Chapter 5 Applications of Integration (積分的應用)

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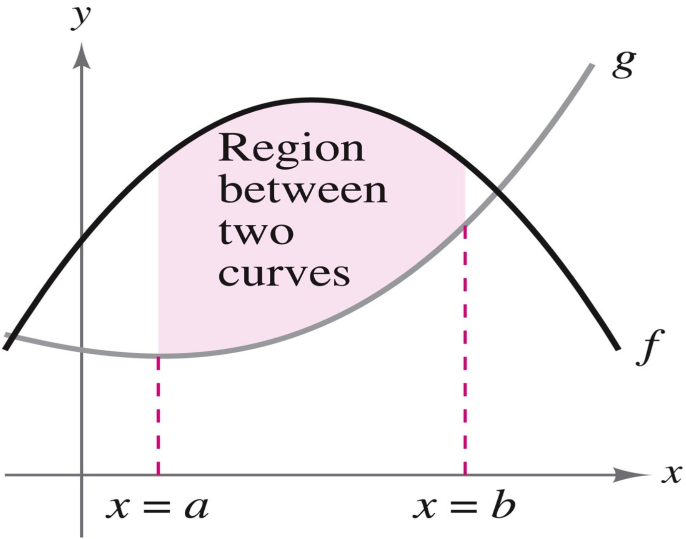
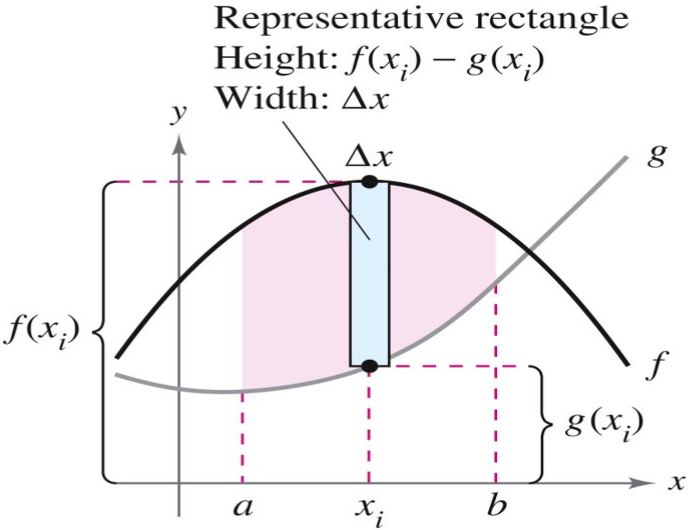
本章預定授課範圍

Section 5.1
Area of a Region Between Two Curves
(兩曲線所夾區域的面積)

Type I 的示意圖 (1/2)



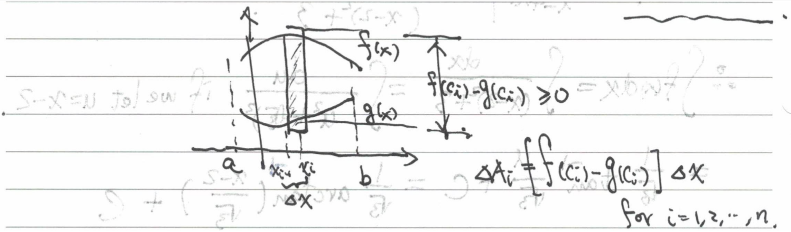
Type I 的示意圖 (2/2)



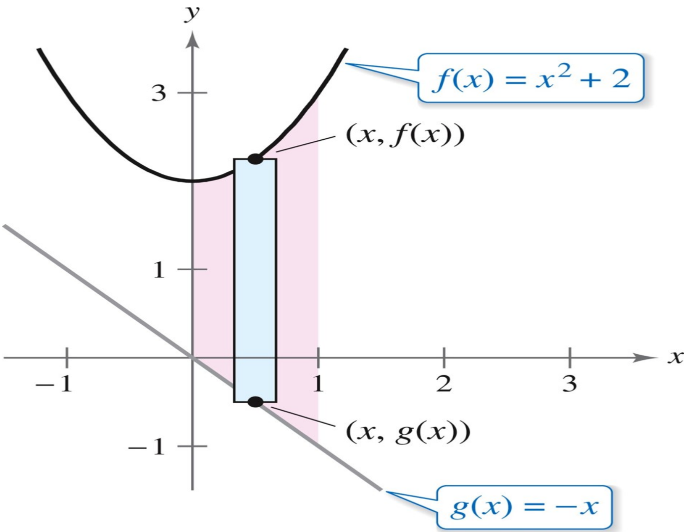
Type I (第一型面積公式)

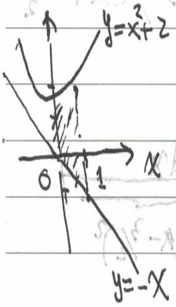
If f and g are conti. on $I = [a, b]$ with $g(x) \leq f(x) \quad \forall x \in I$, then the area of $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is given by

$$A = \text{area}(\mathcal{R}) = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\Delta x = \int_a^b [f(x) - g(x)]\, dx \geq 0.$$



Example 1: Find the area of the region R bounded by the graphs of $y = x^2 + 2$, $y = -x$, $x = 0$ and $x = 1$.


y
3
1
−1
x
−1
1
2
3
$f(x) = x^2 + 2$
$(x, f(x))$
$(x, g(x))$
$g(x) = -x$



Sol: Let $g(x) = -x$ and $f(x) = x^2 + 2$ $\forall x \in [0,1]$.

$\Rightarrow$ $g(x) \le f(x)$ $\forall x \in [0,1]$ and f and g are conti. on $[0,1]$.

$\Rightarrow$ $A = \text{area}(R_1) = \int_0^1 [(x^2+2) - (-x)]\, dx$

$= \int_0^1 (x^2 + x + 2)\, dx = \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\right)\Big|_0^1 = \frac{17}{6}$

#

Remark (兩曲線所圍成的區域面積)

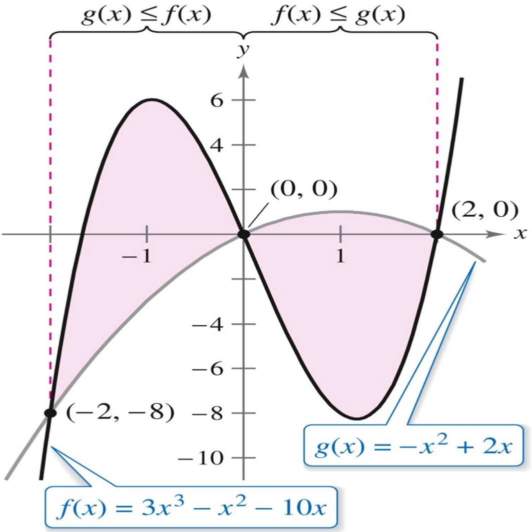
If f and g intersect (相交) at the points $c_1, c_2, \ldots, c_k$ for some $k \in \mathbb{N}$, then the area of the region between curves $y = f(x)$ and $y = g(x)$ is given by

$$\int_{c_1}^{c_2} |f(x) - g(x)|\, dx + \int_{c_2}^{c_3} |f(x) - g(x)|\, dx + \cdots + \int_{c_{k-1}}^{c_k} |f(x) - g(x)|\, dx.$$

Example 4: (兩函數曲線有交點的情形)

Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.

Example 4 的示意圖 (承上頁)



Sol: Solve $f(x)=g(x) \Leftrightarrow 3x^3-x^2-10x=-x^2+2x$.

$\Leftrightarrow 3x^3-12x=3x(x^2-4)=3x(x-2)(x+2)=0$.

$\Leftrightarrow x=-2,0,2$.

$\because g(x)\le f(x)\ \forall x\in[-2,0]$ and $f(x)\le g(x)\ \forall x\in[0,2]$.

$$\therefore A=\text{area}(R)=\int_{-2}^{0}[f(x)-g(x)]\,dx+\int_0^2[g(x)-f(x)]\,dx.$$

$$=\int_{-2}^{0}(3x^3-12x)\,dx+\int_0^2(12x-3x^3)\,dx.$$

$$=\left(\frac{3}{4}x^4-6x^2\right)\Big|_{-2}^{0}+\left(6x^2-\frac{3}{4}x^4\right)\Big|_0^2=\cdots=24.\ \#$$

Type II (第二型面積公式)

If f and g are conti. on $I = [c, d]$ with $g(y) \leq f(y) \quad \forall\, y \in I$, then the area of the region between the graphs of $x = f(y)$ and $x = g(y)$

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, g(y) \leq x \leq f(y)\}$$

is given by

$$A = \text{area}(\mathcal{R}) = \int_c^d [f(y) - g(y)]\, dy \geq 0.$$

Example 5 : (Type II の例子)

Find the area of the region between the graphs of

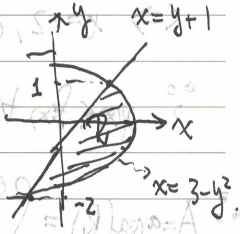
$x = f(y) = 3 - y^2$ and $x = g(y) = y + 1$.

Sol: Need to solve $f(y)=g(y) \iff 3-y^2=y+1$

$\iff y^2+y-2=(y+2)(y-1)=0 \iff y=-2, 1.$

$\because$ $f(y)$ and $g(y)$ are conti. on $[-2,1]$ and

$f(y)\geq g(y)$ $\forall y\in[-2,1]$.



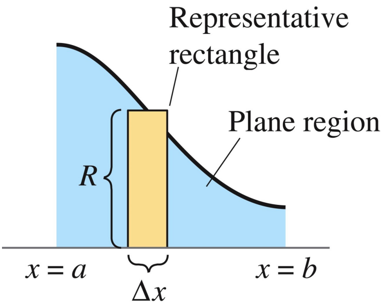
$\therefore A=\text{area}(R)=\int_{-2}^{1}[f(y)-g(y)]\,dy$

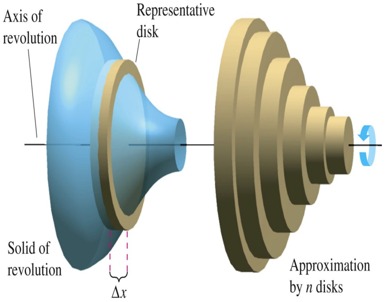
$=\int_{-2}^{1}(3-y^2-y-1)\,dy=\int_{-2}^{1}(2-y^2-y)\,dy$

$=\left(2y-\frac{1}{3}y^3-\frac{1}{2}y^2\right)\Big|_{-2}^{1}=\cdots=\frac{9}{2}.$ #

Section 5.2
Volume: The Disk Method
(體積: 圓盤法)

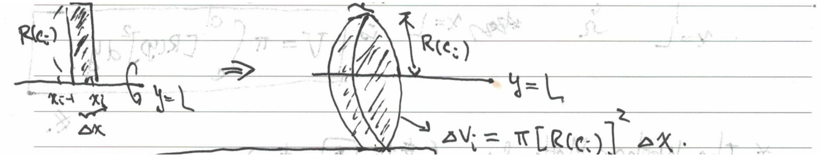
A Solid of Revolution (旋轉體)





The Volume of a Representative Disk

- Given a partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with equal width $\Delta x = \frac{b-a}{n}$, where $x_0 = a$ and $x_n = b$.
- For each $i = 1, 2, \ldots, n$, choose $c_i \in [x_{i-1}, x_i]$, the volume of a disk of width Δx is $\Delta V_i = \pi[R(c_i)]^2 \Delta x$.



Type I: Horizontal Axis of Revolution (水平旋轉軸)

Given the plane region Ω and a horizontal axis of revolution $y = L$. If the radius function $R(x) \geq 0$ is conti. on $[a, b]$, then the volume of a solid formed by revolving Ω about $y = L$ is

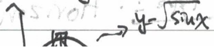
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi [R(c_i)]^2 \Delta x = \int_a^b \pi [R(x)]^2 \, dx \geq 0.$$

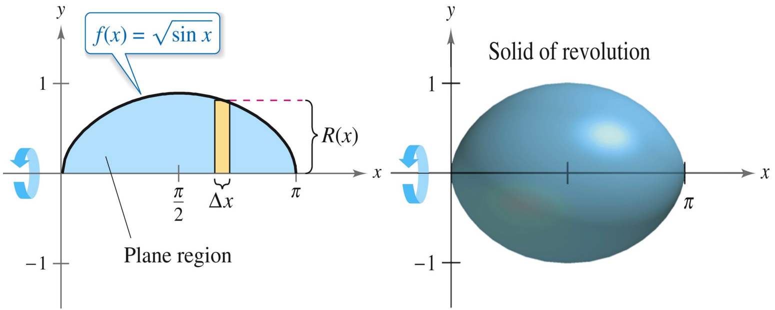
DATE: / /

Example 1: Let Ω be the plane region defined by

$$\Omega = \{ (x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi,\ 0 \le y \le \sqrt{\sin x} \}.$$

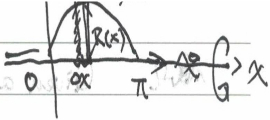
Find the volume of the solid formed by revolving Ω about the x-axis.





y
$f(x) = \sqrt{\sin x}$
1
−1
$R(x)$
x
$\frac{\pi}{2}$
Δx
π
Plane region
y
Solid of revolution
1
−1
x
π

Sol: Since the radius function



$R(x) = \sqrt{\sin x} \geq 0$ is conti. on $[0, \pi]$,

$$V = \pi \int_0^{\pi} [R(x)]^2 \, dx = \pi \int_0^{\pi} \sin x \, dx = \pi(-\cos x) \Big|_0^{\pi}$$

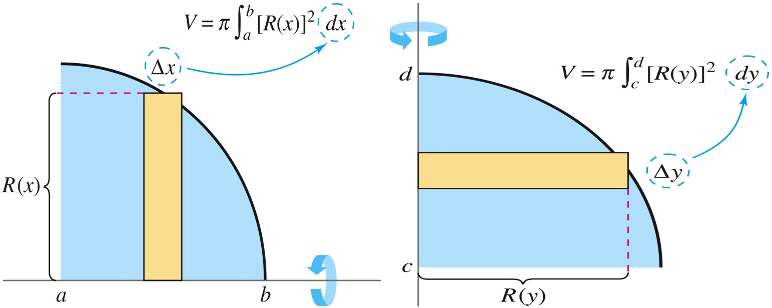
$$= 2\pi. \quad \#$$

Type II: Vertical Axis of Revolution (垂直旋轉軸)

Given the plane region Ω and a vertical axis of revolution $x = L$. If the radius function $R(y) \geq 0$ is conti. on $[c, d]$, then the volume of a solid formed by revolving Ω about $x = L$ is

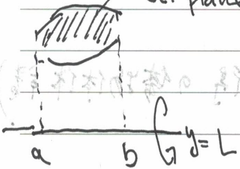
$$V = \int_c^d \pi [R(y)]^2 \, dy \geq 0.$$

示意圖 (承上頁)



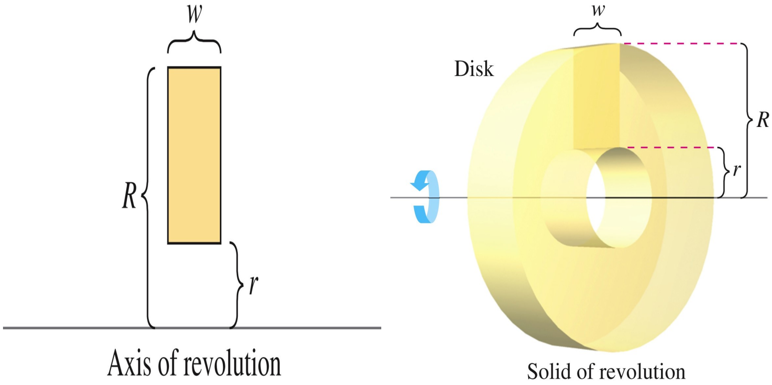
* The Washer Method: (墊圈法)

Ω: plane region



[Q]: What is the volume of a solid formed by revolving Ω about $y=L$?

墊圈 (washer) 的示意圖



The Volume of a Representative Washer

- Given a partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with equal width $\Delta x = \dfrac{b-a}{n}$, where $x_0 = a$ and $x_n = b$.
- For each $i = 1, 2, \ldots, n$, choose $c_i \in [x_{i-1}, x_i]$, the volume of a washer of width Δx is

$$\begin{aligned} \Delta V_i &= \pi[R(c_i)]^2\Delta x - \pi[r(c_i)]^2\Delta x \\ &= \pi\Big([R(c_i)]^2 - [r(c_i)]^2\Big)\Delta x. \end{aligned}$$

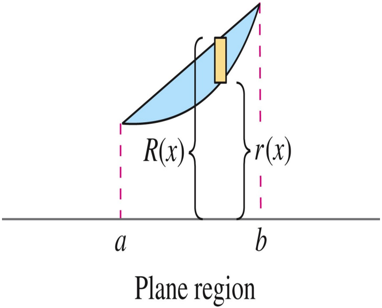
Type I: Horizontal Axis of Revolution $y = L$

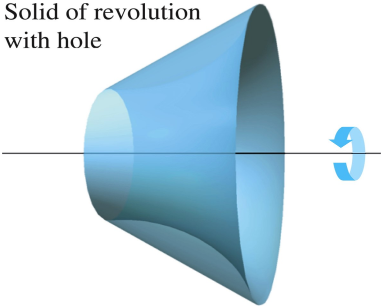
If the outer radius $R(x) \geq 0$ and inner radius $r(x) \geq 0$ are conti. on $[a, b]$, then the volume of a solid formed by revolving Ω about $y = L$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\Big([R(c_i)]^2 - [r(c_i)]^2\Big)\Delta x$$

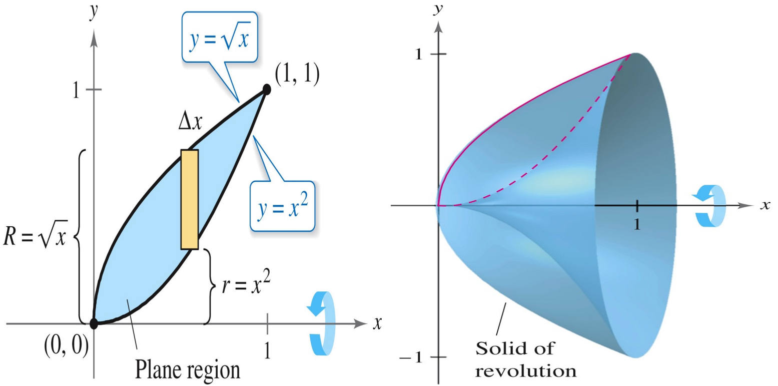
$$= \pi \int_a^b \Big([R(x)]^2 - [r(x)]^2\Big)\, dx.$$

示意圖 (承上頁)

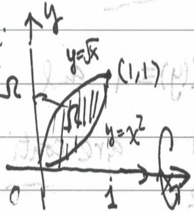




Example 3 : Find the volume of the solid formed by revolving the region Ω bounded by the graphs of $y=\sqrt{x}$ and $y=x^2$ about the x-axis.


y
$y=\sqrt{x}$
(1, 1)
1
Δx
$y=x^2$
$R=\sqrt{x}$
$r=x^2$
x
(0, 0)
1
Plane region
y
1
x
1
−1
Solid of
revolution

Sol.



Sol: Solve $y=\sqrt{x}=x^2$

$\Rightarrow x = x^4 \Rightarrow x(x^3-1)=0.$

$\Rightarrow x=0$ or $x=1$.

Note that $R(x)=\sqrt{x} \geq 0$ and $r(x)=x^2 \geq 0$

are conti. on $[0,1]$. So,

$$V = \pi\int_0^1 \left[(\sqrt{x})^2 - (x^2)^2\right] dx = \pi\int_0^1 (x-x^4)\,dx$$

$$= \pi\left(\frac{1}{2}x^2 - \frac{1}{5}x^5\right)\Big|_0^1 = \frac{3}{10}\pi. \quad \#$$

Type II: Vertical Axis of Revolution $x = L$

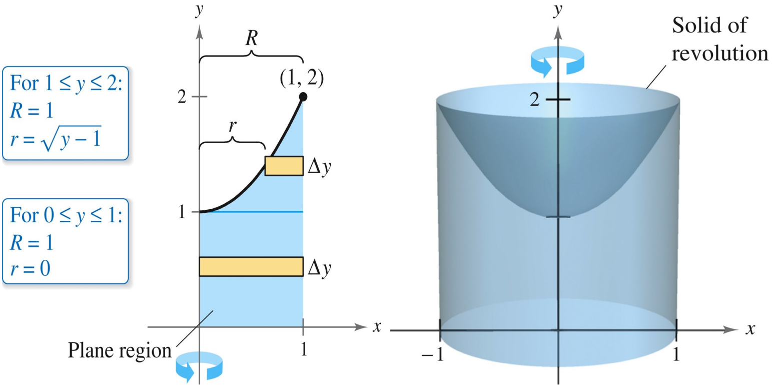
If the outer radius $R(y) \geq 0$ and inner radius $r(y) \geq 0$ are conti. on $[c, d]$, then the volume of a solid formed by revolving Ω about $x = L$ is

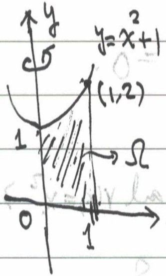
$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2 \right) dy.$$

- **Example 4**: (Type II 的「例子」)

Find the volume of the solid formed by revolving the region Ω bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$, about the y-axis.

Example 4 的示意圖 (承上頁)





Sol: Note that $R(y) \equiv 1$ and

$$r(y)=\begin{cases} 0, & 0 \le y \le 1 \\ \sqrt{y-1}, & 1 \le y \le 2 \end{cases}$$

are conti on $[0,2]$. Then

$$V=\pi\int_0^1 (1^2-0^2)\,dy+\pi\int_1^2 \left[1^2-(\sqrt{y-1})^2\right]dy.$$

$$=\pi+\pi\int_1^2 (2-y)\,dy=\pi+\pi\left(2y-\frac{y^2}{2}\right)\Big|_1^2$$

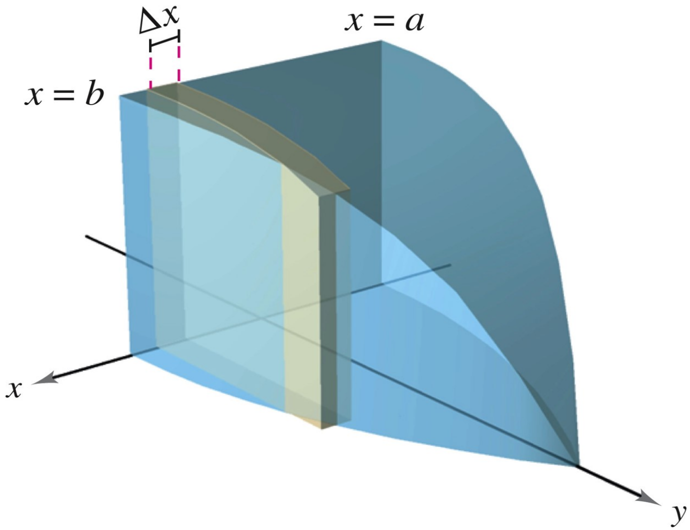
$$=\pi+\pi\left(2-2+\frac{1}{2}\right)=\pi+\frac{\pi}{2}=\frac{3}{2}\pi.$$

#

Def (已知截面積的固體體積)

Let $A(x)$ be the cross-sectional area (截面積) of a solid taken perpendicular to (垂直於) the x-axis at each $x \in [a, b]$. If $A(x)$ is integrable on $[a, b]$, then the volume of the solid is

$$V = \int_a^b A(x)\, dx \geq 0.$$

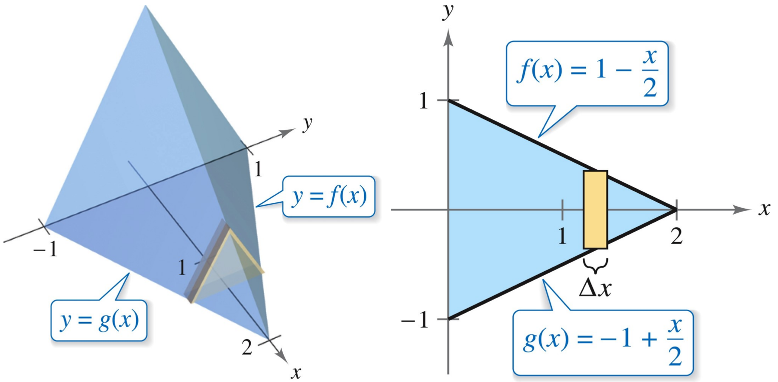


Δx
$x = a$
$x = b$
x
y

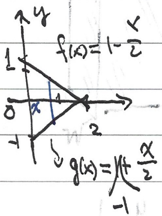
Example 6: The base of the solid is the region bounded by the graphs of $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$ and $x = 0$.

Find the volume of the solid if its cross sections perpendicular to the x-axis are equilateral triangles. (正三角形)

Example 6 的示意圖 (承上頁)



Sol: The graphs of f and g intersect at (2,0),



since $f(x)=g(x) \Leftrightarrow 1-\frac{x}{2}=-1+\frac{x}{2} \Leftrightarrow x=2$.

For each $x\in[0,2]$, the area of the cross section is $A(x)=\frac{\sqrt{3}}{4}(\text{base})^2$

$$=\frac{\sqrt{3}}{4}\left[(f(x)-g(x))\right]^2=\frac{\sqrt{3}}{4}(2-x)^2.$$

So, the volume of the solid is

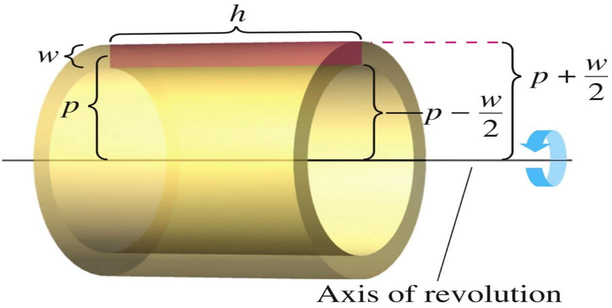
$$V=\int_0^2 A(x)\,dx=\int_0^2\frac{\sqrt{3}}{4}(2-x)^2\,dx=\frac{-\sqrt{3}}{12}(2-x)^3\Big|_0^2$$

$$=0+\frac{\sqrt{3}}{12}\times 8=\frac{2\sqrt{3}}{3}$$

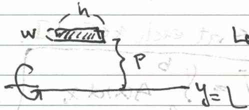
Section 5.3
Volume: The Shell Method
(體積: 殼層法)

The volume of a shell of revolution is given by

$$\Delta V = \pi(p+\frac{w}{2})^2 h - \pi(p-\frac{w}{2})^2 h = 2\pi phw.$$



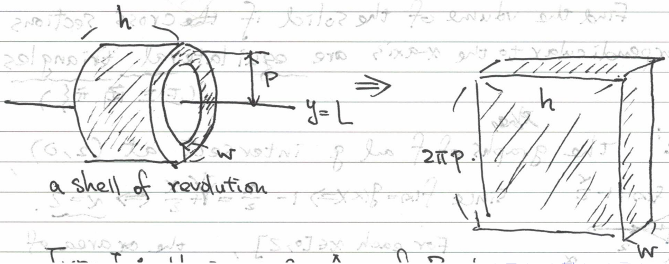
* Observation :



Let Ω be a rectangle in $\mathbb{R}^2$;
$w =$ width of Ω and
$h =$ height of Ω.

$\Rightarrow$ The volume of the shell is

$$\Delta V = \pi\left(p+\frac{w}{2}\right)^2 h - \pi\left(p-\frac{w}{2}\right)^2 h$$
$$= 2\pi p h w.$$



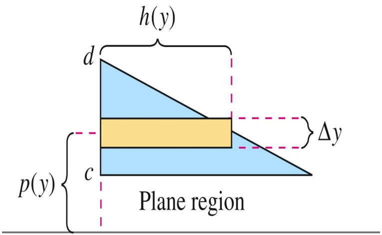
a shell of revolution

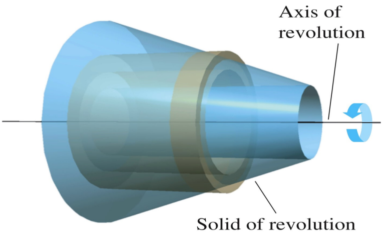
Type I: Horizontal Axis of Revolution $y = L$

If $p(y)$ and $h(y)$ are conti. functions of y on $[c, d]$, then the volume of the solid formed by revolving a plane region Ω about $y = L$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi p(c_i) h(c_i) \Delta y = 2\pi \int_c^d p(y) h(y)\, dy.$$

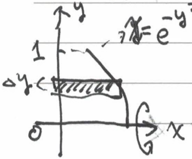
示意圖 (承上頁)


h(y)
d
Δy
c
p(y)
Plane region


Axis of
revolution
Solid of revolution

DATE: / /

Example 2: Find the volume of the solid formed by revolving $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ 0 \le x \le e^{-y^2}\}$ about the x-axis.



Sol: $\because$ $p(y) = y$ and $h(y) = x = e^{-y^2}$ are conti. on $[0,1]$.

$$\therefore V = \int_0^1 2\pi p(y) h(y)\, dy = 2\pi \int_0^1 y e^{-y^2}\, dy.$$

$$= (2\pi)\left(\frac{-1}{2}\right)\int_0^1 e^{-y^2}(-2y)\, dy = (-\pi) e^{-y^2}\Big|_0^1$$

$$= (-\pi)(e^{-1} - 1) = \pi(1 - e^{-1}) \doteq 1.986$$

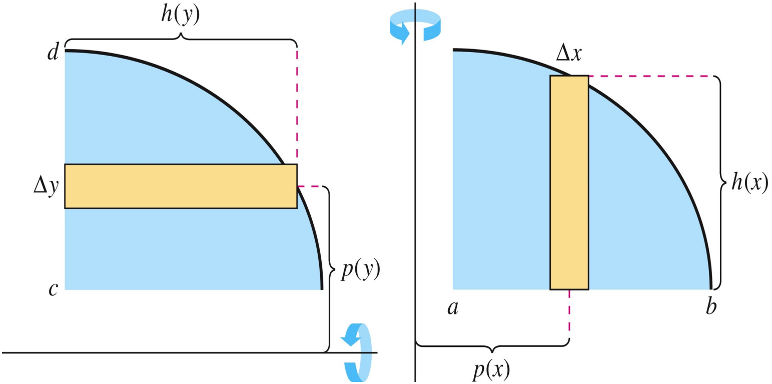
#

Type II: Vertical Axis of Revolution $x = L$

If $p(x)$ and $h(x)$ are conti. functions of x on $[a, b]$, then the volume of the solid formed by revolving a plane region Ω about $x = L$ is

$$V = 2\pi \int_a^b p(x)h(x)\,dx.$$

如何選取函數 $p(\cdot)$ 和 $h(\cdot)$?

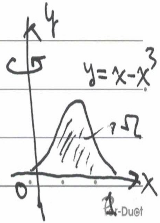


Example 1: (Washer Method 失效或不可行の例子)

Find the volume of the solid formed by revolving

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le x - x^3\}$$

about the y-axis.



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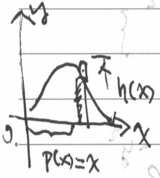
Sol: Note that the Washer Method fails for this example!



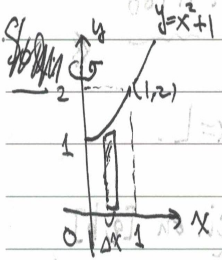
$\because$ $p(x)=x$ and $h(x)=y=x-x^3$ are conti. on $[0,1]$

$\therefore$ $$V=2\pi\int_0^1 x(x-x^3)\,dx=2\pi\int_0^1 (x^2-x^4)\,dx$$

$$=2\pi\left(\frac{1}{3}x^3-\frac{1}{5}x^5\right)\Big|_0^1=2\pi\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4\pi}{15}$$ #

Example 3: (与 Example 4, Sec. 5.2 作比較)

Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1, 0 \le y \le x^2+1\}$. Find the volume of the solid formed by revolving Ω about the y-axis.



Sol: Applying the Shell Method, we consider $p(x) = x$ and $h(x) = x^2+1$ $\forall x \in [0,1]$.

$$\Rightarrow V = 2\pi \int_0^1 p(x)h(x)\,dx = 2\pi \int_0^1 x(x^2+1)\,dx$$

$$= 2\pi \int_0^1 (x^3+x)\,dx = 2\pi\left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_0^1 = \frac{3\pi}{2}$$

#

Note: 此法比 Washer Method 更簡潔!

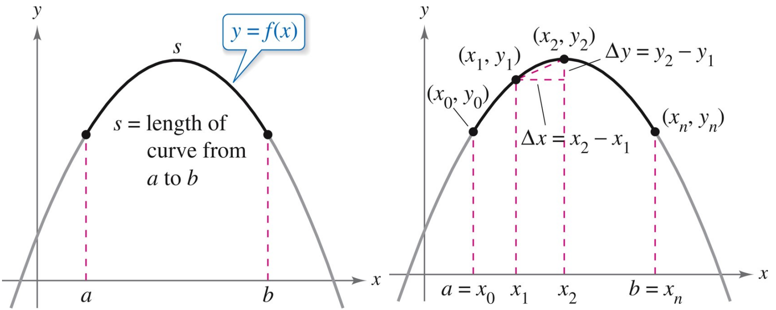
Section 5.4
Arc Length and Surfaces of Revolution
(弧長與旋轉體曲面)

Def (平滑曲線的定義)

Let f be a real-valued function defined on $[a, b]$.

(1) f is continuously differentiable (連續可微) if its first derivative f' is conti. on $[a, b]$. In this case, we denote $f \in C^1[a, b]$.

(2) The graph of $f \in C^1[a, b]$ is called a smooth curve (平滑曲線).


y
y = f(x)
s
s = length of curve from a to b
a
b
x
y
(x2, y2)
(x1, y1)
Δy = y2 − y1
(x0, y0)
(xn, yn)
Δx = x2 − x1
x
a = x0
x1
x2
b = xn

For a smooth curve $y = f(x)$ with $f \in C^1[a, b]$, the arc length of f on the subinterval $[x_{i-1}, x_i]$ is given by

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i)$$

for each $i = 1, 2, \ldots, n$. Moreover, it follows from M.V.T. that $\exists\, c_i \in (x_{i-1}, x_i)$ s.t. $f'(c_i) = \frac{\Delta y_i}{\Delta x_i}$. Thus, we see that

$$\Delta s_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} (\Delta x_i) = \sqrt{1 + [f'(c_i)]^2} (\Delta x_i)$$

for each $i = 1, 2, \ldots, n$.

Type I: 第一型弧長公式

The arc length of a smooth curve $y = f(x)$ on $[a, b]$ is

$$s = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \Delta s_i = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$
$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Example 2: (求 Type I 弧長)

Find the arc length of the curve $y = f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[\frac{1}{2}, 2]$.

Sol: $\because f'(x) = \frac{1}{2}x^2 - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$.

$\therefore \sqrt{1+[f'(x)]^2} = \sqrt{1+\frac{1}{4}\left(x^2-\frac{1}{x^2}\right)^2} = \sqrt{\frac{1}{4}\left(x^4+2+\frac{1}{x^4}\right)}$

$= \sqrt{\frac{1}{4}\left(x^2+\frac{1}{x^2}\right)^2} = \frac{1}{2}\left(x^2+\frac{1}{x^2}\right)$ for $x \neq 0$.

$\Rightarrow$ The arc length of $y=f(x)$ between a and b is

$$S = \int_{1/2}^{2} \frac{1}{2}\left(x^2+\frac{1}{x^2}\right) dx = \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{x}\right)\Bigg|_{1/2}^{2} = \frac{33}{16}.$$

Example 4 (求 Type I 弧長)

Find the arc length of the graph of $y = f(x) = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$.

Sol: $\because \sqrt{1+[f'(x)]^2} = \sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2} = \sqrt{1+\tan^2 x} = \sqrt{\sec^2 x} = \sec x \quad \forall x \in [0, \frac{\pi}{4}]$.

$$\therefore s = \int_0^{\pi/4} \sec x \, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/4} = \ln(\sqrt{2}+1) \doteq 0.881$$

#

Type II: 第二型弧長公式

The arc length of a smooth curve $x = g(y)$ on $[c, d]$ is

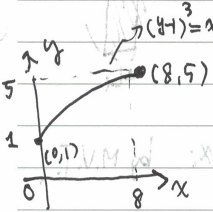
$$s = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \Delta s_i = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{1 + [g'(c_i)]^2}(\Delta y_i)$$

$$= \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 3: (求 Type II 弧長).

Find the arc length of the curve $(y-1)^3=x^2$ on $[0,8]$.

Sol: Consider $x=g(y)=(y-1)^{3/2}$, $\forall y\in[1,5]$. Then



$$g'(y)=\frac{3}{2}(y-1)^{1/2}$$ and hence

$$\sqrt{1+[g'(y)]^2}=\sqrt{1+\frac{9}{4}(y-1)}=\sqrt{\frac{9}{4}y-\frac{5}{4}}.$$

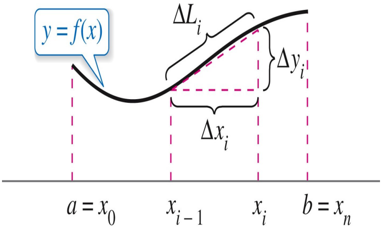
$$=\frac{1}{2}\sqrt{9y-5}.$$

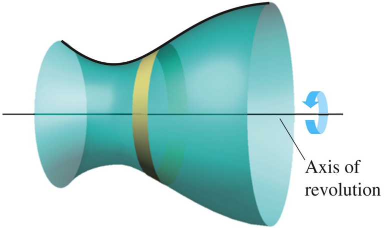
So, the arc length is $s = \int_1^5 \frac{1}{2}\sqrt{9y-5}\, dy$.

$$= \frac{1}{18}\int_1^5 (9y-5)^{1/2}(9)\, dy = \frac{1}{18}\cdot\frac{2}{3}(9y-5)^{3/2}\Big|_1^5$$

$$= \frac{1}{27}(9y-5)^{3/2}\Big|_1^5 = \frac{1}{27}(40^{3/2} - 4^{3/2}) \doteq 9.073$$

#


$y = f(x)$
ΔL_i
Δy_i
Δx_i
$a = x_0$
x_{i-1}
x_i
$b = x_n$


Axis of
revolution

Main Question

What is the surface area S formed by revolving a smooth curve $y = f(x)$ about the horizontal axis of revolution $y = L$?

On the ith subinterval $I_i = [x_{i-1}, x_i]$, choose $c_i, d_i \in I_i$, we let

$$\begin{aligned} r(d_i) &= \text{the redius of revolution at} d_i \in I_i, \\ \Delta L_i &= \text{the arc length of the smooth curve on } I_i \\ &= \sqrt{1 + [f'(c_i)]^2}(\Delta x_i), \end{aligned}$$

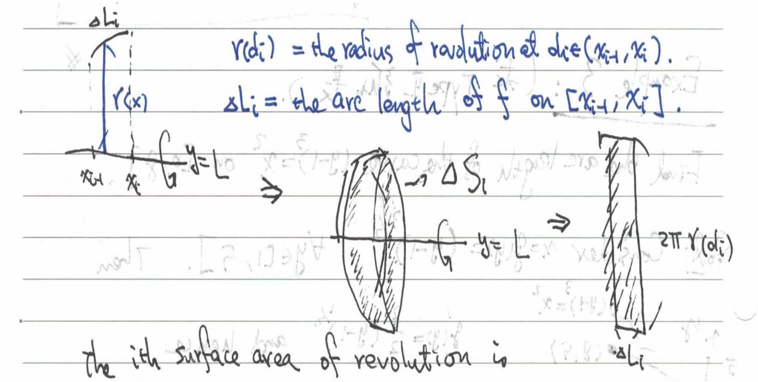
where $r(x) \geq 0$ is conti. on $[a, b]$.

Note (第 i 個旋轉體表面積)

If $d_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, it follows from M.V.T. that $\exists\, c_i \in (x_{i-1}, x_i)$ s.t. the ith surface area of revolution is

$$\Delta S_i \approx 2\pi r(d_i) \cdot \Delta L_i = 2\pi r(d_i)\sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$

for $i = 1, 2, \ldots, n$.


r(di) = the radius of revolution at di∈(xi-1, xi).
ΔLi = the arc length of f on [xi-1, xi].
ΔLi
r(x)
xi-1
xi
y=L
ΔSi
y=L
2π r(di)
ΔLi
The ith surface area of revolution is

Type I: Horizontal Axis of Revolution $y = L$

If $r(x) \geq 0$ is conti. on $[a, b]$ and $f \in C^1[a, b]$, then the area of the surface formed by revolving a curve $y = f(x)$ about $y = L$ is

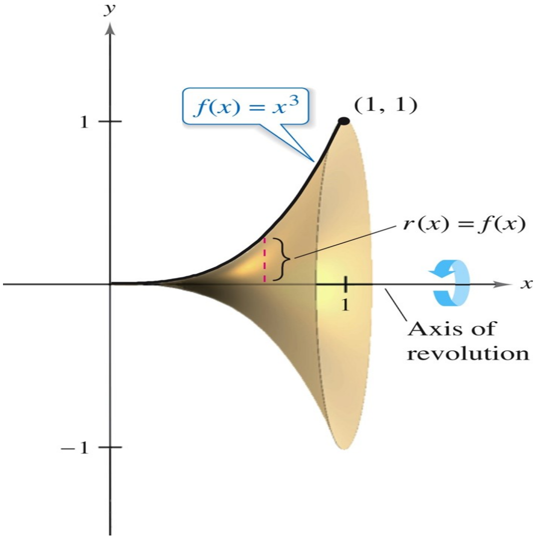
$$S = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi r(d_i)\sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$

$$= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

Example 6: (求 Type I 旋轉表面積). ※

Find the area of surface formed by revolving the curve $y=x^3$ about the x-axis $(0\le x\le 1)$.

Example 6 的示意圖 (承上頁)



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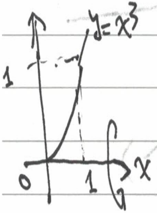
Sol: $\because f(x)=x^3$ and $f'(x)=3x^2\ \forall x\in[0,1]$.



$$\therefore S=\int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx=2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx.$$

$$=\frac{2\pi}{36}\int_0^1(36x^3)(1+9x^4)^{1/2}\,dx=\frac{\pi}{27}(1+9x^4)^{3/2}\Big|_0^1$$

$$=\frac{\pi}{27}(10^{3/2}-1)\doteqdot 3.563.\ \#$$

Type II: Vertical Axis of Revolution $x = L$

If $r(y) \geq 0$ is conti. on $[c, d]$ and $g \in C^1[c, d]$, then the area of the surface formed by revolving a curve $x = g(y)$ about $x = L$ is

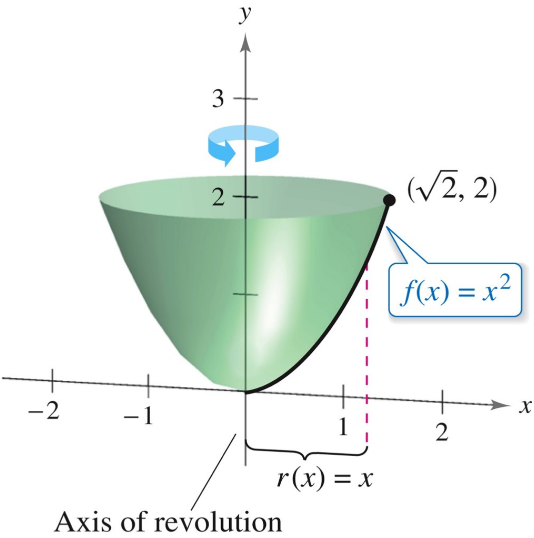
$$S = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi r(d_i) \sqrt{1 + [g'(c_i)]^2} (\Delta y_i)$$

$$= 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} \, dy.$$

Example 7: (求 Type II 旋轉体表面積)

Find the area of surface formed by revolving the curve $y = x^2$ on $[0, \sqrt{2}]$ about the y-axis.

Example 7 的示意圖 (承上頁)



Sol: (使用教科書上的方法)
For the smooth curve $y = f(x) = x^2$ on $[0, \sqrt{2}]$, consider the radius function $r(x) = x$ and then we see that

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + (2x)^2} = \sqrt{1 + 4x^2}.$$

So, the area of surface formed by revolving the curve $y = f(x)$ about the y-axis is given by

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + 4x^2}\, dx$$
$$= \frac{\pi}{4} \int_0^{\sqrt{2}} (8x)(1 + 4x^2)^{1/2}\, dx = \frac{\pi}{6}(1 + 4x^2)^{3/2}\Big|_0^{\sqrt{2}} = \frac{13\pi}{3}.$$

Sol: (使用 Type II 的方法)
From $y = x^2$, we know that $x = g(y) = y^{1/2} \quad \forall\, y \in [0, 2]$. Hence, the radius function $r(y) = g(y) = y^{1/2} \geq 0$ is conti. on $[0, 2]$ and

$$\sqrt{1 + [g'(y)]^2} = \sqrt{1 + (\frac{1}{2} y^{-1/2})^2} = \sqrt{1 + \frac{1}{4} y^{-1}}.$$

So, the surface area of revolution about the y-axis is

$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2}\, dy = 2\pi \int_0^2 y^{1/2} \sqrt{1 + \frac{1}{4} y^{-1}}\, dy$$
$$= \pi \int_0^2 \sqrt{4y + 1}\, dy = \frac{\pi}{6} (4y + 1)^{3/2} \Big|_0^2 = \frac{13\pi}{3}.$$

Thank you for your attention!