Chapter 2
Differentiation
(微分)

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本章預定授課範圍

2.1 The Derivative and the Tangent Line Problem

2.2 Basic Differentiation Rules and Rates of Change

2.3 Product and Quotient Rules and Higher-Order Derivatives

2.4 The Chain Rule

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Section 2.1
The Derivative and the Tangent Line Problem
(導數與切線問題)

Tangent Line Problem (切線問題)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$. What is the slope (斜率) m of the tangent line (切線) to the graph of f at the point $P(c, f(c))$?

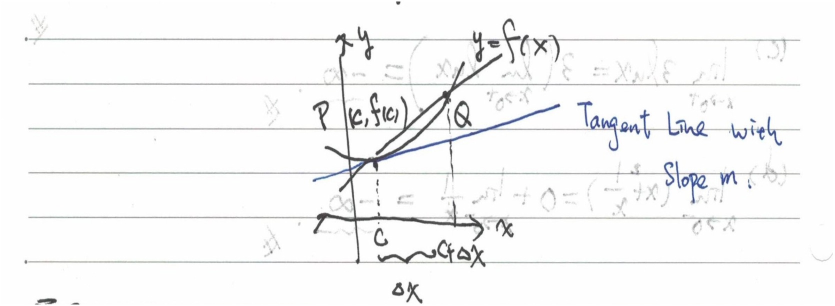
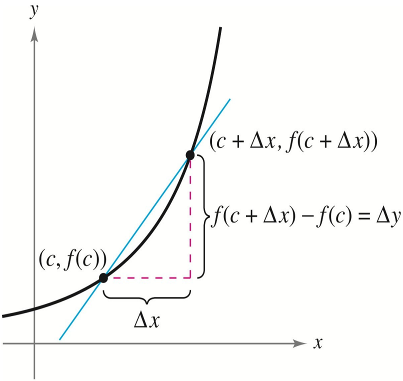
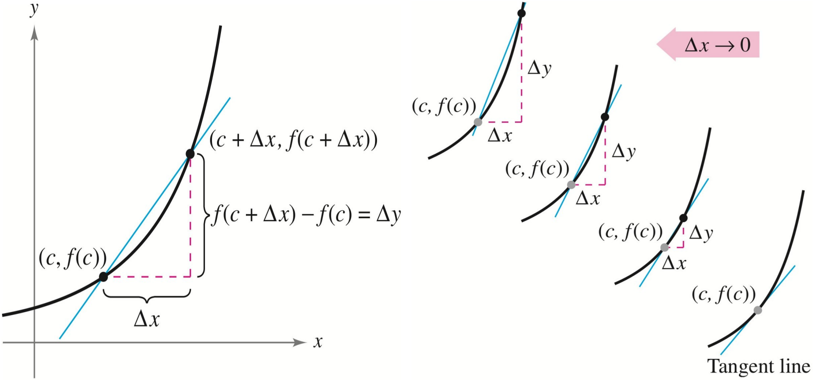


示意圖 (承上頁)




y
x
$(c + \Delta x, f(c + \Delta x))$
$f(c + \Delta x) - f(c) = \Delta y$
$(c, f(c))$
Δx
$\Delta x \to 0$
Δy
$(c, f(c))$
Δx
Δy
$(c, f(c))$
Δx
$(c, f(c))$
Δy
Δx
$(c, f(c))$
Tangent line

Observation

Since the slope of the secant line (割線) passing through $P(c, f(c))$ and $Q(c+\Delta x, f(c+\Delta x))$ is

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x},$$

the slope m of the tangent line at P is determined by considering

$$m = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}.$$

Def (切線的定義)

If $m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ $\exists$, then the line passing through $P(c, f(c))$ with slope m is called a tangent line to the graph of f at P.

Def (垂直切線的定義)

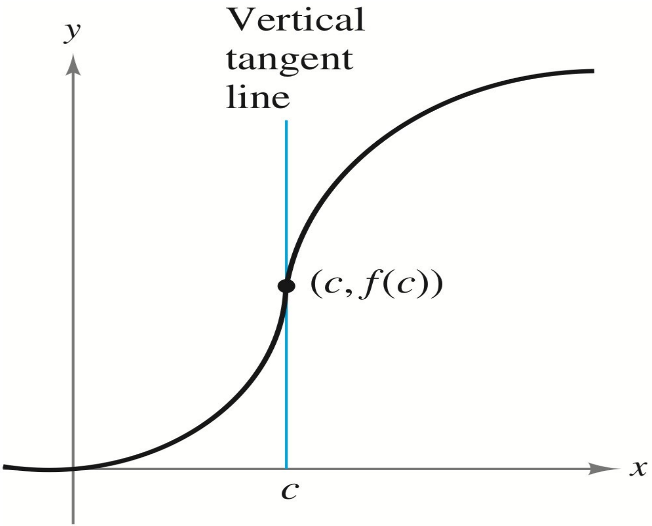
A line $x = c$ is called the vertical tangnet line (垂直切線) to the graph of f at the point $(c, f(c))$ if

$$\lim_{\Delta x \to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \pm\infty$$

or

$$\lim_{\Delta x \to 0^-} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \pm\infty.$$

垂直切線的示意圖



Remark (切線方程式)

Equation of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is given by

$$y - f(c) = m(x - c). \quad \text{(Point-Slope Form; 點斜式)}$$

Example 4

Let $f(x) = \sqrt{x}$ for $x \geq 0$.

(a) Find the tangent line to the graph of f at $(4, 2)$.

(b) Find the slopes of the graph of f at $(1, 1)$ and $(0, 0)$.

Sol: For $c > 0$, the slope of a tangent line passing through $(c, f(c))$ is given by

$$m = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{c + \Delta x} - \sqrt{c}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{c + \Delta x} + \sqrt{c})} = \frac{1}{\sqrt{c + 0} + \sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

(a) At the point $(4, 2)$, $m = \frac{1}{2\sqrt{4}} = \frac{1}{4}$ and the equation of the tangent line at $(4, 2)$ is

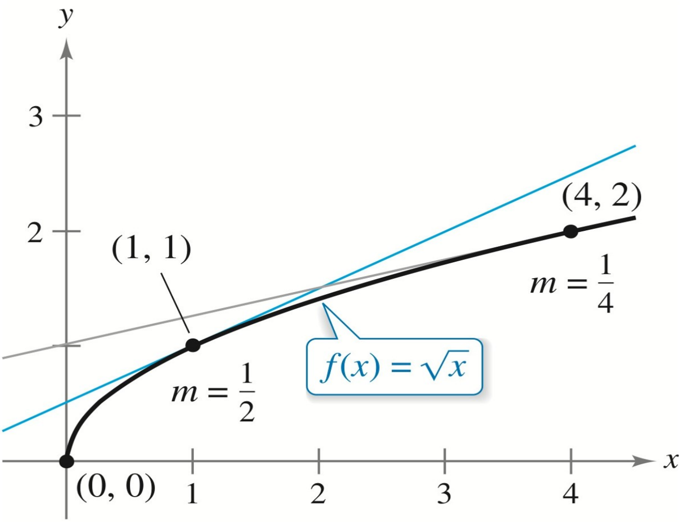
$$y - 2 = \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{1}{4}x + 1.$$

(b) The slope of the graph of f at $(1, 1)$ is $m = \frac{1}{2\sqrt{1}} = \frac{1}{2}$. In addition, the slope of a tangent line at $(0, 0)$ is

$$\lim_{\Delta x \to 0^+} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{1}{\sqrt{\Delta x}} = \infty,$$

and hence the graph of f has a vertical tangent line at $x = 0$.

示意圖 (承上例)



Homework

Read Examples 1 and 2 by yourself.

Def (導數或導函數的定義)

(1) The derivative (導數) of f at $x \in \operatorname{dom}(f)$ is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

(2) f is differentiable (可微分; 簡寫為 diff.) at $x \in \operatorname{dom}(f)$ if the derivative $f'(x)$ $\exists$.

(3) f is diff. on $I = (a, b)$ if it is diff. at each $x \in I$.

Notes

- If $S = \{x \in \text{dom}(f) \mid f'(x) \, \exists\}$, the first derivative f' can be regarded as a function defined on S.
- For any $y = f(x)$, the derivative is often denoted by

$$f'(x) = y'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = D_x f(x) = D_1 f(x).$$

Example 3: Find $f'(x)$ if $f(x) = x^3 + 2x$.

Sol: $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 + 2(x+\Delta x) - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 2(\Delta x)}{\Delta x} = 3x^2 + 2.$$ #

Recall

Sice the following equality

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

holds for any $a, b \in \mathbb{R}$, we immediately obtain

$$(x+\Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3.$$

Thm 2.1 (可微分 $\Longrightarrow$ 連續)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \in X = \mathrm{dom}(f)$. If f is diff. at c, then f is conti. at c.

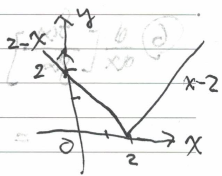
Pf: Suppose that f is diff. at $x=c$.

$$\Rightarrow f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \ \exists.$$

$$\Rightarrow \lim_{x\to c} f(x) = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}(x-c)+f(c)\right]$$

$= f'(c)\cdot 0 + f(c) = f(c)$. So, f is conti. at c. ※

Example 6: (Thm 2.1 of Ref [31])



$f(x) = |x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x < 2 \end{cases}$ is conti. at $x=2$, but it is NOT diff. at $x=2$.

Since $\displaystyle\lim_{x\to 2^+} \frac{|x-2|-0}{x-2} = \lim_{x\to 2^+} \frac{x-2}{x-2} = 1$ and

$\displaystyle\lim_{x\to 2^-} \frac{|x-2|-0}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$, we know that

$$f'(2) = \lim_{x\to 2} \frac{|x-2|-0}{x-2} \quad \nexists$$

So, f is NOT diff. at $x=2$.

HW: See Example 7. (Thm 2.1 of Ref [31]).

Remarks

1. 函數的可微分點必定是連續點，詳見 Thm 2.1。
2. 但是，函數的連續點不一定是可微分點! 詳見 Example 6 和 Example 7。

Sections 2.2 & 2.3
Basic Differentiation Rules and Higher-Order Derivatives
(基本微分法則與高階導數)

Thm (Basic Differentiation Rules)

Let f and g be diff. functions of x and $c \in \mathbb{R}$. Then

(1) $\frac{d}{dx}[c] = 0$.

(2) $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$ for $n \in \mathbb{R}$.

(3) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$.

(4) $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$.

(5) $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) = \text{(前微)(後不微)} + \text{(前不微)(後微)}$.

(6) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{\text{(子微)(母不微) - (子不微)(母微)}}{\text{(母)}^2}$.

Example 5 of Sec. 2.2:

(a) $y = 5x^3 \Rightarrow y' = 5 \cdot 3 \cdot x^{3-1} = 15x^2$.

(b) ~~$y = \frac{2}{x} = x^{-2} \Rightarrow y' = (-2)x^{-2-1} = \frac{-2}{x^3}$~~

$y = \frac{2}{x} = 2x^{-1} \Rightarrow y' = 2(-1)x^{-1-1} = \frac{-2}{x^2}$.

(c) $f(t) = \frac{4}{5}t^2 \Rightarrow f'(t) = \frac{4}{5} \cdot 2 \cdot t^{2-1} = \frac{8}{5}t$.

(d) $y = 2\sqrt{x} = 2x^{1/2} \Rightarrow y' = 2(\frac{1}{2})x^{\frac{1}{2}-1} = x^{-1/2} = \frac{1}{\sqrt{x}}$.

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(e) $y = \frac{1}{2\sqrt[3]{x^2}} = \left(\frac{1}{2}\right)x^{-2/3} \Rightarrow y' = \left(\frac{1}{2}\right)\left(\frac{-2}{3}\right)x^{-5/3} = \frac{-1}{3\,x^{5/3}}$. #

(f) $y = \left(\frac{-3}{2}\right)x \Rightarrow y' = \frac{-3}{2}$. #

Example 7, of Sec 2.2 :

(a) $f(x) = x^3 - 4x + 5 \Rightarrow f'(x) = 3x^2 - 4$

(b) $g(x) = \left(\frac{-1}{2}\right)x^4 + 3x^3 - 2x \Rightarrow g'(x) = -2x^3 + 9x^2 - 2$.

(c) $y = \frac{3x^2 - x + 1}{x} = 3x - 1 + x^{-1} \Rightarrow y' = 3 - x^{-2} = \frac{3x^2 - 1}{x^2}$ #

HW : see Examples 2 and 4 , Sec. 2.2 .

Example 1 of Sec 2.3: Find $h'(x)$ if $h(x) = (3x-2x^2)(5+4x)$

Sol:

$$h'(x) = (3-4x)(5+4x) + (3x-2x^2)(4)$$

$$= 15 - 8x - 16x^2 + 12x - 8x^2.$$

$$= -24x^2 + 4x + 15. \quad ※$$

Example 5 of Sec 2.3: (商法則の例子)

Find the equation of the tangent line to the graph of

$$f(x) = \frac{3-(1/x)}{x+5} \text{ at } (-1, 1).$$

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Sol: Rewrite f as $f(x) = \dfrac{\frac{3x-1}{x}}{x+5} = \dfrac{3x-1}{x^2+5x}$,

$\Rightarrow f'(x) = \dfrac{3(x^2+5x)-(3x-1)(2x+5)}{(x^2+5x)^2} = \dfrac{3x^2+15x-6x^2-13x+5}{(x^2+5x)^2}$

$= \dfrac{-3x^2+2x+5}{(x^2+5x)^2}$.

For $x=-1$, we have $f'(-1)=0$

$\Rightarrow$ $y=1$ is the tangent line to the graph of f at $(-1,1)$.

HW: See Example 4 of Sec. 2.3.

Thm (Derivatives of Elementary Functions)

(1) $\frac{d}{dx}[\sin x] = \cos x, \quad \frac{d}{dx}[\cos x] = -\sin x.$

(2) $\frac{d}{dx}[\tan x] = \sec^2 x, \quad \frac{d}{dx}[\cot x] = -\csc^2 x.$

(3) $\frac{d}{dx}[\sec x] = \sec x \tan x, \quad \frac{d}{dx}[\csc x] = -\csc x \cot x.$

(4) $\frac{d}{dx}[e^x] = e^x, \quad \frac{d}{dx}[\ln|x|] = \frac{1}{x}$ for $x \neq 0$.

Equivalent Def. of Euler number e

The number e is the base number of an exponential function s.t. the slope of the tangent line at $(0, 1)$ is 1, i.e., it satisfies

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - e^0}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

(1)

$$\frac{d}{dx}[\sin x] = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\sin x \cos(\Delta x) + \cos x \sin(\Delta x) - \sin x}{\Delta x}$$
$$= \sin x \left(\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} \right) + \cos x \left(\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} \right)$$
$$= (\sin x)(0) + \cos x \cdot 1 = \cos x.$$

(2) $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$

(3) $\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(0)\cos x - 1\cdot(-\sin x)}{\cos^2 x} = \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) =$
$\sec x \tan x.$

Example 2 of Sec 2.3 : $\frac{d}{dx}[xe^x] = 1\cdot e^x + x\cdot e^x = e^x(x+1)$. #

Example 3 of Sec 2.3 : Find y' if $y = 2x\cos x - 2\sin x$.

Sol: $y' = 2(x\cos x)' - 2(\sin x)' = 2(\cos x - x\sin x) - 2\cos x$

$= -2x\sin x$. #

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Example 8 of Sec 2.3: Find y'.

(a) $y = x - \tan x \Rightarrow y' = 1 - \sec^2 x$. #

(b) $y = x\sec x \Rightarrow y' = \sec x + x(\sec x \tan x)$

$= \sec x(1 + x\tan x)$. #

HW: See Examples 8 and 9 of Sec. 2.2. #

Let $y = f(x)$ be a diff. function of x. Then

- first derivative: $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}[f(x)]$.
- second derivative: $y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}[f(x)] \equiv \frac{d}{dx}(y')$.
- third derivative: $y''' = f'''(x) = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}[f(x)] \equiv \frac{d}{dx}(y'')$.
- fourth derivative: $y^{(4)} = f^{(4)}(x) = \frac{d^4y}{dx^4} = \frac{d^4}{dx^4}[f(x)] \equiv \frac{d}{dx}(y''')$.

  $\vdots$
- nth derivative: $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}[f(x)] \equiv \frac{d}{dx}(y^{(n-1)})$ for $n \in \mathbb{N}$. Here, we denote $y^{(0)} = y = f(x)$.

Section 2.4
The Chain Rule
(連鎖律)

Thm 2.11 (The Chain Rule; 連鎖律)

If $y = f(u)$ is a diff. function of u and $u = g(x)$ is a diff. function of x, then $y = f(g(x))$ is a diff. function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\Big[f(g(x))\Big] = f'(g(x)) \cdot g'(x).$$

Example 3: Find $\frac{dy}{dx}$ for $y=(x^2+1)^3$.

Sol: Let $y=f(u)=u^3$ and $u=g(x)=x^2+1$. Then

$y=f(g(x))=(x^2+1)^3$.

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=(3u^2)(2x)=6x(x^2+1)^2$$ by Chain Rule. ✗

Thm 2.12 (General Power Rule; 廣義冪法則)

If $u(x)$ is a diff. function of x, then $y = \left[u(x)\right]^n$ is also a diff. function for any $n \in \mathbb{R}$ with

$$y' = n \cdot \left[u(x)\right]^{n-1} \cdot u'(x).$$

Example 5: Find all points on the graph of $f(x)=\sqrt[3]{(x^2-1)^2}$ where $f'(x)=0$ and $f'(x) \nexists$.

Sol: $\because f(x)=\sqrt[3]{(x^2-1)^2}=(x^2-1)^{2/3}$. with $u(x)=x^2-1$ and $n=2/3$.

$$\Rightarrow f'(x)=\frac{2}{3}(x^2-1)^{2/3-1}\cdot(x^2-1)'=\frac{2}{3}(x^2-1)^{-1/3}(2x)=\frac{4x}{3\sqrt[3]{x^2-1}}.$$

So, $f'(x)=0$ when $x=0$ and $f'(x) \nexists$ when $x=\pm 1$. #

Per-Duet

Example 8

Find $f'(x)$ if $f(x) = \dfrac{x}{\sqrt[3]{x^2+4}} = \dfrac{x}{(x^2+4)^{1/3}}$.

Sol: From the Quotient Rule of Differentiation and the Chain Rule, we see that

$$
\begin{aligned}
f'(x) &= \frac{1 \cdot (x^2+4)^{1/3} - x \cdot (1/3)(x^2+4)^{-2/3}(2x)}{(x^2+4)^{2/3}} \\
&= \frac{1}{3}(x^2+4)^{-2/3} \cdot \left[\frac{3(x^2+4) - 2x^2}{(x^2+4)^{2/3}}\right] \\
&= \frac{1}{3(x^2+4)^{2/3}} \cdot \frac{x^2+12}{(x^2+4)^{2/3}} = \frac{x^2+12}{3(x^2+4)^{4/3}}.
\end{aligned}
$$

Example 11 (1/2)

Fid the derivative y'.

(a) $y = \cos 3x^2 = \cos(3x^2)$.

(b) $y = (\cos 3)x^2$.

(c) $y = \cos(3x)^2 = \cos(9x^2)$.

Sol: By the Chain Rule, we have

(a) $y' = (-\sin 3x^2) \cdot (6x) = -6x\sin(3x^2)$.

(b) $y' = (\cos 3) \cdot (2x) = (2\cos 3)x$.

(c) $y' = (-\sin 9x^2) \cdot (18x) = -18x\sin(9x^2)$.

Example 11 (2/2)

(d) $y = \cos^2 x = (\cos x)^2$.

(e) $y = \sqrt{\cos x} = (\cos x)^{1/2}$.

Sol: From the Chain Rule, we see that

(d) $y' = 2(\cos x)(-\sin x) = -2 sixx \cos x$.

(e) $y' = \frac{1}{2}(\cos x)^{-1/2}(-\sin x) = \dfrac{-\sin x}{2\sqrt{\cos x}}$.

Thm (Derivatives of e^u and $\ln|u|$)

If $u = u(x)$ is a diff. function of x, then

(1) $\frac{d}{dx}[e^x] = e^x \quad \forall\, x \in \mathbb{R}, \quad \frac{d}{dx}[e^u] = e^u \cdot u\,'$.

(2) $\frac{d}{dx}[\ln x] = \frac{1}{x}$ for $x > 0, \quad \frac{d}{dx}[\ln u] = \frac{u\,'}{u}$ for $u > 0$.

(3) $\frac{d}{dx}[\ln|x|\,] = \frac{1}{x}$ for $x \neq 0, \quad \frac{d}{dx}[\ln|u|\,] = \frac{u\,'}{u}$ for $u \neq 0$.

Example 13

(b) Find $y' = f'(x)$ if $y = f(x) = \ln(x^2 + 1)$.

Sol: If we let $u = u(x) = x^2 + 1 > 0$ for $x \in \mathbb{R}$, then

$$f'(x) = \frac{d}{dx}\left[\ln(x^2 + 1)\right] = \frac{u'}{u} = \frac{2x}{x^2 + 1} \qquad \forall\, x \in \mathbb{R}.$$

Homework

Please read Example 13 of Section 2.4 by yourself.

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Example 14: Differentiate $f(x) = \ln\sqrt{x+1}$ if $x > -1$.

Sol: Note that $f(x) = \ln(x+1)^{1/2} = \frac{1}{2}\ln(x+1)$ for $x > -1$

$$\Rightarrow y' = f'(x) = \frac{1}{2}\frac{(x+1)'}{x+1} = \frac{1}{2(x+1)} \text{ for } x > -1.$$

Example 15

Differentiate the following function

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \forall\, x \in \left(\frac{1}{\sqrt[3]{2}}, \infty\right).$$

Sol: Firstly, it follows from Laws of Logarithm that

$$\begin{aligned} f(x) &= \ln\left[x(x^2+1)^2\right] - \ln\left[(2x^3-1)^{1/2}\right] \\ &= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1). \end{aligned}$$

Thus, the first derivative of f is given by

$$\begin{aligned} f'(x) &= \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \\ &= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} \qquad \forall\, x \in \left(\frac{1}{\sqrt[3]{2}}, \infty\right). \end{aligned}$$

Def (函數 a^x 和 $\log_a x$ 的定義)

Let $0 < a \neq 1$.

(1) The exponential function to the base a (以 a 為底的指數函數) is defined by

$$a^x \equiv e^{x \ln a} \quad \forall x \in \mathbb{R}.$$

(2) The logarithmic function to the base a (以 a 為底的對數函數) is defined by

$$\log_a x \equiv \frac{\ln x}{\ln a} \quad \forall x > 0.$$

Thm (函數 a^u 和 $\log_a |u|$ 的微分公式)

Let $u = u(x)$ be a diff. function of x and $0 < a \neq 1$.

(1) $\frac{d}{dx}[a^x] = (\ln a)a^x \quad \forall\, x \in \mathbb{R}, \quad \frac{d}{dx}[a^u] = (\ln a)a^u \cdot u\,'$.

(2) $\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$ for $x > 0$, $\quad \frac{d}{dx}\left[\log_a u\right] = \frac{u\,'}{(\ln a)u}$ for $u > 0$.

(3) $\frac{d}{dx}\left[\log_a |x|\right] = \frac{1}{(\ln a)x}$ for $x \neq 0$.

(4) $\frac{d}{dx}\left[\log_a |u|\right] = \frac{u\,'}{(\ln a)u}$ for $u \neq 0$.

Example 16 :

(a) $y = 2^x \Rightarrow y' = (\ln 2) 2^x$.

(b) $y = 2^{3x} \Rightarrow y' = (\ln 2) 2^{3x} (3) = 3(\ln 2) 2^{3x}$.

(c) $y = \log_{10}(\cos x) \Rightarrow y' = \dfrac{-\sin x}{(\ln 10)\cos x} = \dfrac{-\tan x}{\ln 10}$.

(d) $y = \log_3\left(\dfrac{\sqrt{x}}{x+5}\right) = \dfrac{1}{2}\log_3 x - \log_3(x+5)$.

$$\Rightarrow y' = \frac{1}{2(\ln 3)x} - \frac{1}{(\ln 3)(x+5)} = \frac{5-x}{2(\ln 3)x(x+5)}.$$

#

Section 2.5
Implicit Differentiation
(隱微分)

Representations of a function

- Explicit Form (顯式):

$$y = f(x),$$

  where x is the independent variable (自變量) and y is the dependent variable (應變量).

- Implicit Form (隱式):

$$F(x, y) = 0,$$

  where we assume that $y = y(x)$ is a diff. function of x.

  [Q]: How to fined $y' = \frac{dy}{dx}$ under the implicit form?
  Ans: apply the technique of Implicit Differentiation!

Example (補充題)

Find $\frac{dy}{dx}$ if $xy = 1$.

Sol: Assume that $y = y(x)$ is a diff. function of x. Thaen x and y satisfy the following equation

$$F(x, y) \equiv xy - 1 = x \cdot y(x) - 1 = 0.$$

Differentiating w.r.t. x on both sides of equality gives that

$$\frac{d}{dx}[x \cdot y(x) - 1] = y + x \cdot \frac{dy}{dx} = \frac{d}{dx}(0) = 0.$$

So, we have $\frac{dy}{dx} = \frac{-y}{x}$ for $x \neq 0$.

Example 8 (利用隱微分求切線方程式)

Find the tangent line to the graph of

$$x^2(x^2+y^2)=y^2 \quad \text{or} \quad F(x,y)\equiv x^4+x^2y^2-y^2=0$$

at the point $(\sqrt{2}/2, \sqrt{2}/2)$.

Sol: Assume that $y = y(x)$ is a diff. function of x. It follows from the Chain Rule that

$$4x^3 + 2xy^2 + 2x^2y \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} = 0.$$

Thus, the first derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{-2x(2x^2 + y^2)}{2y(x^2 - 1)} = \frac{x(2x^2 + y^2)}{y(1 - x^2)}.$$

At the given point, $m = \left.\frac{dy}{dx}\right|_{x=y=\frac{\sqrt{2}}{2}} = \frac{3/2}{1/2} = 3$, and hence the equation of the tangent line at this point is

$$y - \frac{\sqrt{2}}{2} = 3\left(x - \frac{\sqrt{2}}{2}\right) \quad \text{or} \quad y = 3x - \sqrt{2}.$$

Example 1: Find $\frac{dy}{dx}$ given that $y^3+y^2-5y-x^2=-4$

Sol: Assume that $y=y(x)$ is a diff. function of x.

$$\frac{d}{dx}\left[y^3+y^2-5y-x^2\right]=\frac{d}{dx}[-4]=0$$

$$\Rightarrow 3y^2\frac{dy}{dx}+2y\frac{dy}{dx}-5\frac{dy}{dx}-2x=0 \quad \text{by Chain Rule}$$

$$\Rightarrow (3y^2+2y-5)\frac{dy}{dx}=2x \Rightarrow \frac{dy}{dx}=\frac{2x}{3y^2+2y-5}.$$

#

Example 5: Determine the slope of the graph of

$$3(x^2+y^2)^2 = 100xy$$

at the point $(3,1)$.

Sol: Assume that $y=y(x)$ is a diff. function of x.

$$\frac{d}{dx}\left[3(x^2+y^2)^2\right] = \frac{d}{dx}\left[100xy\right].$$

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$$\Rightarrow 6(x^2+y^2)\left(2x+2y\cdot\frac{dy}{dx}\right) = 100y + 100x\frac{dy}{dx}$$. by Chain Rule

$$\Rightarrow \left[-100x + 12y(x^2+y^2)\right]\frac{dy}{dx} = 100y - 12x(x^2+y^2).$$

$$\Rightarrow \frac{dy}{dx} = \frac{100y - 12x(x^2+y^2)}{-100x + 12y(x^2+y^2)}$$

$$= \frac{25y - 3x(x^2+y^2)}{-25x + 3y(x^2+y^2)}$$

So, the slope at $(3,1)$ is $m = \left.\frac{dy}{dx}\right|_{\substack{x=3\\y=1}} = \frac{-65}{-45} = \frac{13}{9}$. #

(*) HW: see Example 4 of Sec. 2.5!

Example 7: (求二階導數)

Given $x^2+y^2=25$, find $\frac{d^2y}{dx^2}=y''$.

Sol: Assume that $y=y(x)$ is a diff. function of x.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(25) \Rightarrow 2x+2y\frac{dy}{dx}=0.$$

$$\Rightarrow \frac{dy}{dx}=\frac{-x}{y}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{(-1)y-(-x)\cdot\frac{dy}{dx}}{y^2}=\frac{-y+x\cdot\left(\frac{-x}{y}\right)}{y^2}=\frac{-(y^2+x^2)}{y^3}$$

$$=\frac{-25}{y^3} \text{ because } x^2+y^2=25. \quad \#$$

Example 9:

Find y' if $y = \frac{(x-2)^2}{\sqrt{x^2+1}}$ for $x \neq 2$.

Sol:

$\because \ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1)$

$\therefore$ By Chain Rule $\Rightarrow \frac{d}{dx}[\ln y] = \frac{d}{dx}\left[2\ln(x-2) - \frac{1}{2}\ln(x^2+1)\right]$.

$\Rightarrow \frac{y'}{y} = \frac{2}{x-2} - \frac{2x}{2(x^2+1)} = \frac{x^2+2x+2}{(x-2)(x^2+1)}$

$\Rightarrow y' = \frac{y(x^2+2x+2)}{(x-2)(x^2+1)} = \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}}$.

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Section 2.6
Derivatives of Inverse Functions
(反函數的微分法則)

Thm 2.17 (反函數的可微分性)

Let f be diff. on an open interval I. If f has an inverse function $g = f^{-1}$, then g is diff. at any $x \in \text{range}(f)$ for which $f'(g(x)) \neq 0$, with the derivative

$$g'(x) = \frac{1}{f'(g(x))}.$$

$f'(g(x))$. #

Example 1: (Thm 2.17 a 例子)

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

(a) What is $f^{-1}(3)$?

(b) What is $(f^{-1})'(3)$?

Sol: $\because$ f is one-to-one on $\mathbb{R}$

$\therefore$ $f^{-1} = g$ $\exists$ on $\mathbb{R}$.

(a) Since $f(2) = 3$, $f^{-1}(3) = 2 = g(3)$.

(b) Note that $f'(x) = \frac{3}{4}x^2 + 1 \Rightarrow f'(2) = 4$

So, $(f^{-1})'(3) = g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(2)} = \dfrac{1}{4}$ #

by Thm 2.17 #

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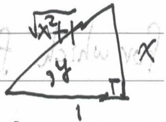
Example 3: Find y' for $y = \arctan x$ $\forall x \in \mathbb{R}$.

Sol: Note that $y = \arctan x \Leftrightarrow \tan y = x$

By Chain Rule $\Rightarrow \frac{d}{dx}[\tan y] = \frac{d}{dx}(x) = 1$

$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow y' = \frac{1}{\sec^2 y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\Rightarrow y' = \frac{1}{(\sqrt{x^2+1})^2} = \frac{1}{1+x^2}$ $\forall x \in \mathbb{R}$.



$(\tan y = x)$

Thm 2.18 (

Thm 2.18 (反三角函數的微分公式)

Let $u = u(x)$ be a diff. function of x. Then

(1) $\frac{d}{dx}\sin^{-1}u = \frac{u'}{\sqrt{1-u^2}}$, $\quad \frac{d}{dx}\cos^{-1}u = \frac{-u'}{\sqrt{1-u^2}}$.

(2) $\frac{d}{dx}\tan^{-1}u = \frac{u'}{1+u^2}$, $\quad \frac{d}{dx}\cot^{-1}u = \frac{-u'}{1+u^2}$.

(3) $\frac{d}{dx}\sec^{-1}u = \frac{u'}{|u|\sqrt{u^2-1}}$, $\quad \frac{d}{dx}\csc^{-1}u = \frac{-u'}{|u|\sqrt{u^2-1}}$.

Example 4:

(a) $\frac{d}{dx}\sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}$.

(b) $\frac{d}{dx}\tan^{-1}(3x) = \frac{3}{1+9x^2}$.

(c) $\frac{d}{dx}\sin^{-1}\sqrt{x} = \frac{(1/2)x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$.

(d) $\frac{d}{dx}\sec^{-1}(e^{2x}) = \frac{2e^{2x}}{|e^{2x}|\sqrt{e^{4x}-1}} = \frac{2}{\sqrt{e^{4x}-1}}$. #

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Thank you for your attention!