Chapter 3
Applications of Differentiation
(微分的應用)

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本章預定授課範圍

Section 3.1
FExtrema on an Interval
(區間上的極值)

Def. of Relative Extrema (相對極值)

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$ with $c \in D$.

(1) f has a relative maximum (相對極大值; rel. max.) at the point $(c, f(c))$ if $\exists$ open interval I s.t. $f(x) \leq f(c) \quad \forall x \in I$.

(2) f has a relative minimum (相對極小值; rel. min.) at the point $(c, f(c))$ if $\exists$ open interval I s.t. $f(x) \geq f(c) \quad \forall x \in I$.

(3) Rel. max. and rel. min. are called the relative extrema of f.

Some Questions

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$.

- Does f *always* have a relative extremum on D?
- How to find the relative extrema of f?
- What is $f'(c)$ if $f(c)$ is a relative extremum?

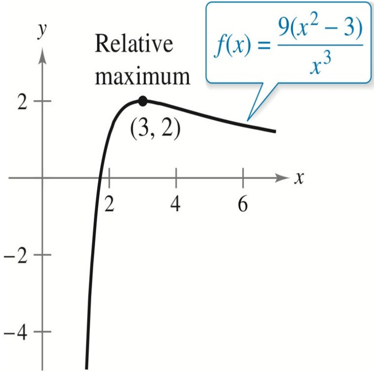
Example 1 (極值發生處的導數)

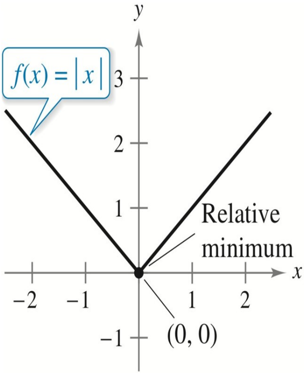
(a) The rational function

$$f(x) = \frac{9(x^2-3)}{x^3} \quad \text{with} \quad f'(x) = \frac{9(9-x^2)}{x^4}$$

has a rel. max. at the point $(3,2)$, and $f'(3) = 0$ in this case.

(b) The function $f(x) = |x|$ has a rel. min. value $f(0) = 0$ at the origin $(0,0)$, but $f'(0) \nexists$. (Why?)


y
Relative
maximum
$f(x) = \frac{9(x^2 - 3)}{x^3}$
2
(3, 2)
2
4
6
x
−2
−4


y
$f(x) = |x|$
3
2
1
Relative
minimum
x
−2
−1
1
2
−1
(0, 0)

.

Def. of Critical Numbers (臨界數)

If $f'(c) = 0$ or $f'(c) \nexists$ for some $c \in \mathrm{dom}(f)$, then the value c is called a critical number of f.

Thm 3.2 (發生相對極值的必要條件)

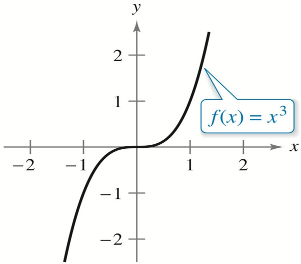
If f has a relative extremum at the point $(c, f(c))$ with $c \in D = \mathrm{dom}(f)$, then

$$f'(c) = 0 \quad \text{or} \quad f'(c) \nexists,$$

i.e. $x = c$ must be a critical number of f.

Example (Thm 3.2 的反例)

For $f(x) = x^3$, $x = 0$ is the only critical number of f, since $f'(x) = 3x^2 = 0 \iff x = 0$. But, $f(0) = 0$ is NOT a relative extremum of f.



Cubing function

Proof of Thm 3.2

Suppose that $f(c)$ is a relative extremum with $f'(c) \neq 0$ $\exists$. Without loss of generality, we may assume that $f'(c) > 0$.

For $\varepsilon = \dfrac{f'(c)}{2} > 0$, $\exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \frac{f'(c)}{2} \quad \text{or} \quad \frac{f(x) - f(c)}{x - c} > \frac{f'(c)}{2} > 0.$$

Thus, we know that

$$f(x) > f(c) \quad \forall\, x \in (c, c+\delta) \quad \text{and} \quad f(x) < f(c) \quad \forall\, x \in (c-\delta, c).$$

This contradicts to the assumption and hence completes the proof.

Def. of Absolute Extrema (絕對極值)

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$ with $c \in D$.

(1) $f(c)$ is the absolute maximum (絕對極大值; abs. max.) of f on D if $f(x) \leq f(c) \quad \forall x \in D$.

(2) $f(c)$ is the absolute minimum (絕對極小值; abs. min.) of f on D if $f(x) \geq f(c) \quad \forall x \in D$.

(3) Abs. max. and abs. min. are called the absolute extrema of f on D.

Thm 3.1 (Extreme Value Theorem; E.V.T. 極值定理)

If f is conti. on $I = [a, b]$, then $\exists\, c_1, c_2 \in I$ s.t.

$$f(c_1) \le f(x) \le f(c_2) \quad \forall\, x \in I,$$

i.e., $f(c_1)$ is the abs. min. value of f on I and $f(c_2)$ is the abs. max. value of f on I, respectively.

How to find the points c_1 and c_2 in Thm 3.1?

Step 1 find all critical numbers $c_1, c_2, \ldots, c_k$ of f in the open interval (a, b), where $k \in \mathbb{N}$.

Step 2 evaluate $f(a)$, $f(b)$ and $f(c_i)$ for $i = 1, 2, \ldots, k$.

Step 3 compare the function values obtained in Step 2.

Example 2: (Thm 3.1 の例子)

Find the (absolute) extrema of $f(x)=3x^4-4x^3$ on $I=[-1,2]$.

Sol: $f'(x)=12x^3-12x^2=12x^2(x-1)\ \forall x\in\mathbb{R}$.

So, $f'(x)=0 \iff x=0$ and $x=1$. are critical numbers

x	-1	0	1	2
$f(x)$	7	0	-1	16

So, $f(1)=-1$ is the absolute min. value of f

and $f(2)=16$ " " max. " " " .

✗

Example 3 : (Thm 3.1 の例子)

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on $I = [-1, 3]$.

Sol: $\because f'(x) = 2 - 2x^{-1/3} = 2x^{-1/3}(x^{1/3} - 1)$

$$= \frac{2(x^{1/3}-1)}{x^{1/3}}.$$

$\therefore f'(x) = 0 \iff x^{1/3} - 1 = 0 \iff x = 1.$

and $f'(x) \nexists$ when $x = 0$.

$\Rightarrow x = 0$ and $x = 1$ are critical numbers of f.

x	-1	0	1	3
$f(x)$	-5	0	-1	$6 - 3\sqrt[3]{9} \doteqdot -0.24$

So, $f(-1) = -5$ is the absolute min value of f

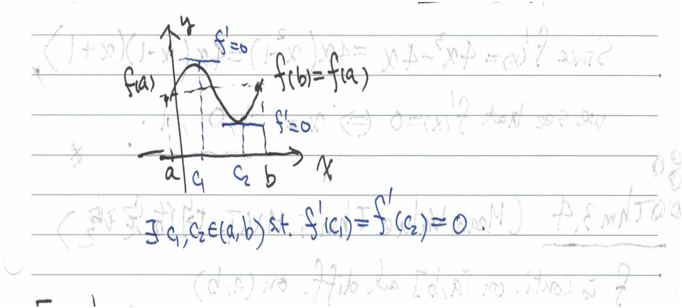
and $f(0) = 0$ " " " max. " " "

#

Section 3.2
Rolle's Theorem and the Mean Value Theorem
(洛爾定理與均值定理)

Thm 3.3 (Rolle's Theorem)

Suppose that f is conti. on $[a, b]$ and diff. on (a, b). If $f(a) = f(b)$, then $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$.



Proof of Thm 3.3

- Since f is conti. on $[a, b]$ and $f(a) = f(b)$, it follows from E.V.T. that $\exists\, c \in (a, b)$ s.t. $f(c)$ is a relative extremum. Otherwise, f must be a constant function on $[a, b]$ and hence $f'(x) = 0 \quad \forall\, x \in (a, b)$.
- Next, we shall claim that $f'(c) = 0$. If not, say $f'(c) > 0$, then it foyws from the ε-δ Def. of a limit that $\exists\, \delta > 0$ s.t. $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(c) < f(x)$ for $x \in (c, c + \delta)$. Thus, $f(c)$ is NOT a relative extremum and this gives a contradiction!
- Similarly, we can deduce that $f'(c) < 0 \Longrightarrow f(c)$ is NOT a relative extremum. Consequently, we must have $f'(c) = 0$ for some $c \in (a, b)$.

Example 1: (Rolle's Thm 的例子)

Let $f(x) = x^2 - 3x + 2 \quad \forall x \in \mathbb{R}$.

(a) Find two x-intercepts a and b.

(b) Show that $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$.

Sol: (a) ∵ $f(x) = x^2 - 3x + 2 = (x-1)(x-2) = 0$

$\Leftrightarrow$ $x=1$ and $x=2$ are zeros of f.

i.e. f has two x-intercepts at $x=1$ and $x=2$. #

(b) ∵ f is conti. on $[1,2]$, $f'(x) = 2x-3$ $\forall x \in (1,2)$ and $f(1) = f(2) = 0$.

∴ $\exists\, c \in (1,2)$ s.t. $f'(c) = 0$ by Rolle's Thm.

In fact, $c = 3/2$ #

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Example 2: (求 Rolle's Thm 中的 c 值)

Let $f(x) = x^4 - 2x^2$. Find all $c \in (-2,2)$ s.t. $f'(c) = 0$.

Sol: ∵ f is conti. on $[-2,2]$, diff. on $(-2,2)$

and $f(-2) = f(2) = 8$.

∴ By Rolle's Thm $\Rightarrow \exists\, c \in (-2,2)$ s.t. $f'(c) = 0$.

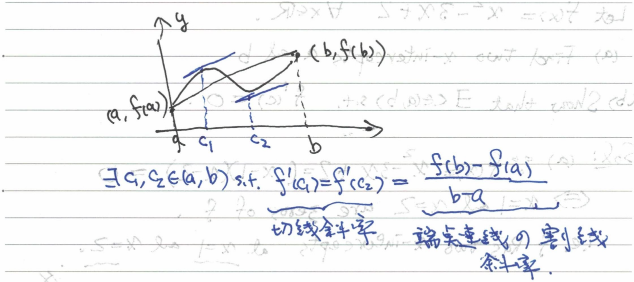
Since $f'(x) = 4x^3 - 4x = 4x(x^2-1) = 4x(x-1)(x+1)$,

we see that $f'(x) = 0 \Leftrightarrow x = -1, 0, 1$. #

Thm 3.4 (Mean Value Theorem; M.V.T. 均值定理)

If f is conti. on $[a, b]$ and diff. on (a, b), then $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a).$$



Proof of Thm 3.4

- Let $g : [a, b] \to \mathbb{R}$ be a function defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \quad \forall\, x \in [a, b].$$

Since f is conti. on $[a, b]$ and diff. on (a, b), we know that g is conti. on $[a, b]$, diff. on (a, b) and $g(a) = 0 = g(b)$.

- Since $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a} \quad \forall\, x \in (a, b)$, it follows from Thm 3.3 (Rolle's Thm) that $\exists\, c \in (a, b)$ s.t.

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

or, equivalently, we prove that $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

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Example 3: (求 MVT 中的 c 值)

For $f(x) = 5 - \frac{4}{x}$, find all $c \in (1, 4)$ s.t.

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Sol: $\because$ f is conti. on $[1,4]$ and diff. on $(1,4)$

$\therefore$ $\exists c \in (1,4)$ s.t. $f'(c) = \dfrac{f(4)-f(1)}{4-1} = \dfrac{4-1}{3} = 1$ by Mean Value Thm.

$\because$ $f'(x) = \dfrac{d}{dx}(5-4x^{-1}) = 4x^{-2} = \dfrac{4}{x^2} > 0 \quad \forall x \in [1,4]$.

$\therefore$ $f'(c) = \dfrac{4}{c^2} = 1 \iff c^2 = 4 \iff c = -2$ or 2.
(不合)

$\Rightarrow c = 2$ because we need $c \in (1,4)$.

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Example (M.V.T. 的補充題)

For any $a, b \in \mathbb{R}$, prove the following inequality

$$|\sin a - \sin b| \leq |a - b|.$$

Proof: Let $a, b \in \mathbb{R}$. Without loss of generality, we may assume that $a < b$. Since $f(x) = \sin x$ is conti. on $[a, b]$ and diff. on (a, b), it follows from M.V.T. that $\exists\, c \in (a, b)$ s.t.

$$\sin b - \sin a = f'(c) \cdot (b - a) = (\cos c) \cdot (b - a).$$

So, we immediately see that

$$|\sin a - \sin b| = |\cos c| \cdot |a - b| \leq |a - b|$$

because $|\cos c| \leq 1$, and hence this completes the proof.

Section 3.3
Increasing and Decreasing Functions and the First Derivative Test
(遞增、遞減函數與一階導數測試)

Def (單調函數的定義)

Let f be a real-valued function defined on an interval I.

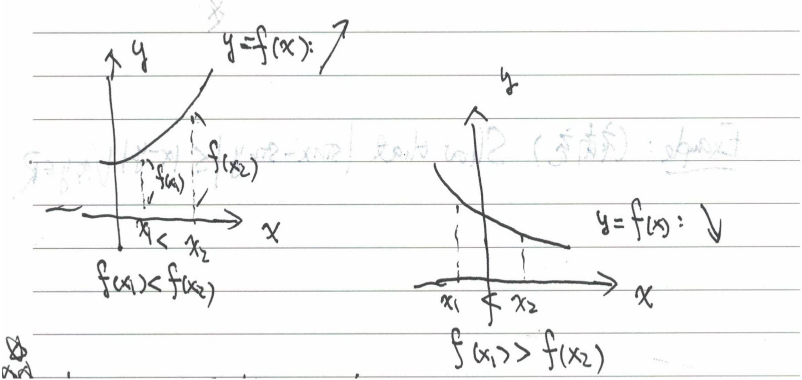
(1) f is increasing (遞增; $\nearrow$) on I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

(2) f is decreasing (遞減; $\searrow$) on I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

(3) The increasing or decreasing functions are called monotonic functions (單調函數).

Note: Monotonic functions are one-to-one, but one-to-one functions are NOT necessarily monotonic!

單調函數的示意圖



Thm 3.5 (單調函數的充分條件)

(1) $f'(x) > 0 \quad \forall x \in (a, b) \Longrightarrow f$ is increasing ($\nearrow$) on $[a, b]$.

(2) $f'(x) < 0 \quad \forall x \in (a, b) \Longrightarrow f$ is decreasing ($\searrow$) on $[a, b]$.

(3) $f'(x) = 0 \quad \forall x \in (a, b) \Longrightarrow f$ is constant on $[a, b]$.

Example (Thm 3.5 的反例)

The function $f(x) = x^{1/3}$ is increasing on $\mathbb{R}$, but its first derivative satisfies $f'(x) = \frac{1}{3x^{2/3}} > 0 \quad \forall x \in \mathbb{R} \backslash \{0\}$.

For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, since f is conti. on $[x_1, x_2]$ and diff. on (x_1, x_2), it follows from M.V.T. that $\exists\, c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

(1) If $f'(x) > 0 \quad \forall\, x \in (a, b)$, then $f'(c) > 0$ and hence $f(x_2) - f(x_1) > 0$ or $f(x_2) > f(x_1)$. This implies that f is increasing ($\nearrow$) on (a, b).

(2) If $f'(x) < 0 \quad \forall\, x \in (a, b)$, then $f'(c) < 0$ and hence $f(x_2) - f(x_1) < 0$ or $f(x_2) < f(x_1)$. This implies that f is decreasing ($\searrow$) on (a, b).

(3) If $f'(x) = 0 \quad \forall\, x \in (a, b)$, then $f(x_2) - f(x_1) = 0$ or $f(x_1) = f(x_2) \quad \forall\, x_1, x_2 \in (a, b)$, i.e., f is constant on (a, b).

Example 1: (Thm 3.5 的例子)

Find open intervals where ~~f 遞增 or 遞減~~ $f(x) = x^3 - \frac{3}{2}x^2$ is $\nearrow$ or $\searrow$:

Sol: $\because f'(x) = 3x^2 - 3x = 3x(x-1)$

$\therefore f'(x) = 0 \Leftrightarrow x = 0$ or $x = 1$

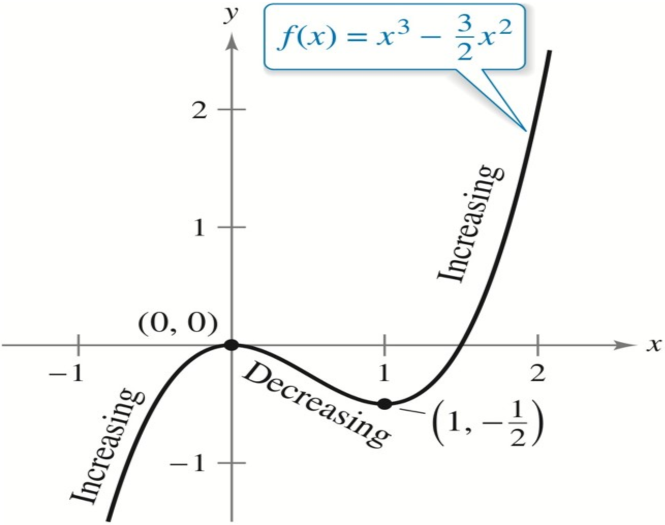
So, $x=0$ and $x=1$ are two critical numbers of f.

x		0		1	
$f'(x)$	$+$		$-$		$+$
I/D	↗		↘		↗

Hence, f is ↗ on $(-\infty, 0)$ and $(1, \infty)$, and

f is ↘ on $(0,1)$ by Thm 3.5. #

Example 1 的示意圖



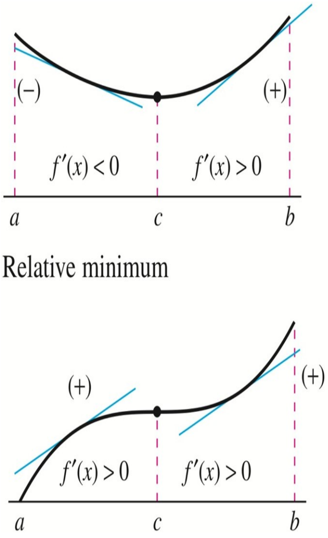
Thm 3.6 (First Derivative Test; 一階導數測試)

Let f be diff. on an open interval except possibly at c. If $x = c$ is a critical number of f, then

(1) sign of f' changes from $(+)$ to $(-)$ at $c \Longrightarrow f(c)$ is a rel. max. value of f.

(2) sign of f' changes from $(-)$ to $(+)$ at $c \Longrightarrow f(c)$ is a rel. min. value of f.

(3) sign of f' does not change on both sides of $c \Longrightarrow f(c)$ is not a relative extremum.


(−)
(+)
$f'(x) < 0$
$f'(x) > 0$
a
c
b
Relative minimum
(+)
(−)
$f'(x) > 0$
$f'(x) < 0$
a
c
b
Relative maximum
(+)
(+)
$f'(x) > 0$
$f'(x) > 0$
a
c
b
(−)
(−)
$f'(x) < 0$
$f'(x) < 0$
a
c
b

Example 3: (Thm 3.6 的例子)

Find all relative extrema for $f(x) = (x^2-4)^{2/3}$.

Sol: Note that $f'(x) = \frac{2}{3}(x^2-4)^{-1/3}(2x) = \frac{4x}{3\sqrt[3]{x^2-4}}$. Then

$f'(0) = 0 \Leftrightarrow x = 0$ and $f'(x) \nexists$ when $x = \pm 2$.

So, $x = -2, 0, 2$ are critical numbers of f.

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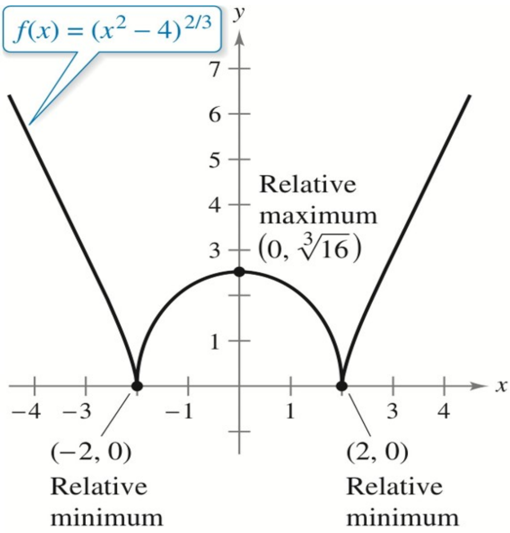
x		-2		0		2	
$f'(x)$	$-$		$+$		$-$		$+$
I/D	↘		↗		↘		↗

So, from Thm 3.6, we know that

$f(-2) = f(2) = 0$ is a relative min. and

$f(0) = \sqrt[3]{16}$ is a relative max. #

Example 3 的示意圖



Example 4 : (Thm 3.6 的例子)

Find all relative extrema for $f(x)=\dfrac{x^4+1}{x^2}$ for $x\neq 0$.

Sol: Note that $f(x)=\dfrac{x^4+1}{x^2}=x^2+x^{-2}$ for $x\neq 0$.

$\Rightarrow f'(x)=2x-2x^{-3}=2x^{-3}(x^4-1)$

$=\dfrac{2(x^4-1)}{x^3}=\dfrac{2(x^2+1)(x-1)(x+1)}{x^3}$.

$\Rightarrow f'(x)=0$ when $x=-1$ and $x=1$, and

$f'(x)$ 不存在 when $x=0$.

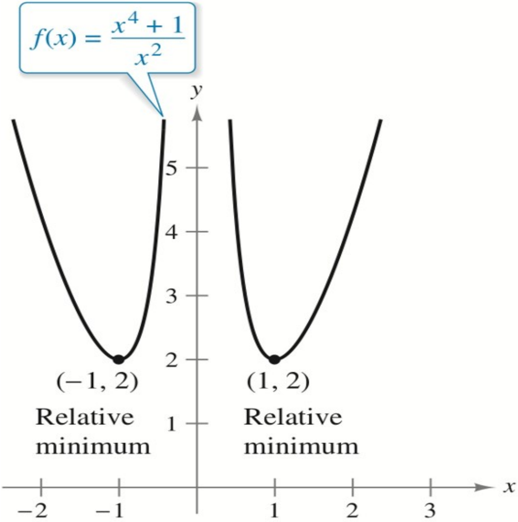
So, $x = -1$ and $x = 1$ are critical numbers of f.

x		-1		0		1	
$f'(x)$	$-$		$+$		$-$		$+$
I/D	$\searrow$		$\nearrow$		$\searrow$		$\nearrow$

So, $f(-1) = f(1) = 2$ is a relative min. value.

Note: $f(0)$ is NOT well-defined

Example 4 的示意圖



Section 3.4
Concavity and the Second Derivative Test
(凹性與二階導數測試)

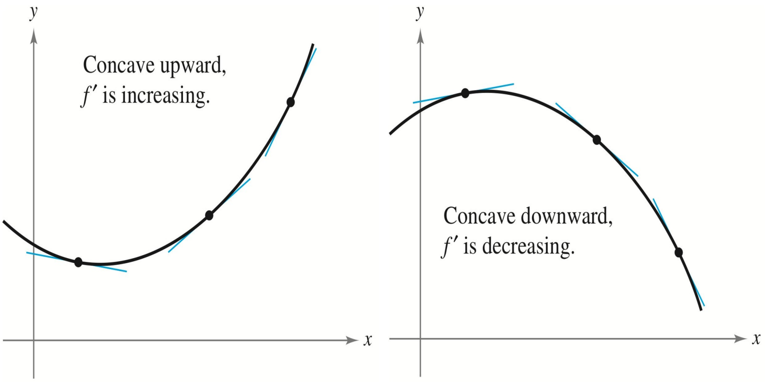
Def. of Concavity (凹性)

Let f be diff. on an open interval $I = (a, b)$.

(1) The graph of f is concave upward (凹向上; C.U.) on I if its first derivative f' is $\nearrow$ on I.

(2) The graph of f is concave downward (凹向下; C.D.) on I if its first derivative f' is $\searrow$ on I.

凹性的示意圖 (承上頁)


y
Concave upward,
f′ is increasing.
x
y
Concave downward,
f′ is decreasing.
x

Thm 3.7 (Test for Concavity; 凹性測試法)

Suppose that $f''(x)$ exists on an open interval I.

(1) $f''(x) > 0 \quad \forall x \in I \implies$ the graph of f is C.U. on I.

(2) $f''(x) < 0 \quad \forall x \in I \implies$ the graph of f is C.D. on I.

pf: It follows immediately from the Def. of Concavity and $f''(x) = \frac{d}{dx}[f'(x)]$ that

(1) $f''(x) > 0 \quad \forall x \in I \implies f'$ is increasing on $I \implies$ the graph of f is C.U. on I.

(2) $f''(x) < 0 \quad \forall x \in I \implies f'$ is decreasing on $I \implies$ the graph of f is C.D. on I.

Example 1: (判斷函數圖形の凹性)

Determine open intervals where the graph of $f(x)=e^{-x^2/2}$ is C.U. or C.D.

Sol: $\because f(x)=e^{-x^2/2}\ \forall x\in\mathbb{R}$.

$\therefore f'(x)=-xe^{-x^2/2}$ by Chain Rule.

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$$\Rightarrow f''(x) = -e^{-x^2/2} - xe^{-x^2/2}(-x) = e^{-x^2/2}(x^2-1)$$

$$= e^{-x^2/2}(x+1)(x-1). \quad \forall x \in \mathbb{R}.$$

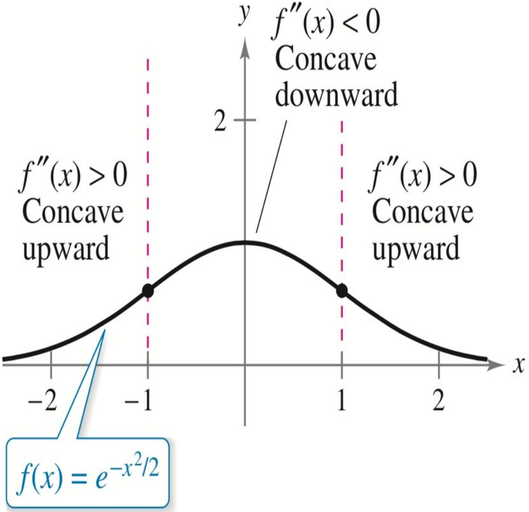
So, $f''(x) = 0 \iff x = -1$ or $x = 1$.

x		-1		1	
f''	$+$		$-$		$+$
C.U./C.D.	C.U.		C.D.		C.U.

Hence, the graph of is C.U. on $(-\infty, -1)$ and $(1, \infty)$.

" " " , " C.D. " $(-1, 1)$. #

Example 1 的示意圖



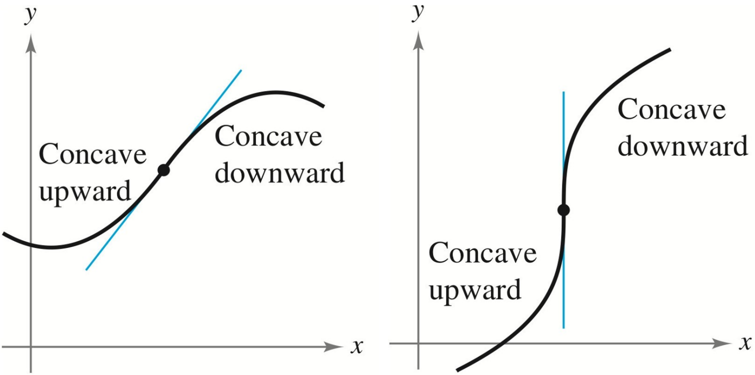
Def. of Points of Inflection (反曲點; P.I.)

Let f be conti. on an open interval containing c. If the graph of f

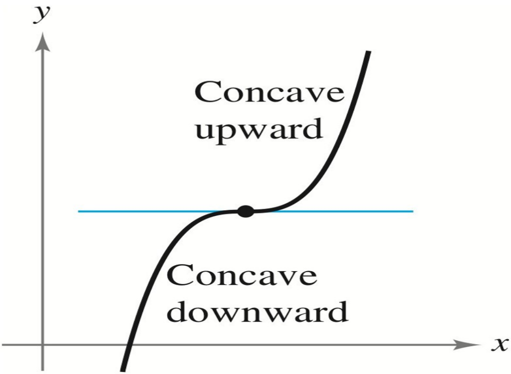
1. has a (vertical) tangent line at $(c, f(c))$, and
2. its concavity changes on both sides of c,

then $(c, f(c))$ is called a point of inflection of the graph of f.
(函數圖形凹性改變的轉折點即為反曲點!)

反曲點的示意圖 (1/2)


y
Concave
upward
Concave
downward
x
y
Concave
downward
Concave
upward
x

反曲點的示意圖 (2/2)



Thm 3.8 (反曲點的必要條件)

Suppose that $f''(x)$ exists on an open interval containing c. If $(c, f(c))$ is a point of inflection of the graph of f, then

$$f''(c) = 0 \quad \text{or} \quad f''(c) \nexists.$$

Without loss of generality, we assume that $f''(c) > 0$ $\exists$. Since f'' exists at c, we know that,

$$\lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c).$$

Thus, for $\varepsilon = \frac{f''(c)}{2} > 0$, $\exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then

$$\left| \frac{f'(x) - f'(c)}{x - c} - f''(c) \right| < \frac{f''(c)}{2} \quad \text{or} \quad \frac{f'(x) - f'(c)}{x - c} > \frac{f''(c)}{2} > 0.$$

Then $f'(x) < f'(c)$ for $x \in (c - \delta, c)$ and $f'(c) < f'(x)$ for $x \in (c, c + \delta)$. Thus, it follows from the Def. of concavity that the graph of f is C.U. on $I = (c - \delta, c + \delta)$. This contradicts to our assumption that $(c, f(c))$ is a point of inflection of f, and hence we complete the proof.

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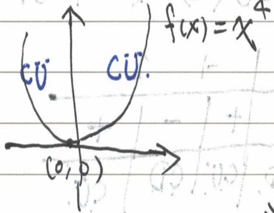
Example: For $f(x) = x^4$, $f'(x) = 4x^3$ and $f''(x) = 12x^2$

$\Rightarrow f''(0) = 0$, but $(0,0)$ is NOT a point of inflection of the graph of f.

x		0	
f''		+	+
CU/CD		CU	CU



#

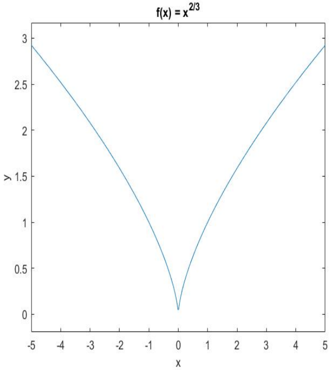
Example (Thm 3.8 的另一個反例)

Consider $f(x) = \sqrt[3]{x^2} = x^{2/3} \quad \forall\, x \in \mathbb{R}$. Then its first and second derivatives are given by

$$f'(x) = \frac{2}{3}x^{-1/3} \quad \text{and} \quad f''(x) = \frac{-2}{9}x^{-4/3} < 0$$

for all $x \neq 0$. In this case, we see that $f''(0) \nexists$ and the origin $(0,0)$ is NOT a point of inflection!

示意圖 (承上例)



Example 3: (反曲點の例子)

Find all P.I. and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.

Sol: Note that $f'(x) = 4x^3 - 12x^2$

$$\Rightarrow f''(x) = 12x^2 - 24x = 12x(x-2).$$

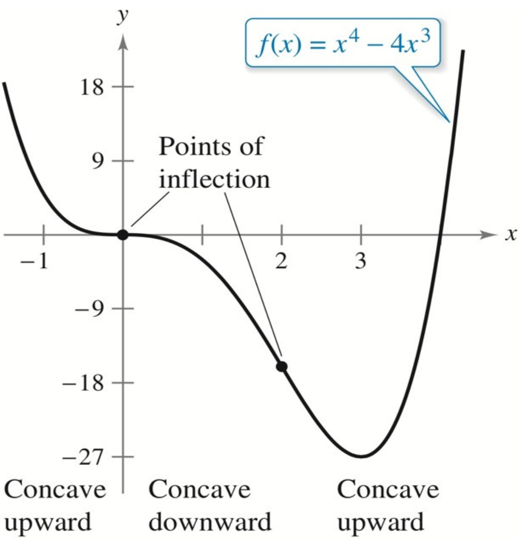
So, $f''(x)=0 \Leftrightarrow x=0$ or $x=2$.

x		0		2	
f''	+		−		+
CU/CD	CU		CD		CU

$\Rightarrow$ The graph of f is C.U. on $(-\infty,0)$ and $(2,\infty)$, and C.D. on $(0,2)$.

$\Rightarrow (0,0)$ and $(2,-16)$ are points of inflection. #

Example 3 的示意圖



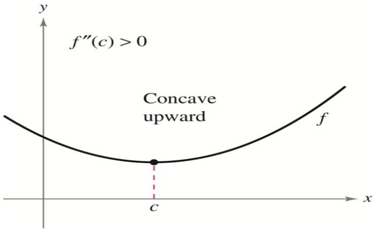
Thm 3.9 (Second Derivative Test; 二階導數測試)

Suppose that $f'(c) = 0$ and f'' $\exists$ on an open interval containing c.

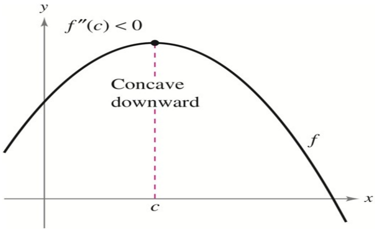
(1) $f''(c) > 0 \Longrightarrow f(c)$ is a rel. min. value.

(2) $f''(c) < 0 \Longrightarrow f(c)$ is a rel. max. value.

(3) $f''(c) = 0 \Longrightarrow$ the test is inconclusive.



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.



Example 4 (Thm 3.9 的例子)

Find all relative extrema of the (polynomial) function

$$f(x) = -3x^5 + 5x^3.$$

Sol: $\because f'(x) = -15x^4 + 15x^2 = -15x^2(x^2-1)$

$= -15x^2(x+1)(x-1).$

$\therefore f'(x) = 0 \Leftrightarrow x = -1, 0, 1.$

$\because f''(x) = -60x^3 + 30x = -30x(2x^2 - 1)$.

$\because f''(-1) = 30 > 0 \Rightarrow f(-1) = -2$ is a relative min.

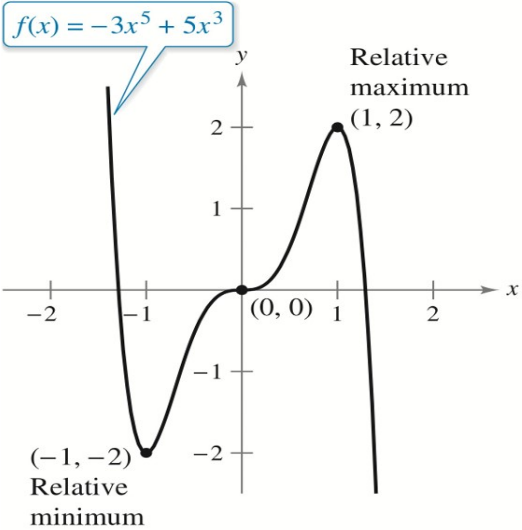
$f''(1) = -30 < 0 \Rightarrow f(1) = 2$ is a relative max.

$f''(0) = 0 \Rightarrow$ Second Derivative Test fails!

In fact, $(0, 0)$ is a point of inflection for f.

x	$-1/\sqrt{2}$		0		$1/\sqrt{2}$
f''		$-$		$+$	
CU/CD		CD		CU	

#


$f(x) = -3x^5 + 5x^3$
y
Relative
maximum
(1, 2)
2
1
−2
−1
(0, 0)
1
2
x
−1
−2
(−1, −2)
Relative
minimum

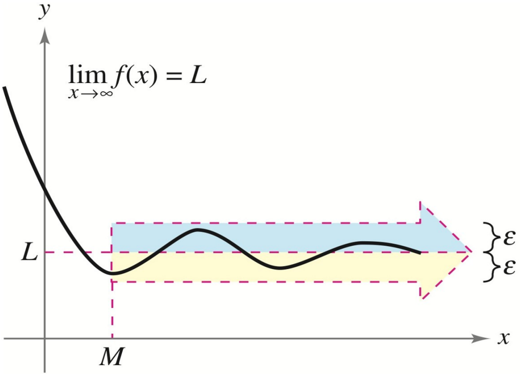
Section 3.5
Limits at Infinity
(無窮遠處的極限)

Def (Limits at Infinity)

(1) $\lim_{x\to\infty} f(x) = L \iff \forall\, \varepsilon > 0$, $\exists\, M > 0$ s.t. if $x > M$, then $|f(x) - L| < \varepsilon$.

(2) $\lim_{x\to-\infty} f(x) = L \iff \forall\, \varepsilon > 0$, $\exists\, N < 0$ s.t. if $x < N$, then $|f(x) - L| < \varepsilon$.

示意圖 (承上頁)



Thm 3.10 (重要的極限法則)

(1) If $r > 0$ is a rational number and $c \in \mathbb{R}$, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 = \lim_{x\to-\infty} \frac{c}{x^r}.$$

(2) $\lim_{x\to\infty} e^{-x} = 0 = \lim_{x\to-\infty} e^{x}.$

Proof of (1) in Thm 3.10

Let $\varepsilon > 0$ be given arbitrarily. Choose $M, N \in \mathbb{R}$ with

$$M > \left(\frac{|c|}{\varepsilon}\right)^{1/r} > 0 \quad \text{and} \quad N < -\left(\frac{|c|}{\varepsilon}\right)^{1/r} < 0.$$

Thus we have the following inequalities

$$\frac{|c|}{M^r} < \varepsilon \quad \text{and} \quad \frac{|c|}{(-N)^r} < \varepsilon. \qquad \text{(Check!)}$$

If $x > M(> 0)$ or $x < N(< 0)$, then

$$\left|\frac{c}{x^r} - 0\right| = \frac{|c|}{x^r} < \frac{|c|}{M^r} < \varepsilon \quad \text{or} \quad \left|\frac{c}{x^r} - 0\right| = \frac{|c|}{|x|^r} = \frac{|c|}{(-x)^r} < \frac{|c|}{(-N)^r} < \varepsilon.$$

So, it follows from the Def. that

$$\lim_{x\to\infty} \frac{c}{x^r} = 0 = \lim_{x\to-\infty} \frac{c}{x^r}.$$

Example 1 :

(a) $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)=5-0=5.$

(b) $\lim_{x\to\infty}\frac{3}{e^x}=\lim_{x\to\infty}3e^{-x}=3(0)=0.$ #

Example 2: (分子和分母為同階多項式)

$$\lim_{x\to\infty}\frac{2x-1}{x+1}=\lim_{x\to\infty}\frac{x(2-\frac{1}{x})}{x(1+\frac{1}{x})}=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{1+\frac{1}{x}}$$

$$=\frac{2-0}{1+0}=2. \quad \#$$

Example 3: (分子階數增加且分母不變)

(a) $\lim_{x\to\infty} \frac{2x+5}{3x^2+1} = \lim_{x\to\infty} \frac{\frac{2x+5}{x^2}}{\frac{3x^2+1}{x^2}} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{1}{x^2}} = \frac{0+0}{3+0} = 0.$ ※

(b) $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1} = \lim_{x\to\infty} \frac{2+5/x^2}{3+1/x^2} = \frac{2}{3}.$ ※

(c) $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1} = \lim_{x\to\infty} \frac{x^3(2+\frac{5}{x^3})}{x^2(3+\frac{1}{x^2})} = \lim_{x\to\infty} x\left(\frac{2}{3}\right) = \infty.$ ※

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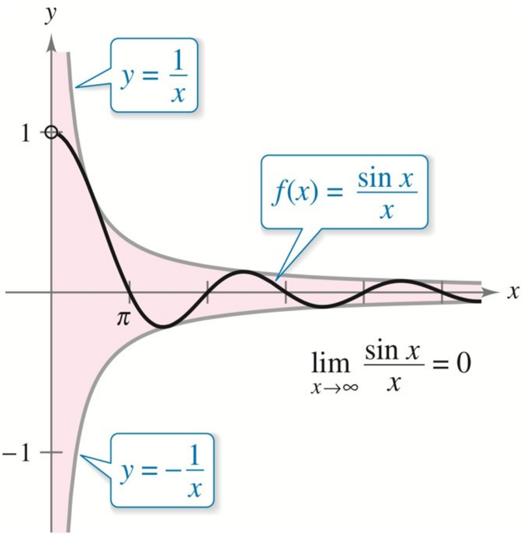
Example 5:

(a) $\lim_{x\to\infty} \sin x \nexists$ because $f(x)=\sin x$ oscillates between -1 and 1 as $x\to\infty$.

(b) $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ because $\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ for $x>0$.

and $\lim_{x\to\infty}\left(\frac{-1}{x}\right) = \lim_{x\to\infty}\frac{1}{x} = 0$. #

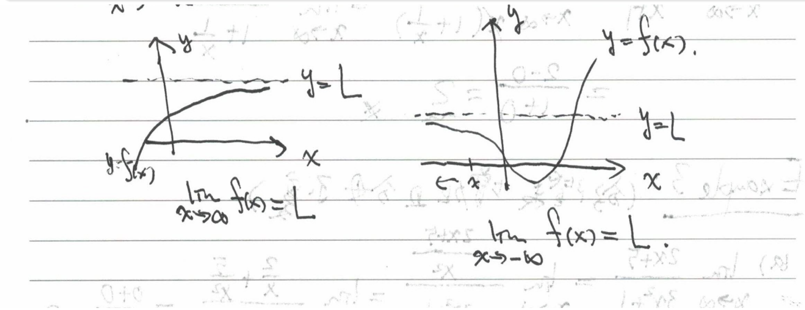
Example 5 的示意圖



Def (水平漸近線的定義)

A line $y = L$ is callled a horizontal asymptote (水平漸近線) of the graph of f if

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$



Example 4: Find all horizontal asymptotes for ~~asymptotes for~~ f

$$f(x) = \frac{1}{1+e^{-x}} \quad \forall x \in \mathbb{R}.$$

Sol:

$\because \lim_{x\to\infty} \frac{1}{1+e^{-x}} = \frac{1}{1+0} = 1$ and $\lim_{x\to-\infty} \frac{1}{1+e^{-x}} = \left(\frac{1}{\infty}\right) = 0$

$\therefore$ $y=1$ and $y=0$ are two horizontal asymptotes to the graph of f $\#$

Per-Duet

Def (無窮遠處的無窮極限值)

(1) $\lim_{x\to\infty} f(x) = \infty\ (-\infty) \iff \forall\, M > 0\ (M < 0),\ \exists\, N > 0$ s.t. if $x > N$, then $f(x) > M\ (f(x) < M)$.

(2) $\lim_{x\to-\infty} f(x) = \infty\ (-\infty) \iff \forall\, M > 0\ (M < 0),\ \exists\, N < 0$ s.t. if $x < N$, then $f(x) > M\ (f(x) < M)$.

Def (斜漸近線的定義)

A line $y = mx + b$ with slope $m \neq 0$ is called a slant (or oblique) asymptote (斜漸近線) of the graph of f if

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x \to -\infty} [f(x) - (mx + b)] = 0.$$

Note: If $p(x)/q(x)$ is a rational function with $\deg(p) = \deg(q) + 1$, applying the method of long division (長除法), we obtain

$$\frac{p(x)}{q(x)} = (mx + b) + \frac{r(x)}{q(x)},$$

where $r(x)$ is a polynomial with $\deg(r) < \deg(q)$.

Example 8 (斜漸近線的例子)

Let $f(x) = \dfrac{2x^2-4x}{x+1}$ for $x \neq -1$.

(a) Evaluate $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

(b) Find a slant asymptote for f.

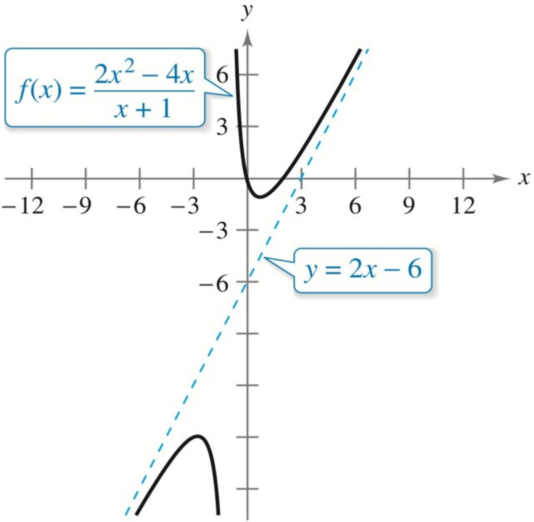
Sol: By Long Division $\Rightarrow f(x) = (2x-6) + \dfrac{6}{x+1}$ for $x \neq -1$.

(a) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} (2x-6) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (2x-6) = -\infty$. #

(b) $\because f(x) - (2x-6) = \dfrac{6}{x+1}$ for $x \neq -1$.

$\therefore \lim\limits_{x\to\pm\infty} [f(x) - (2x-6)] = \lim\limits_{x\to\pm\infty} \dfrac{6}{x+1} = 0 \Rightarrow y = 2x-6$ is a slant asymptote for f. #

Example 8 的示意圖



Section 3.7
Differentials
(全微分或是微分)

Def. of Differentials

Let f be diff. on an open interval I with $x \in I$.

(1) The differential of x, denoted by dx, is any nonzero real number.

(2) The differential of $y = f(x)$ is defined by $dy = f'(x)dx$.

Main Question

If $dx = \Delta x \approx 0$ is sufficiently small, how to estimate

$$\Delta y = f(x + \Delta x) - f(x) = f(x + dx) - f(x)$$

using the differential dy directly?

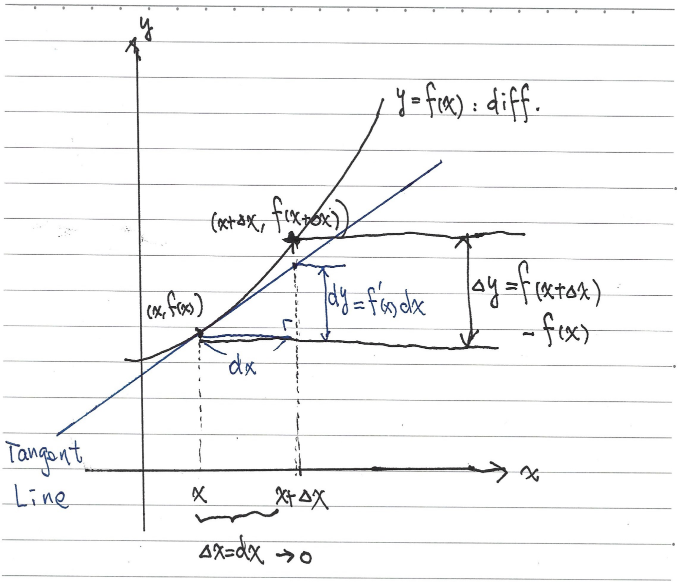
Thm (利用 dy 估計 Δy)

If $dx = \Delta x \approx 0$ is sufficiently small, then

(1) $f(x+\Delta x) - f(x) = \Delta y \approx dy = f'(x)dx$.

(2) $f(x+dx) = f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)dx$.

Δy 與 dy 的示意圖 (承上頁)



Example 2 (比較 Δy 和 dy).

For $y=f(x)=x^2$, $dy=f'(x)dx=2x\,dx$ and $\Delta y=f(x+\Delta x)-f(x)$

$$=(x+\Delta x)^2-x^2=2x\Delta x+(\Delta x)^2.$$

If $\Delta x=dx=0.01$ and $x=1$, then

$dy=2(1)(0.01)=0.02$. and $\Delta y=2(1)(0.01)+(0.01)^2$

$=0.0201$. #

Equivalent Def. of Differentiability (可微分性的等價定義)

Let f be a real-valued function defined on $D = \text{dom}(f)$. Then

$$f \text{ is diff. at } x \in D.$$

$$\Longrightarrow \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) \quad \exists.$$

$$\Longrightarrow \exists \text{ a function } \varepsilon_1 = \varepsilon_1(\Delta x) \text{ with } \lim_{\Delta x \to 0} \varepsilon_1(\Delta x) = 0 \text{ s.t.}$$

$$\Delta y = f(x+\Delta x) - f(x) = f'(x) \cdot \Delta x + \varepsilon_1(\Delta x) \cdot \Delta x.$$

Note: In Example 2, $f(x) = x^2$ is diff. at any $x \in \mathbb{R}$, since we have

$$\Delta y = f(x+\Delta x) - f(x) = (2x) \cdot \Delta x + \varepsilon_1(\Delta x) \cdot \Delta x,$$

where $\varepsilon_1(\Delta x) \equiv \Delta x \to 0$ as $\Delta x \to 0$.

Example 4: (求 dy)

Find the differential dy of the given function.

(a) $y = x^2 \Rightarrow dy = 2x\,dx$.

(b) $y = \sqrt{x} = x^{1/2} \Rightarrow dy = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\,dx$.

(c) $y = 2\sin x \Rightarrow dy = 2\cos x \, dx.$

(d)

$y = xe^x \Rightarrow dy = (1 \cdot e^x + x \cdot e^x)\, dx = e^x (x+1)\, dx.$

(e) $y = \frac{1}{x} = x^{-1} \Rightarrow dy = (-1) x^{-2}\, dx = \frac{-1}{x^2}\, dx.$

Example 7 (使用dy 估計函數值)

Use the differential to approximate $\sqrt{16.5}$.

Sol: Let $f(x) = \sqrt{x}$, $x = 16$ and $\Delta x = dx = 0.5$.

From (*) $\Rightarrow f(x+\Delta x) \approx f(x) + f'(x)\,dx$ when $dx = \Delta x \approx 0$.

$\Rightarrow \sqrt{x+\Delta x} \approx \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$ when $dx = \Delta x \approx 0$.

Since $x = 16$ and $\Delta x = dx = 0.5$, we see that

$$\sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625.$$

In fact, the true value is $\sqrt{16.5} \approx 4.0620$. #

Thank you for your attention!