的連續性)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$.

(1) f is continuous (連續的; 簡寫為 conti.) at c if $\lim_{x \to c} f(x) = f(c)$.

(2) f is conti. on I if it is conti. at each $c \in I$.

(3) f is everywhere conti. (處處連續) if it is conti. on $\mathbb{R} = (-\infty, \infty)$.

Def (函數的不連續性)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$.

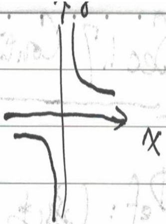
(1) f has a discontinuity (不連續點; 簡寫為 disconti.) at c if it is NOT conti. at c.

(2) A disconti. of f at c is called removable (可移除的) if f can be made conti. at c by redefining $f(c)$. Otherwise, the disconti. at c is called nonremovable (不可移除的).

Example 1: Discuss the continuity of each function.

(a) $f(x)=\frac{1}{x}$ is conti. on $(-\infty,0)\cup(0,\infty)=\text{dom}(f)$, since $\lim_{x\to c} f(x)=\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$ for $c\neq 0$.

$\because \lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{1}{x} \nexists$



$\therefore$ The disconti. at $x=0$ is nonremovable.

(b) $g(x)=\dfrac{x^2-1}{x-1}$ is conti. on $(-\infty,1)\cup(1,\infty)=\text{dom}(g)$. $\qquad f(x)=\frac{1}{x}\ \forall x\neq 0$

$\because \lim_{x\to 1} g(x)=\lim_{x\to 1}\dfrac{(x+1)(x-1)}{x-1}=\lim_{x\to 1}(x+1)=2$

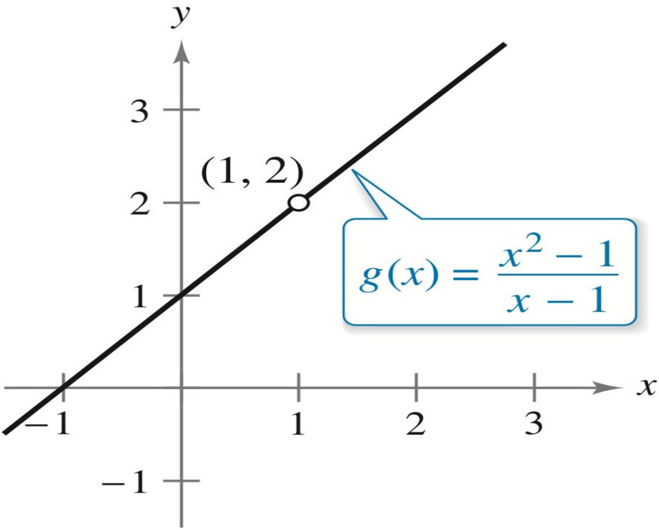
$\therefore$ The disconti at $x=1$ is removable by redefining

$g(1)=2=\lim_{x\to 1} g(x)$

$$\Rightarrow \tilde{g}(x)=\begin{cases}\dfrac{x^2-1}{x-1}=x+1 & \text{for } x\neq 1\\ 2 & \text{for } x=1\end{cases}$$ is conti. on $\mathbb{R}$.

※

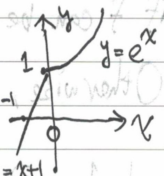
示意圖 (承上例)



(c)
$$h(x) = \begin{cases} x+1, & x \le 0 \\ e^x, & x > 0 \end{cases}$$ is conti. on $(-\infty, 0)$ and $(0, \infty)$.

Since $\lim_{x \to 0} h(x) = 1 = h(0)$, h is conti. at $x = 0$.

$\Rightarrow$ h is ~~everywhere~~ everywhere conti. #



(d) $y = \sin x$ is everywhere conti., #

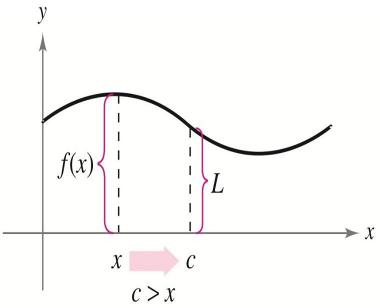
Since $\lim_{x \to c} \sin x = \sin c$ $\forall c \in \mathbb{R}$. #

Def (單邊極限值的定義)

(1) f has the limit L from the right (or the right-hand limit L; 右極限值) at c, denoted by $\lim_{x \to c^+} f(x) = L$, if $f(x) \to L$ as $x \to c$ from the right.

(2) f has the limit L from the left (or the left-hand limit L; 左極限值) at c, denoted by $\lim_{x \to c^-} f(x) = L$, if $f(x) \to L$ as $x \to c$ from the left.

單邊極限值的示意圖



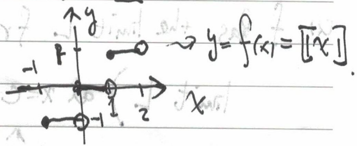
Example 3 (最大整數函數)

The greatest integer function is defined by

$$f(x) = [\![x]\!] = \text{the greatest } n \in \mathbb{Z} \text{ s.t. } n \leq x.$$

$\therefore \lim_{x \to n^+} [\![x]\!] = n$ and $\lim_{x \to n^-} [\![x]\!] = n-1$ $\forall n \in \mathbb{Z}$.

$\therefore \lim_{x \to n} [\![x]\!] \nexists$



$\Rightarrow f(x) = [\![x]\!]$ has a nonremovable discontinuity at each $x = n \in \mathbb{Z}$. #

Existence of a Limit

Thm 1.10 (函數極限值存在的等價條件)

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x).$$

(f 在 c 點的極限值為 $L \iff$ 左右極限值均為 L)

Def (單邊連續的定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \in X$.

(1) f is conti. from the right (右連續) at c if $\lim_{x \to c^+} f(x) = f(c)$.

(2) f is conti. from the left (左連續) at c if $\lim_{x \to c^-} f(x) = f(c)$.

Remark

f is conti. at $c \iff f$ is conti. from the right and from the left at c.
(f 在 c 點連續 $\iff$ f 在 c 點右連續且左連續)

Def (在閉區間上的連續性)

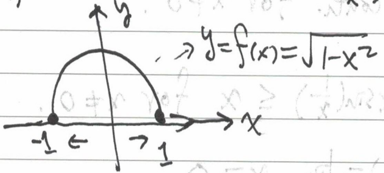
We say that f is conti. on $I = [a, b]$ if the following conditions hold:

1. f is conti. on the open interval (a, b).
2. f is conti. from the right at a, i.e., $\lim_{x \to a^+} f(x) = f(a)$.
3. f is conti. from the left at b, i.e., $\lim_{x \to b^-} f(x) = f(b)$.

Example 4: $f(x)=\sqrt{1-x^2}$ is conti. on $I=[-1,1]=dom(f)$

because f is conti. on $(-1,1)$, and

$$\lim_{x\to(-1)^+}\sqrt{1-x^2}=0=f(-1) \text{ and } \lim_{x\to 1^-}\sqrt{1-x^2}=0=f(1).$$



Thm (連續函數的性質)

1. If f and g are conti. at c and $b \in \mathbb{R}$, then $f \pm g$, bf, fg and f/g with $g(c) \neq 0$ are conti. at c, respectively.
2. If g is conti. at c and f is conti. at $g(c)$, then $(f \circ g)(x) = f(g(x))$ is conti. at c.
3. All elementary functions are conti. on their domains.

Note: the above properties are also true for one-sided continuity!

Example 7 :

(a) $f(x) = \tan x$ is conti. on $\text{dom}(f) = \{x \in \mathbb{R} \mid x \neq (n+\frac{1}{2})\pi,\ n \in \mathbb{Z}\}$.

(b) $g(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is conti. on $(-\infty, 0)$ and $(0, \infty)$

because $\frac{1}{x}$ is conti. for $x \neq 0$ and $\sin x$ is conti. on $\mathbb{R}$.

Since $\lim_{x \to 0} g(x) \nexists$, g has a (nonremovable) disconti.

at $x = 0$

(c) $h(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is everywhere conti.

Reason: ① h is conti. for $x \neq 0$.

② $\because -|x| \leq x\sin(\frac{1}{x}) \leq |x|$ for $x \neq 0$.

and $\lim_{x\to 0}(-|x|) = \lim_{x\to 0}|x| = 0$

$\therefore \lim_{x\to 0} h(x) = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0$ by the Squeeze Thm.
$= h(0)$

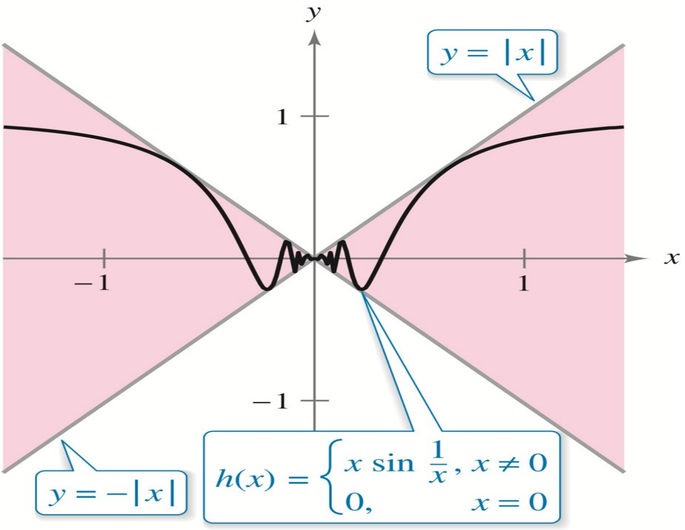
$\Rightarrow$ h is conti. at $x=0$.

From ① and ② $\Rightarrow$ h is conti. on $\mathbb{R}$. #

Thm:

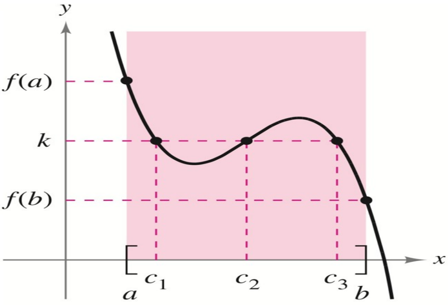
$\lim_{x\to c} f(x) = L \Leftrightarrow \lim_{x\to c}|f(x)-L| = 0 \Rightarrow$ ~~$\Leftarrow$~~ $\lim_{x\to c}|f(x)| = |L|$.

示意圖 (承上例)



Thm 1.13 (I.V.T.; 中間值定理)

If f is conti. on $[a, b]$, $f(a) \neq f(b)$ and k is any number between $f(a)$ and $f(b)$, then $\exists\, c \in [a, b]$ s.t. $f(c) = k$.



Example 8 : (Application of I.V.T.)

Show that $f(x) = x^3 + 2x - 1$ has a zero in $[0,1]$.

pf: $\because$ f is conti. on $[0,1]$ ~~f(0) = ...~~

and $f(0) = -1 < 0 < 2 = f(1)$

$\therefore$ By I.V.T. $\Rightarrow$ $\exists\, c \in (0,1)$ s.t. $f(c) = 0$.

So, f has a zero c in $[0,1]$. #

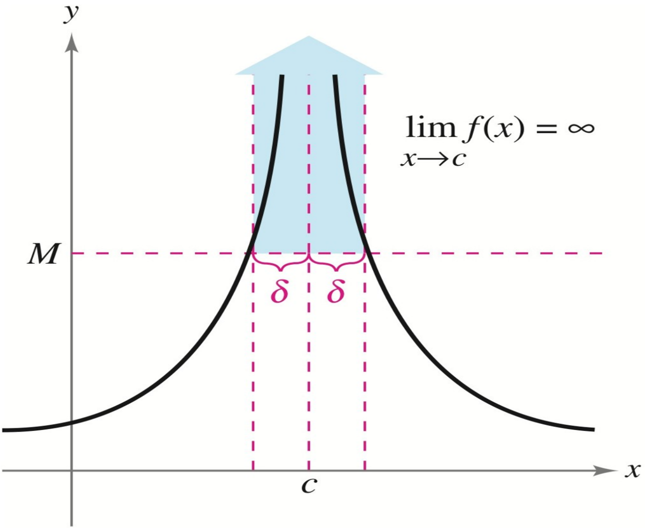
Section 1.8
Infinite Limits
(無窮極限)

Def (無窮極限值的定義; 1/2)

(1) $\lim_{x \to c} f(x) = \infty \iff \forall\, M > 0,\ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $f(x) > M$.

(2) $\lim_{x \to c} f(x) = -\infty \iff \forall\, N < 0,\ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $f(x) < N$.

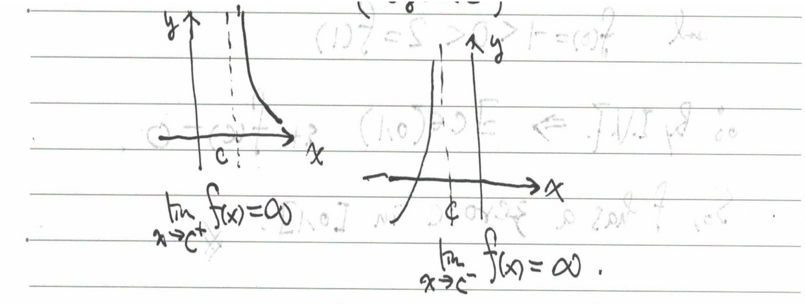
示意圖 (承上頁)



Def (無窮極限值的定義; 2/2)

(3) $\lim_{x \to c^+} f(x) = \infty \left(\text{or } \lim_{x \to c^-} f(x) = \infty \right) \iff \forall\, M > 0,\ \exists\, \delta > 0$ s.t. if $c < x < c + \delta$ (or $c - \delta < x < c$), then $f(x) > M$.

(4) $\lim_{x \to c^+} f(x) = -\infty \left(\text{or } \lim_{x \to c^-} f(x) = -\infty \right) \iff \forall\, N < 0,\ \exists\, \delta > 0$ s.t. if $c < x < c + \delta$ (or $c - \delta < x < c$), then $f(x) < N$.


y
x
c
$\lim_{x \to c^+} f(x) = \infty$
y
x
c
$\lim_{x \to c^-} f(x) = \infty$.

重要口訣 (切記!)

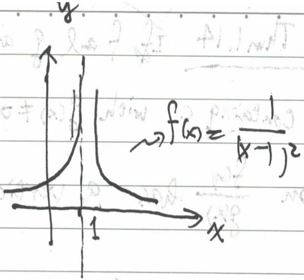
若 $+0$ 與 -0 分別代表接近零的正數與負數, 則

- $\frac{1}{+0} = \infty, \qquad \frac{1}{-0} = -\infty$
- $\frac{1}{\infty} = \frac{1}{-\infty} = 0,$

其中，$+\infty = \infty$ 和 $-\infty$ 分別為正負無窮遠處的符號。

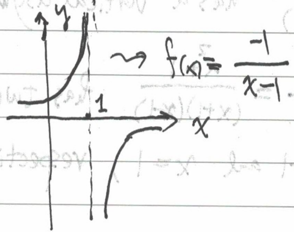
Example 1:

(a) $\lim\limits_{x \to 1} \frac{1}{(x-1)^2} \left(\cancel{= \frac{1}{+0}}\right) = \infty$



(b) If $f(x) = \frac{-1}{x-1}$ for $x \neq 1$,

then $\lim\limits_{x \to 1^+} \frac{-1}{x-1} \left(\cancel{= \frac{-1}{+0}}\right) = -\infty$ and $\lim\limits_{x \to 1^-} \frac{-1}{x-1} \left(\cancel{= \frac{-1}{-0}}\right) = \infty$



Example 4: Find the limits.

(a) $\lim_{x \to 1^-} \frac{x^2-3x}{x-1} \left(= \frac{-2}{-0} \right) = \infty$

(b) $\lim_{x \to 1^+} \frac{x^2-3x}{x-1} \left(= \frac{-2}{+0} \right) = -\infty$ #

Def (鉛直漸近線或垂直漸近線)

If $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$, then the line $x = c$ is a vertical asymptote (垂直漸近線) of the graph of f.

Thm 1.14 (判斷垂直漸近線的位置)

If f and g are conti. on an open interval I containing c, where $g(x) \neq 0 \quad \forall\, x \in I \setminus \{c\}$. If $f(c) \neq 0$ and $g(c) = 0$, then $\dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

Example 2: (Finding Vertical asymptotes)

(a) $f(x) = \frac{1}{2(x+1)}$ has a vertical asymptote at $x = -1$.

(b) $f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x+1)(x-1)}$ has two vertical asymptotes at $x = -1$ and $x = 1$, respectively.

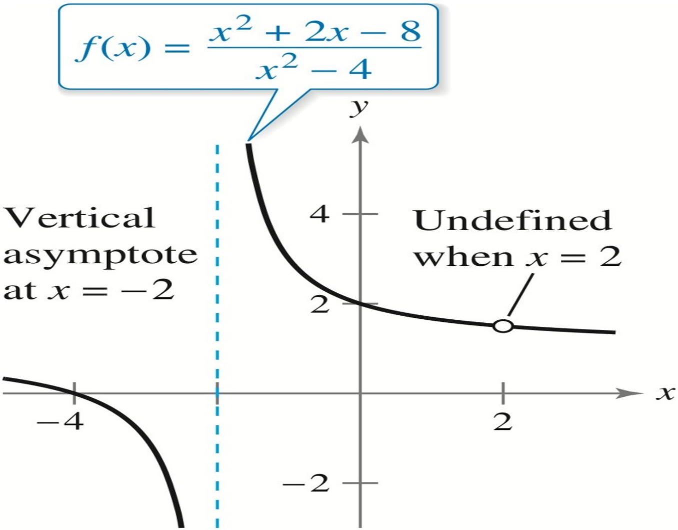
(c) $f(x) = \cot x = \frac{\cos x}{\sin x}$ has infinitely many vertical asymptotes at $x = n\pi$ $\forall n \in \mathbb{Z}$.

Example 3: Let $f(x) = \dfrac{x^2+2x-8}{x^2-4}$. Then

$$f(x) = \frac{(x+4)(x-2)}{(x+2)(x-2)} = \frac{x+4}{x+2} \text{ for } x \neq -2 \text{ and } x \neq 2.$$

$\Rightarrow$ f has a vertical asymptote at $x = -2$. #

Example 3 的示意圖 (承上頁)



Thm 1.15 (Properties of Infinite Limits)

Suppose that $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = L \neq 0$.

1. $\lim_{x\to c}[f(x) \pm g(x)] = \pm\infty$.
2. $\lim_{x\to c}[f(x)g(x)] = \pm\infty$ if $L > 0$.
3. $\lim_{x\to c}[f(x)g(x)] = \mp\infty$ if $L < 0$.
4. $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$.

Example 5 Find the limits.

(a) $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right) = 1+\lim_{x\to 0}\frac{1}{x^2} = \infty$ because $\lim_{x\to 0}\frac{1}{x^2}=\infty$. #

(b) $\lim_{x\to 1^-}\frac{x^2+1}{\cot \pi x} = 0$ because $\lim_{x\to 1^-}\cot\pi x = \lim_{x\to 1^-}\frac{\cos \pi x}{\sin \pi x} \left(= \frac{-1}{+0}\right) = -\infty$. #

(c) $\lim_{x\to 0^+} 3\ln x = 3\left(\lim_{x\to 0^+}\ln x\right) = -\infty$. #

(d) $\lim_{x\to 0^-}\left(x^2+\frac{1}{x}\right) = 0+\lim_{x\to 0^-}\frac{1}{x} = -\infty$. #

Thank you for your attention!