Chapter 4 Integration (積分)

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Section 4.1
Antiderivatives and Indefinite Integration
(反導函數與不定積分)

Def (反導函數的定義)

A function F is an antiderivative (反導函數) of f on the interval I if $F'(x) = f(x) \quad \forall\, x \in I$.

Example

The functions $F_1(x) = x^3$, $F_2(x) = x^3 - 5$ and $F_3(x) = x^3 + 97$ are antiderivatives of $f(x) = 3x^2$ on $\mathbb{R}$.

Thm 4.1 (最廣反導函數的表示式)

If F is an antiderivative of f on the interval I, then

$$G \text{ is an antiderivative of } f \text{ on } I \Longleftrightarrow G(x) = F(x) + C,$$

where C is a constant.

Pf: ($\Leftarrow$) Suppose that $G(x) = F(x) + C$ and $F'(x) = f(x)$ $\forall x \in I$.

Then $G'(x) = F'(x) = f(x)$ $\forall x \in I$.

So, G is an antiderivative of f on I.

($\Rightarrow$) Suppose that G is an antiderivative of f on I, and let $H(x) = G(x) - F(x)$ $\forall x \in I$.

$\Rightarrow H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$ $\forall x \in I$.

$\Rightarrow H(x) \equiv C$ $\forall x \in I$, where C is a constant.

$\Rightarrow G(x) = F(x) + C$, where C is a constant.

#

Def. of Indefinite Integrals

If $F(x)$ is an antiderivative of f on the interval I, the indefinite integral of f w.r.t. x (函數 f 對 x 的不定積分) is defined by

$$\int f(x)\,dx = F(x) + C,$$

where f is called the integrand (被積函數) and C is called the constant of integration. (積分常數)

Remark (積分與微分的關係)

If $F'(x) = f(x) \quad \forall\, x \in I$, then

$$\int F'(x)\, dx = \int f(x)\, dx = F(x) + C$$

and

$$\frac{d}{dx}\left[\int f(x)\, dx\right] = \frac{d}{dx}\left[F(x) + C\right] = F'(x) = f(x),$$

i.e., the integration and differentiation are inverse operations to each other! (積分與微分互為反運算!)

Thm (Basic Integration Rules; 1/4)

Suppose that the antiderivatives of f and g exist and let $0 \neq k \in \mathbb{R}$.

(1) $\displaystyle\int k\,dx = kx + C.$

(2) $\displaystyle\int [k \cdot f(x)]\,dx = k \cdot \left[\int f(x)\,dx\right].$

(3) $\displaystyle\int [f(x) \pm g(x)]\,dx = \left[\int f(x)\,dx\right] \pm \left[\int g(x)\,dx\right].$

(4) $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$.

Thm (Basic Integration Rules; 2/4)

(5) $\displaystyle\int \cos x\,dx = \sin x + C.$

(6) $\displaystyle\int \sin x\,dx = -\cos x + C.$

(7) $\displaystyle\int \sec^2 x\,dx = \tan x + C.$

(8) $\displaystyle\int \sec x \tan x\,dx = \sec x + C.$

(9) $\displaystyle\int \csc^2 x\,dx = -\cot x + C.$

(10) $\displaystyle\int \csc x \cot x\,dx = -\csc x + C.$

Thm (Basic Integration Rules; 3/4)

(11) $$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C.$$

(12) $$\int a^{kx}\,dx = \frac{1}{k(\ln a)}a^{kx} + C \text{ for } 0 < a \neq 1.$$

(13) $$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C.$$

Thm (Basic Integration Rules; 4/4)

(14) $$\int \frac{1}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k}\sin^{-1}(kx) + C.$$

(15) $$\int \frac{1}{1+k^2x^2}\,dx = \frac{1}{k}\tan^{-1}(kx) + C.$$

(16) $$\int \frac{1}{x\sqrt{k^2x^2-1}}\,dx = \sec^{-1}(kx) + C.$$

Example 3: Find the indefinite integral.

(a) $\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{1}{-2}x^{-2} + C = \frac{-1}{2x^2} + C.$

(b) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx = \frac{1}{3/2}x^{3/2} + C = \frac{2}{3}x^{3/2} + C.$

(c) $\int 2\sin x\,dx = 2\left(\int \sin x\,dx\right) = -2\cos x + C.$

(d) $\int \frac{3}{x}dx = 3\left(\int \frac{1}{x}dx\right) = 3\ln|x| + C.$ #

Example 6:

$$\int \frac{\sin x}{\cos^2 x}\,dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)dx$$

$$= \int \sec x \tan x\,dx = \sec x + C.$$

#

Example 7: Find the indefinite integral.

(a) $\displaystyle\int \frac{2}{\sqrt{x}}\,dx = 2\int x^{-1/2}\,dx = \frac{2}{1/2}x^{1/2} + C = 4\sqrt{x} + C.$

(b) $\displaystyle\int (t^2+1)^2\,dt = \int (t^4+2t^2+1)\,dt = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C.$

(c) $\displaystyle\int \frac{x^3+3}{x^2}\,dx = \int (x+3x^{-2})\,dx = \int x\,dx + 3\left(\int x^{-2}\,dx\right)$

$\displaystyle= \frac{1}{2}x^2 + 3\left(\frac{1}{-1}x^{-1}\right) + C = \frac{1}{2}x^2 - \frac{3}{x} + C.$

(d) $\int \sqrt[3]{x}(x-4)\,dx = \int (x^{4/3} - 4x^{1/3})\,dx$

$$= \frac{x^{7/3}}{7/3} - 4\left(\frac{1}{4/3}x^{4/3}\right) + C = \frac{3}{7}x^{7/3} - 3x^{4/3} + C$$

If g is conti. on the interval $[x_0, \infty)$, consider the initial-value problem (I.V.P.; 初值問題)

$$\begin{cases} F'(x) = g(x), & \text{(differential equation; D.E. 微分方程)} \\ F(x_0) = y_0. & \text{(initial condition; I.C. 初值條件)} \end{cases} \tag{1}$$

Main Questions

- What is the general solution (通解) $F(x)$ of the D.E.in (1)?
- How to solve the particular solution (特解) of the I.V.P. (1)?

Integrating the D.E. in (1) w.r.t. $x \Longrightarrow$

$$F(x) = \int F'(x)\, dx = \int g(x)\, dx = G(x) + C$$

is the general solution of the D.E., where C is a constant.

Solutions of the I.V.P.

Integrating the D.E. in (1) w.r.t. $x \Longrightarrow$

$$F(x) = \int F'(x)\,dx = \int g(x)\,dx = G(x) + C$$

is the general solution of the D.E., where C is a constant.

Substituting the I.C. into the general solution $\Longrightarrow$
$y_0 = F(x_0) = G(x_0) + C$ or $C = y_0 - G(x_0)$. So, the particular solution to I.V.P. (1) is given by

$$F(x) = F_p(x) = G(x) + y_0 - G(x_0).$$

Example 8: (求解 I.V.P.)

Consider $F'(x) = \frac{dF(x)}{dx} = e^x$. — (*)

(a) Find the general solution to (*).

(b) Find a particular solution to (*) satisfying the I.C. $F(0) = 3$.

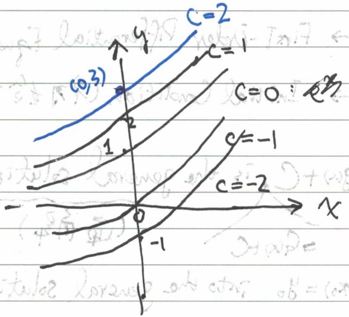
Sol: (a) The general solution to (*) is

$$F(x) = \int F'(x)\,dx = \int e^x\,dx = e^x + C$$

(b) With $F(0)=3$, we obtain $3 = F(0) = e^0 + C = 1 + C$

$\Rightarrow C = 2$. So, the particular solution to (*) is

given by $F(x) = e^x + 2$ #.



Section 4.2
Area
(面積)

Def (Σ Notation)

The sum of real numbers $a_1, a_2, \ldots, a_n$ is denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

where i is the index (足碼) of the summation and a_i is the ith term (第 i 項) of the sum.

Thm 4.2 (Summation Formulas)

(1) $\sum_{i=1}^{n} c = c + c + \cdots + c = cn$ for any constant c.

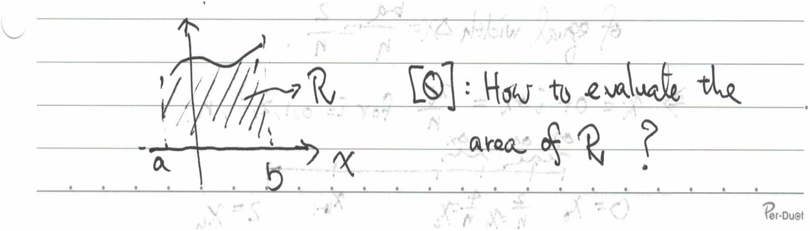
(2) $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

(3) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(4) $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$.

Let $f \geq 0$ be conti. on $[a, b]$, and consider the plane region defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad 0 \leq y \leq f(x)\}.$$



Dividing $[a, b]$ into n subintervals

$$[x_0, x_1], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$$

of equal width $\Delta x = \frac{b-a}{n}$, where $a = x_0 < x_1 < \cdots < x_n = b$.

Dividing $[a, b]$ into n subintervals

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of equal width $\Delta x = \frac{b-a}{n}$, where $a = x_0 < x_1 < \cdots < x_n = b$.

For each $1 \leq i \leq n$, since f is conti. on each $[x_{i-1}, x_i]$, it follows from E.V.T. that $\exists\, m_i, M_i \in [x_{i-1}, x_i]$ s.t.

$$f(m_i) \leq f(x) \leq f(M_i) \quad \forall\, x \in [x_{i-1}, x_i].$$

Def (下和與上和的定義)

(1) Lower sum (下和): $s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$.

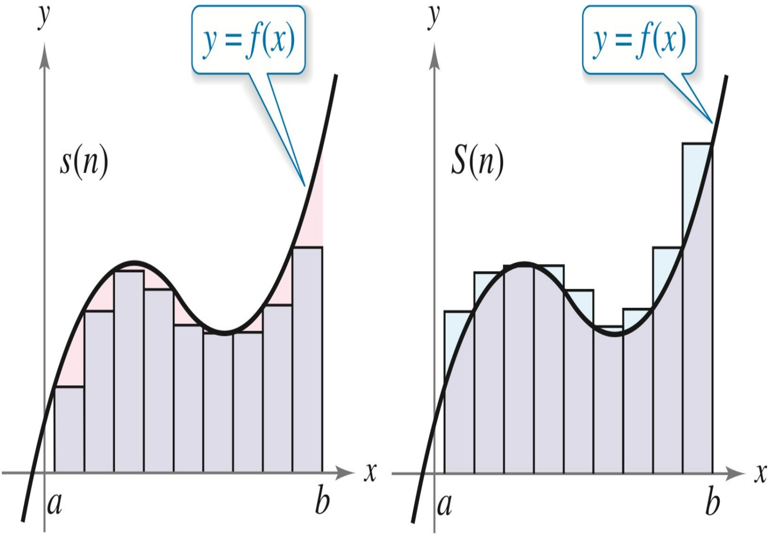
(2) Upper sum (上和): $S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$.

Remark

From the definitions of $s(n)$ and $S(n)$, we see that

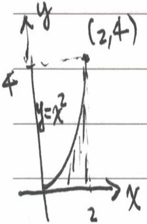
$$s(n) \leq \text{area}(\mathcal{R}) \leq S(n) \quad \forall\, n \in \mathbb{N}.$$

示意圖 (承上頁)



Example 4: (求上和與下和)

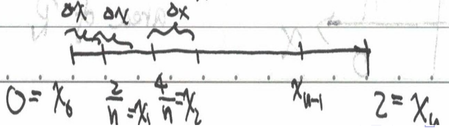
Find the upper and lower sums ~~for~~ of $y = f(x) = x^2$ on $[0, 2]$.



Sol: Let $R_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2$ and $0 \le y \le x^2\}$. be the region bounded by the graph of $y = f(x) = x^2$ and the x-axis between $x=0$ and $x=2$.

Dividing $[0,2]$ into n subintervals $[x_{i-1}, x_i]$ $(i=1,2,\cdots,n)$ of equal width $\Delta x = \frac{b-a}{n} = \frac{2}{n}$.

$\Rightarrow x_i = 0 + i\Delta x = \frac{2i}{n}$ for $i = 0,1,2,\cdots,n$.



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$\because$ f is conti. and $\nearrow$ on each $[x_{i-1}, x_i]$.

$\therefore$ $f(m_i) = f(x_{i-1}) = \left[\frac{2(i-1)}{n}\right]^2$ and $f(M_i) = f(x_i) = \left(\frac{2i}{n}\right)^2$.

for $i = 1, 2, \cdots, n$.

So, the lower ~~limit~~ sum is

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \sum_{i=1}^{n} \frac{4(i-1)^2}{n^2} \cdot \frac{2}{n} = \frac{8}{n^3}\sum_{i=1}^{n}(i^2 - 2i + 1)$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\cdot\frac{n(n+1)}{2} + n\right]$$

$$= \frac{4}{3n^3}(2n^3 - 3n^2 + n) = \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.$$

Similarly, the upper sum is given by

$$S(n) = \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right) = \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2\right)$$

$$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{4}{3n^3}\left(2n^3 + 3n^2 + n\right)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} \quad \#$$

Thm 4.3 (下和與上和的極限值)

If $f \geq 0$ is conti. on $[a, b]$, then

$$\lim_{n\to\infty} s(n) = A = \lim_{n\to\infty} S(n) \quad \exists.$$

In this case, we know that $\text{area}(\mathcal{R}) = A \quad \exists$ by the Squeeze Thm.

Def (非負連續函數與 x-軸所夾面積)

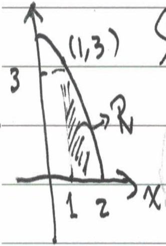
If $f \geq 0$ is conti. on $[a, b]$, then the area of the region $\mathcal{R}$ bounded by the graph of f, the x-axis, $x = a$ and $x = b$ is

$$A = \text{area}(\mathcal{R}) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$ and $\Delta x = \frac{b-a}{n}$.

Example 6: (求曲線與 x 軸所圍成の面積)

Find the area of the region R bounded by the graph of $y = f(x) = 4 - x^2$, the x-axis, $x = 1$ and $x = 2$.



Sol: Let $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ and let $c_i = x_i = 1 + i\Delta x$

$= 1 + \frac{i}{n}$ for $i = 1, 2, \cdots, n$.

$$\Rightarrow f(c_i) = 4 - \left(1 + \frac{i}{n}\right)^2 = 3 - \frac{2i}{n} - \frac{i^2}{n^2}.$$

$$\Rightarrow S(n) = \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right)\cdot\left(\frac{1}{n}\right).$$

$$= \frac{1}{n}\sum_{i=1}^{n} 3 - \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2. = 3 - \frac{2}{n^2}\,\frac{n(n+1)}{2} - \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

$$= 3 - \left(1+\frac{1}{n}\right) - \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

So, the area is $A = \text{area}(R) = \lim_{n\to\infty} S(n) = 3 - 1 - \left(\frac{1}{6}\right)(2) = 2 - \frac{1}{3} = \frac{5}{3}$ #

Section 4.3
Riemann Sums and Definite Integrals
(黎曼和與定積分)

Def (黎曼和的定義)

Let f be defined on a closed interval $I = [a, b]$.

(1) The set $\Delta = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ is called a partition (分割) of I if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

(2) The width of the ith subinterval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1}$ for each $i = 1, 2, \ldots, n$.

(3) The norm (範數) of a partition Δ is defined by $\|\Delta\| = \max_{1 \le i \le n} \Delta x_i$.

(4) If $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann sum (黎曼和) of f for the partition Δ.

Remarks

(1) For a general partition Δ of $[a, b]$, we see that

$$\frac{b-a}{n} \leq \|\Delta\| \Longrightarrow \frac{b-a}{\|\Delta\|} \leq n.$$

So, $\|\Delta\| \to 0 \Rightarrow n \to \infty$, but $n \to \infty \not\Rightarrow \|\Delta\| \to 0$!

(2) If $\Delta x_i = \frac{b-a}{n} \quad \forall\, i$, then $\|\Delta\| \to 0 \Longleftrightarrow n \to \infty$.

Def (定積分的定義)

Let f be defined on a closed interval $I = [a, b]$.

(1) If the limit $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ $\exists$ for any partition Δ of I, then f is integrable (可積分的) on I. In this case, the limit

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called the definite integral (定積分) of f from a to b.

(2) The number a is called the lower limit of integration (積分下限) and b is called the upper limit of integration (積分上限).

Remarks

(1) $\int f(x)\,dx = F(x) + C$ denotes a family of functions, but $\int_a^b f(x)\,dx$ is a real number.

(2) In general, the following notations

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(w)\,dw = \cdots$$

denote the same definite integral of f from a to b.

Thm 4.4 (定積分的存在性)

If f is conti. on a closed interval $I = [a, b]$, then f is integrable on I, i.e., the definite integral of f from a to b

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \quad \exists$$

for any partition Δ of the interval I.

General Version of Thm 4.4

f has at most finitely many discontinuities on $[a, b] \Longrightarrow f$ is (Riemann) integrable on $[a, b]$.

(函數 f 在 $[a, b]$ 上最多僅有限個不連續點 $\Longrightarrow f$ 在 $[a, b]$ 上必定是一個黎曼可積的函數!)

Example 2：(利用 Thm 4.4 求積分)

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$

Sol: $\because$ $f(x)=2x$ is conti. on $[-2,1]$

$\therefore$ f is integrable on $[-2,1]$ by Thm 4.4.

Take $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $c_i = -2 + i(\Delta x) = -2 + \frac{3i}{n}$ for $i=1,2,\cdots,n$,

$$\Rightarrow \int_{-2}^{1} 2x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)\left(\frac{3}{n}\right).$$

$$= \lim_{n\to\infty}\left[\frac{6}{n}\left(\sum_{i=1}^{n} -2 + \frac{3}{n}\sum_{i=1}^{n} i\right)\right]$$

$$= \lim_{n\to\infty} \frac{6}{n}\left(-2n + \frac{3}{n}\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(-12+9+\frac{9}{n}\right) = -3$$

Thm 4.5 (非負連續函數與 x-軸所圍的區域面積)

If $f \geq 0$ is conti. on $[a, b]$, then the area of the region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$$

is given by $A = \text{area}(\mathcal{R}) = \int_a^b f(x)\,dx \geq 0$.

Def (Two Special Definite Integrals)

(1) $\int_a^a f(x)\,dx = 0$ for any $a \in \mathbb{R}$.

(2) If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

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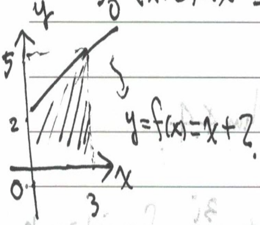
Example 4 :

(a) $\int_{-\pi}^{\pi} \sin x \, dx = 0$

(b) Since $f(x) = x+2 > 0$ is conti. on $[0,3]$, we see that

$$\int_0^3 (x+2)dx = 梯形面積 = \frac{(2+5)\times 3}{2} = \frac{21}{2}.$$



$$\Rightarrow \int_3^0 (x+2)dx = -\int_0^3 (x+2)dx = \frac{-21}{2}.$$

※

Thm (定積分的性質)

Suppose that f and g are integrable on $[a, b]$, and let $k \in \mathbb{R}$.

(1) $$\int_a^b [k \cdot f(x)]\, dx = k \cdot \left(\int_a^b f(x)\, dx \right).$$

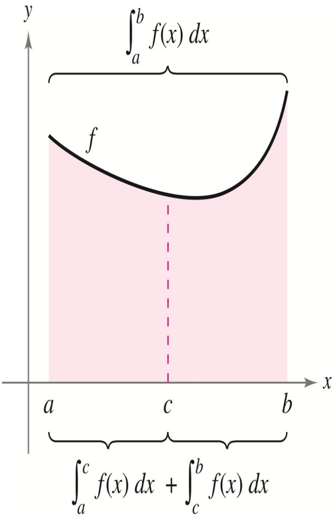
(2) $$\int_a^b [f(x) \pm g(x)]\, dx = \left(\int_a^b f(x)\, dx \right) \pm \left(\int_a^b g(x)\, dx \right).$$

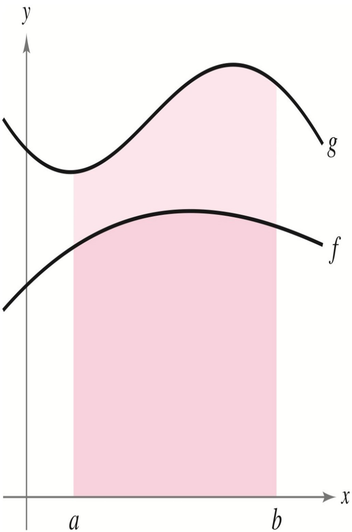
(3) Additivity (可加性): $\displaystyle\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ for $a < c < b$.

(4) Preservation of Inequality (不等號的維持):

$$f(x) \leq g(x) \quad \forall\, x \in [a, b] \Longrightarrow \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx.$$

示意圖 (承上頁)





Note

In general, we know that

$$\int_a^b [f(x)g(x)]\,dx \neq \Big(\int_a^b f(x)\,dx\Big)\Big(\int_a^b g(x)\,dx\Big) \quad \text{and}$$

$$\int_a^b \frac{f(x)}{g(x)}\,dx \neq \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}.$$

Example 5: $\int_{-1}^{1}|x|dx = \int_{-1}^{0}|x|dx + \int_{0}^{1}|x|dx$

$= \int_{-1}^{0}(-x)dx + \int_{0}^{1}x\,dx = \frac{1}{2} + \frac{1}{2} = 1.$ #

Example 6:

Given $\int_1^3 x^2 dx = \frac{26}{3}$, $\int_1^3 x dx = 4$, $\int_1^3 dx = 2$,

evaluate $\int_1^3 (-x^2 + 4x - 3) dx$

Sol: $\int_1^3 (-x^2 + 4x - 3) dx$

$= -\int_1^3 x^2 dx + 4\int_1^3 x dx - 3\int_1^3 dx$

$= \frac{-26}{3} + 4(4) - 3(2) = \frac{-26}{3} + 10 = \frac{4}{3}$. ※

Section 4.4
The Fundamental Theorem of Calculus
(微積分基本定理; F.T.C.)

Thm 4.9 (The First F.T.C.)

If f is conti. on $I = [a, b]$ and $F'(x) = f(x) \quad \forall x \in I$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \equiv F(x)\Big|_a^b \equiv F(x)\Big]_a^b.$$

Note: the definite integral of f from a to b can be evaluated by the function values of antiderivative $F(a)$ and $F(b)$ directly!

Example 2 of Section 4.3 (Revisited)

Applying the First F.T.C. (Thm 4.9) directly, it is easily seen that

$$\int_{-2}^{1} 2x\,dx = x^2\Big|_{-2}^{1} = 1^2 - (-2)^2 = 1 - 4 = -3,$$

since $F(x) = x^2$ is an antiderivative of the integrand $f(x) = 2x$.

Example 2 : Find $\int_0^2 |2x-1|\,dx$.

Sol: Note that $f(x) = |2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x, & \text{if } x < \frac{1}{2}. \end{cases}$

So, $\int_0^2 |2x-1|\,dx = \int_0^{1/2} (1-2x)\,dx + \int_{1/2}^2 (2x-1)\,dx$

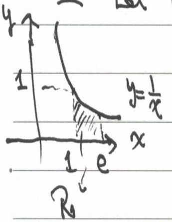
$= (x - x^2)\Big|_0^{1/2} + (x^2 - x)\Big|_{1/2}^2$

$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left[2 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] = \frac{1}{4} + 2 + \frac{1}{4} = \frac{5}{2}.$ ※

Example 3: (利用 F.T.C. 求面積)

Find the area of the region bounded by the graph of $y = f(x) = \frac{1}{x}$, the x-axis, $x=1$ and $x=e$.

Sol: Let $R_1 = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le e,\ 0 \le y \le f(x) = \frac{1}{x}\}$.



$$\Rightarrow A = \text{area}(R_1) = \int_1^e \frac{1}{x}\,dx$$

$$= \ln|x|\Big|_1^e = \ln e - \ln 1 = 1 - 0 = \underline{1}.$$

Thm 4.10 (M.V.T. for Integrals; 積分的均值定理)

If f is conti. on $I = [a, b]$, then $\exists\, c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or} \quad f(c)\cdot(b-a) = \int_a^b f(x)\,dx.$$

Proof of Thm 4.10

Since f is conti. on $I = [a, b]$, it follows that $\int_a^b f(x)\,dx\ \exists$ and $\exists\, m, M \in I$ s.t. $f(m) \le f(x) \le f(M) \quad \forall\, x \in I$. We thus obtain

$$f(m)(b-a) = \int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx = f(M)(b-a)$$

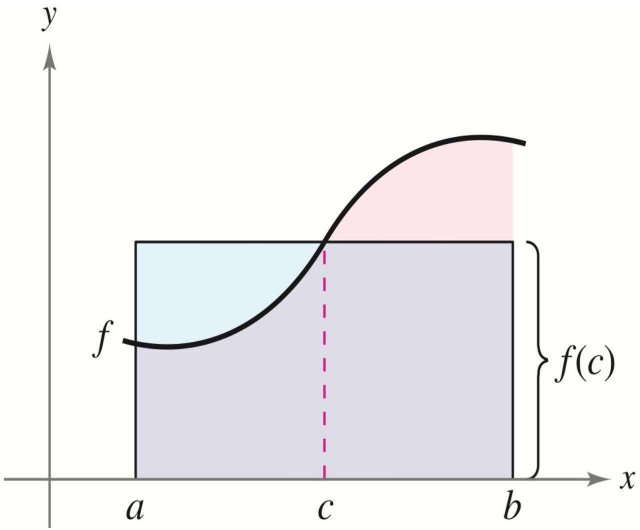
or, equivalently, we see that

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M).$$

Now, from I.V.T., $\exists\, c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \equiv f_{av}.$$

示意圖 (承上頁)



Def (函數在閉區間上的平均值)

If f is conti. on $I = [a, b]$, then the average value of f on I is given by

$$f_{av} \equiv \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example 4: (求 f_{av} 值)

Find the average value of $f(x) = 3x^2 - 2x$ on $I = [1, 4]$.

Sol:

$$f_{av} = \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx = \frac{1}{3}(x^3 - x^2)\Big|_1^4$$

$$= \frac{1}{3}\left[(64-16)-(1-1)\right] = \frac{48}{3} = 16 \quad \#$$

The Definite Integral as a Function

If f is conti. on an open interval I containing a, define a real-valued function by

$$F(x) = \int_a^x f(t)\,dt \qquad \forall\, x \in I.$$

Some Questions

- Is F differentiable on the open interval I?
- If yes, what is the derivative of F?
- Furthermore, is F an antiderivative of f on I?

Thm 4.11 (The Second F.T.C.)

If f is conti. on an open interval I containing a, and define a real-valued function by

$$F(x) = \int_a^x f(t)\,dt \qquad \forall\, x \in I,$$

then F is diff. on I with the derivative

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \forall\, x \in I,$$

i.e. F is an antiderivative of f on the open interval I.

For any $x \in I$, we shall prove that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = f(x).$$

By M.V.T., $\exists\, c_1 \in [x, x+h]$ s.t. $f(c_1) = \frac{1}{h}\int_x^{x+h} f(t)\,dt$ for $h > 0$, and $\exists\, c_2 \in [x+h, x]$ s.t. $f(c_2) = \frac{1}{-h}\int_{x+h}^{x} f(t)\,dt$ for $h < 0$. Thus,

$$\lim_{h\to 0+} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0+} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0+} f(c_1) = f(x)$$

and

$$\lim_{h\to 0-} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0-} \frac{\int_{x+h}^{x} f(t)\,dt}{-h} = \lim_{h\to 0-} f(c_2) = f(x),$$

since f is conti. on I. This completes the proof.

Remark (Thm 4.11 的變形版本)

If f is conti. on an open interval I containing a, then it follows from Thm 4.11 that

$$\frac{d}{dx}\left[\int_x^a f(t)\,dt\right] = \frac{d}{dx}\left[-\int_a^x f(t)\,dt\right] = -f(x)$$

for all $x \in I$. (積分上下限顛倒會差了一個負號喔!)

Example 7: (Thm 4.11 的例子)

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1} \quad \forall x \in \mathbb{R}.$$

General Version of Thm 4.11

If f is conti. on an open interval I containing a, and $u(x), v(x)$ are diff. functions of x with range(u), range$(v) \subseteq I$, then

(1) $$\frac{d}{dx}\left[\int_a^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x).$$

(2) $$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x).$$

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Example 8 Find $\frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos t \, dt\right]$.

Sol. (Method 1) $\because \int_{\pi/2}^{x^3} \cos t \, dt = \sin t \Big|_{\pi/2}^{x^3}$

$= \sin(x^3) - 1$

$\therefore \frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos t \, dt\right] = \frac{d}{dx}(\sin x^3 - 1) = 3x^2 \cos x^3$.

(Method 2) If we let $u(x) = x^3$, then from Thm 4.11

$\Rightarrow \frac{d}{dx}\left[\int_{\pi/2}^{u(x)} \cos t \, dt\right] = \cos(u(x)) \cdot u'(x) = 3x^2 \cos x^3$.

#

Section 4.5
Integration by Substitution
(積分代換法)

Main Goal

Applying a substitution $u = g(x)$ to deal with the integral

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du,$$

and its associated definite integrals.

Thm 4.13 (合成函數的反導函數)

Let $g : D \to I$ be a function with range$(g) = I$ being an interval, and let f be conti. on I. If g is diff. on D and $F'(x) = f(x) \quad \forall\, x \in I$, then

$$\int f(g(x))g'(x)\, dx = F(g(x)) + C,$$

where C is a constant of integration.

Remark (The method of u-substitution; u-代換法)

If we let $u = g(x)$, then $du = g'(x)\,dx$ and it follows from Thm 4.13 that

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C,$$

where C is a constant of integration.

Example 2: Find $\int 5e^{5x}\,dx$.

Sol: Let $u = 5x = g(x)$. Then $du = 5\,dx$.

$$\Rightarrow \int e^{5x} \cdot 5\,dx = \int e^u\,du = e^u + C = e^{5x} + C.$$ ※

Example 5 : Find $\int x\sqrt{2x-1}\,dx$

Sol: Let $u = 2x-1$. Then $du = 2dx \Rightarrow dx = \frac{1}{2}du$, So, $\int x\sqrt{2x-1}\,dx = \int \frac{u+1}{2}\cdot\sqrt{u}\cdot\left(\frac{1}{2}\right)du$

$$= \frac{1}{4}\int \left(u^{3/2}+u^{1/2}\right)du = \frac{1}{4}\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C$$

$$= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C$$

Thm 4.14 (General Power Rule for Integration)

If $u = g(x)$ is a diff. function of x and $n \neq -1$, then

$$\int [g(x)]^n g'(x)\, dx = \int u^n\, du = \frac{u^{n+1}}{n+1} + C = \frac{[g(x)]^{n+1}}{n+1} + C,$$

where C is a constant of integration.

Example 7 : (Thm 4.14 の例子)

(a) $\int 3(3x-1)^4 dx = \int (3x-1)^4 \cdot 3dx = \frac{(3x-1)^{4+1}}{4+1} + C$

$= \frac{(3x-1)^5}{5} + C$. #

(b) $\int (e^x+1)(e^x+x)dx = \int (e^x+x)\cdot(e^x+1)dx$

$= \frac{1}{1+1}(e^x+x)^{1+1} + C = \frac{1}{2}(e^x+x)^2 + C$. #

(c) $\int (3x^2)\sqrt{x^3-2}\, dx = \int (x^3-2)^{1/2}(3x^2)dx = \frac{1}{1+1/2}(x^3-2)^{\frac{1}{2}+1} + C$

$= \frac{2}{3}(x^3-2)^{3/2} + C$ #

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(d) $\int \frac{-4x}{(1-2x^2)^2}dx = \int (1-2x^2)^{-2}(-4x)dx$

$= \frac{1}{-2+1}(1-2x^2)^{-2+1} + C = \frac{-1}{1-2x^2} + C$ ※

(e) $\int \cos^2 x \sin x dx = -\int (\cos x)^2(-\sin x)dx$

$= \frac{-1}{3}(\cos x)^3 + C = \frac{-\cos^3 x}{3} + C$ ※

Thm 4.15 (定積分的 u-代換法)

If $u = g(x)$ has a conti. derivative on $I = [a, b]$ and f is conti. on range(g), then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

(將對 x 的定積分變數變換成對 u 的定積分!)

Example 9: (Thm 4.15 的例子)

Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$.

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Sol: Let $\boxed{u = g(x) = \sqrt{2x-1}}$. Then $du = \frac{2}{2\sqrt{2x-1}}\,dx$

$= \frac{1}{\sqrt{2x-1}}\,dx$, $x = \frac{u^2+1}{2}$ and

x	1	5
$g(x)$	1	3

So, $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx = \int_1^3 \left(\frac{u^2+1}{2}\right) du = \frac{1}{2}\left(\frac{1}{3}u^3 + u\right)\Big|_1^3$

$= \frac{1}{2}\left[(9+3) - \left(\frac{1}{3}+1\right)\right] = \frac{1}{2}\left(12 - \frac{4}{3}\right) = 6 - \frac{2}{3} = \frac{16}{3}$

Thm 4.16 (奇偶函數在 $[-a, a]$ 上的定積分)

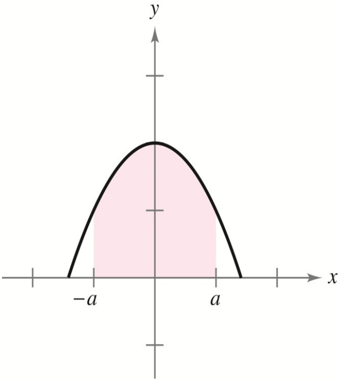
Let f be integrable on the closed interval $I = [-a, a]$ with $a > 0$.

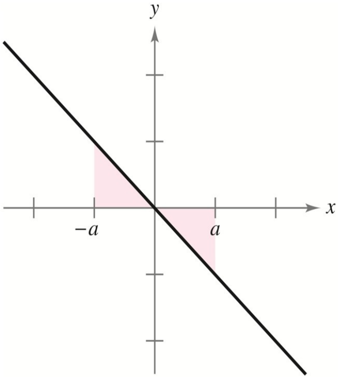
(1) If f is even, i.e., $f(-x) = f(x) \quad \forall x \in I$, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$$

(2) If f is odd, i.e., $f(-x) = -f(x) \quad \forall x \in I$, then $\int_{-a}^{a} f(x)\, dx = 0$.

示意圖 (承上頁)





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pf of (1): Since f is even, $f(-x) = f(x)$ $\forall x \in [-a, a]$.

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{a}^{0} f(-x)(-1)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{a}^{0} f(u)du + \int_{0}^{a} f(x)dx \text{ by Thm 4.15.}$$

$$= \int_{0}^{a} f(u)du + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

#

Example 10: Find $\int_{-\pi/2}^{\pi/2} (\sin^3 x + \sin x)\cos x \, dx$.

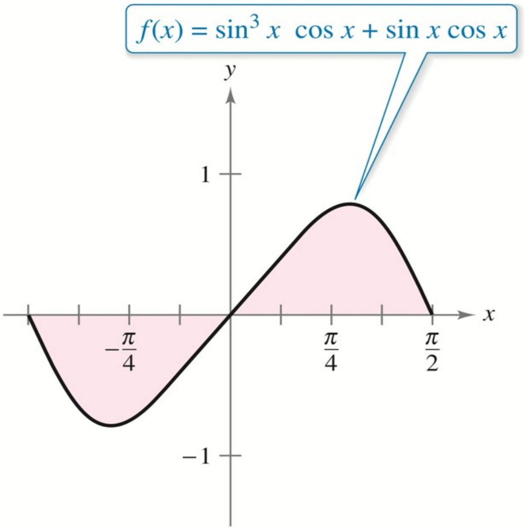
Sol: Let $f(x) = (\sin^3 x + \sin x)\cos x \quad \forall x \in \mathbb{R}$.

Then $f(-x) = [\sin^3(-x) + \sin(-x)]\cos(-x) = -f(x) \quad \forall x \in \mathbb{R}$.

$\Rightarrow f$ is an odd function on $\mathbb{R}$.

By Thm 4.16 $\Rightarrow \int_{-\pi/2}^{\pi/2} f(x)\,dx = 0$. ※

示意圖 (承上例)



Section 4.6
Indeterminate Forms and L'Hôpital's Rule
(不定型與羅必達法則)

Types of Indeterminate Forms

For the limit of a quotient $f(x)/g(x)$ as $x \to c$, we may have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}, \ \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad 1^{\infty}, \ \infty^{0}, \ 0^{0}, \quad \infty - \infty,$$

which are called the indeterminate forms (不定型).

Thm 4.18 (L'Hôpital's Rule; L'H Rule)

Assume that f and g are diff. on $I = (a, c) \cup (c, b)$ and $g'(x) \neq 0 \quad \forall\, x \in I$. If the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty},$$

then we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}. \qquad \text{(分子和分母各別微分喔!)}$$

Note: Thm 4.18 also holds for the one-sided limits. (羅必達法則也適用於求解單邊極限值!)

Remark (與 Thm 1.9 比較)

Thm 1.9 of Section 1.6 can be derived immediately from the L'Hôpital's Rule, since we see that the limit is an indeterminate form of type $\frac{0}{0}$ and hence

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=\lim_{\theta\to 0}\frac{\cos\theta}{1}=\cos(0)=1.$$

Example 1: (Type $\frac{0}{0}$)

Find $\lim_{x\to 0} \frac{e^{2x}-1}{x}$.

Sol: By L'Hôpital's Rule $\Rightarrow \lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2$.

※

Example 2: Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$. (Type $\frac{\infty}{\infty}$)

Sol: $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$ by L'Hôpital's Rule.

※

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Example 3: (使用兩次 L'Hôpital's Rule)

Find $\lim_{x\to -\infty} \frac{x^2}{e^{-x}}$. (Type $\frac{\infty}{\infty}$)

Sol:

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} \overset{L'H}{=} \lim_{x\to -\infty} \frac{2x}{-e^{-x}}. \text{ (Type } \frac{\infty}{\infty}\text{)}$$

$$\overset{L'H}{=} \lim_{x\to -\infty} \frac{2}{e^{-x}} \left(= \cancel{\frac{2}{\infty}}\right) = 0 \text{ by using L'H Rule twice!}$$

Indeterminate Forms of Type $0 \cdot \infty$

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = \pm\infty$, then consider

$$\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} \frac{f(x)}{1/g(x)} \quad (\text{Type } \frac{0}{0})$$

or

$$\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} \frac{g(x)}{1/fx)} \quad (\text{Type } \frac{\infty}{\infty}).$$

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Example 4: Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$. (Type $0\cdot\infty$)

Sol: $\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$ (Type $\frac{\infty}{\infty}$)

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty} \frac{1}{(2\sqrt{x})e^x} = 0$ by L'H Rule.

Indeterminate Forms of Type 1^∞, ∞^0 or 0^0

Assume that $\lim_{x\to c}[f(x)]^{g(x)} = 1^\infty, \infty^0, 0^0$. If we know that

$$\lim_{x\to c} g(x)\ln[f(x)] = L \quad (\text{Type } \frac{0}{0} \text{ or } \frac{\infty}{\infty})$$

using the L'Hôpital's Rule, then

$$\lim_{x\to c}[f(x)]^{g(x)} = \lim_{x\to c} e^{g(x)\ln[f(x)]} = e^L.$$

Example 5: Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. (Type 1^∞)

Sol: Note that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+\frac{1}{x})}$. —(*)

$\because \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x} = \lim_{t\to 0^+}\frac{\ln(1+t)}{t}$ (Type $\frac{0}{0}$)

by letting $t=\frac{1}{x}$.

$\therefore \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{t\to 0^+}\frac{\ln(1+t)}{t} = \lim_{t\to 0^+}\frac{\frac{1}{1+t}}{1} = 1$ —(**)

by L'Hôpital's Rule.

From (*) and (**) $\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$. ✧

Example (補充題)

Evaluate

$$\lim_{x\to\infty} x^{\frac{1}{x}}. \quad (\text{Type } \infty^0)$$

Sol: From Example 2, we see that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

by applying L'Hôpital's Rule. So, we immediately obtain

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^0 = 1.$$

Example 6: Evaluate $\lim_{x\to 0^+} (\sin x)^x$. (Type 0^0).

Sol: Note that $\lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{x\ln(\sin x)}$. —(*)

$\because$ $\lim_{x\to 0^+} x\ln(\sin x) = \lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{-1/x^2} = \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x}$.

$= \lim_{x\to 0^+} \left[\left(\frac{x}{\sin x}\right)(-x)\cos x\right] = (1)(0)(1) = 0$. —(**)

$\therefore$ From (*) and (**) $\Rightarrow \lim_{x\to 0^+} (\sin x)^x = e^0 = \underline{1}$. ※

Indeterminate Forms of Type $\infty - \infty$

The original limit will become an indeterminate form of type $\dfrac{0}{0}$, if we apply the technique of reduction to common denominator.

(使用通分技巧將原極限問題變成標準不定型!)

Example 7 (使用兩次羅必達法則)

Evaluate the limit

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right). \quad (\text{Type } \infty-\infty)$$

Solution of Example 7

Applying the L'Hôpital's Rule twice, we see that

$$\begin{aligned}
\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) &= \lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x} \quad (\text{Type } \tfrac{0}{0}; \text{通分!}) \\
&= \lim_{x\to 1^+}\frac{1-1/x}{\ln x+(x-1)(1/x)} \quad (\text{使用 L'H Rule!}) \\
&= \lim_{x\to 1^+}\left(\frac{1-1/x}{\ln x+(x-1)(1/x)}\cdot\frac{x}{x}\right) \\
&= \lim_{x\to 1^+}\frac{x-1}{x\ln x+x-1} \quad (\text{Type } \tfrac{0}{0}) \\
&= \lim_{x\to 1^+}\frac{1}{\ln x+x(1/x)+1} \quad (\text{再次使用 L'H Rule!}) \\
&= \frac{1}{0+1+1}=\frac{1}{2}. \quad (\text{直接代入求極限喔!})
\end{aligned}$$

Section 4.7
The Natural Logarithmic Function: Integration
(自然對數函數的積分)

Thm 4.19 (Log Rule for Integration)

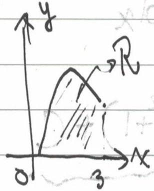
Let $u = u(x)$ be a diff. function of x. Then

(1) $$\int \frac{1}{x}\,dx = \ln|x| + C,$$

(2) $$\int \frac{u'(x)}{u(x)}\,dx = \int \frac{1}{u}\,du = \ln|u(x)| + C,$$

where C is a constant of integration.

Example 3: Find the area of the region R bounded by the graph of $f(x) = \frac{x}{x^2+1}$, the x-axis and $x = 3$.



Sol: $\because f(x) = \frac{x}{x^2+1} \geq 0$ is conti. on $[0,3]$.

$\therefore A = \text{area}(R) = \int_0^3 \frac{x}{x^2+1}\,dx$

Let $u = x^2+1$. Then $du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$.

So, $A = \int_0^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^{10} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_1^{10} = \frac{\ln 10}{2} \doteqdot 1.151$ #

Example 4:

Example 4: Use Log Rule to find the integral.

(a) $\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x^3+x| + C$, if we let $u = x^3+x$

(b) $\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C$ if we let $u = \tan x$.

(c) $\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x}\,dx = \frac{1}{2}\ln|x^2+2x| + C$

(d) $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{3}{3x+2}\,dx = \frac{1}{3}\ln|3x+2| + C$

Example 5: (長除法の例子)

Find $\int \frac{x^2+x+1}{x^2+1}\,dx$.

Sol: By Long Division $\Rightarrow$ $f(x) = \frac{x^2+x+1}{x^2+1} = 1 + \frac{x}{x^2+1}$.

So, $\int f(x)\,dx = \int \left(1+\frac{x}{x^2+1}\right) dx$

$= \int 1\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = x + \frac{1}{2}\ln(x^2+1) + C$

Example 6 Find $\int \frac{2x}{(x+1)^2}\,dx$ using $u = x+1$.

Sol: Let $u = x+1$. Then $du = dx$ and $x = u-1$.

So, $\int \frac{2x}{(x+1)^2}\,dx = \int \frac{2(u-1)}{u^2}\,du = 2\int \left(\frac{1}{u} - \frac{1}{u^2}\right)du$

$= 2\ln|u| + 2u^{-1} + C = 2\ln|x+1| + \frac{2}{x+1} + C$ ※

Per-Duet

Example 8: Find $\int \tan x \, dx$.

Sol: $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx$

$= -\ln|\cos x| + C = \ln|\sec x| + C$ ※

Example 9: Find $\int \sec x \, dx$.

Sol: $\int \sec x \, dx = \int \left(\sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$

$= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \, dx = \ln|\sec x + \tan x| + C$

if we let $u = \sec x + \tan x$. ※

Thm (三角函數的積分公式)

Let $u = u(x)$ be a diff. function of x. Then

(1) $\displaystyle\int \sin u\, du = -\cos u + C$,

(2) $\displaystyle\int \cos u\, du = \sin u + C$,

(3) $\displaystyle\int \tan u\, du = -\ln|\cos u| + C$,

(4) $\displaystyle\int \cot u\, du = \ln|\sin u| + C$,

(5) $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$,

(6) $\displaystyle\int \csc u\, du = -\ln|\csc u + \cot u| + C$,

where C is a constant of integration.

Section 4.8
Inverse Trigonometric Functions: Integration
(反三角函數的積分)

Thm 4.20 (Integrals Involving Inverse Trig. Functions)

Let $u = u(x)$ be a diff. function of x and $a > 0$. Then

(1) $$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin(\frac{u}{a}) + C,$$

(2) $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan(\frac{u}{a}) + C,$$

(3) $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\text{arcsec}(\frac{|u|}{a}) + C,$$

where C is a constant.

Example 2: Find $\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Sol: Let $u = e^x$. Then $du = e^x dx = u\,dx \Rightarrow dx = \frac{1}{u} du$.

Hence, $\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{\frac{1}{u} du}{\sqrt{u^2-1}} = \int \frac{du}{u\sqrt{u^2-1}}$

$= \sec^{-1}|u| + C = \text{arcsec}(e^x) + C$. #

Example 3: Find $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Sol: $\int \frac{x+2}{\sqrt{4-x^2}}\,dx = \int \frac{x}{\sqrt{4-x^2}}\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= \frac{-1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx + 2\sin^{-1}\left(\frac{x}{2}\right) + C = -\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right) + C$. #

Per-Duet

Note (配出完全平方項)

For the integrals involving $x^2 + bx + c$, we shall compete the square of the quadratic term as

$$x^2 + bx + c = \left[x^2 + bx + \left(\frac{b}{2}\right)^2\right] + c - \left(\frac{b}{2}\right)^2$$
$$= \left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2.$$

Example 4 Find $\int \frac{1}{x^2-4x+7}\,dx$.

Sol: $\because f(x) = \frac{1}{x^2-4x+7} = \frac{1}{(x-2)^2+3}$.

$\therefore \int f(x)\,dx = \int \frac{dx}{(x-2)^2+3} = \int \frac{du}{u^2+(\sqrt{3})^2}$ if we let $u = x-2$.

$= \frac{1}{\sqrt{3}} \tan^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{\sqrt{3}} \arctan\left(\frac{x-2}{\sqrt{3}}\right) + C$ #

Example 5: Find the area of the region R_1 bounded by the graph of $f(x) = \frac{1}{\sqrt{3x - x^2}}$, the x-axis, and $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Sol: $A = \text{area}(R_1) = \int_{3/2}^{9/4} \frac{dx}{\sqrt{3x-x^2}} = \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - (x-3/2)^2}}$

$= \sin^{-1} \frac{x-(3/2)}{3/2} \Big|_{3/2}^{9/4} = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \doteq 0.524$. #

$(u = x - 3/2)$

Note

In Example 5, completing the square for $3x - x^2$, we obtain

$$3x - x^2 = -(x^2 - 3x) = -\left[x^2 - 3x + \left(\frac{3}{2}\right)^2\right] + \left(\frac{3}{2}\right)^2$$
$$= \left(\frac{3}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2.$$

Thank you for your attention!