Chapter 6
Integration Techniques and Improper Integrals
(積分技巧與瑕積分)

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本章預定授課範圍

Section 6.1
Integration by Parts
(分部、分步或部分積分)

If u and v are diff. functions of x, then

$$(uv)' = u'v + uv' = uv' + vu'.$$

Thus, we immediately obtain

$$\int (uv)'\,dx = \int uv'\,dx + \int vu'\,dx$$
$$\Longrightarrow uv = \int u\,dv + \int v\,du$$
$$\Longrightarrow \int u\,dv = uv - \int v\,du.$$

Thm 6.1 (I.B.P. 公式)

If $u = u(x)$ and $v = v(x)$ are functions of x havingg conti. derivatives, then we have

$$\int u\,dv = uv - \int v\,du.$$

How to choose u and dv?

- $u = u(x)$: easily differentiable function of x. (u: 好微分函數)
- $v = v(x)$: easily integrable function of x. (v: 好積分函數)

Type I of I.B.P.

The integrals of the form

$$\int x^n e^{ax}\,dx, \quad \int x^n \sin(ax)\,dx, \quad \int x^n \cos(ax)\,dx$$

with $n \in \mathbb{R}$ and $a \neq 0$.

How to select u and dv for this case?

- $u = x^n$.
- $dv = e^{ax}dx, \quad \sin(ax)dx, \quad \cos(ax)dx$.

Example 1: Find $\int xe^x\,dx$

Sol: Let $u=x$ and $dv=e^x\,dx$

$\Rightarrow du=dx$ and $v=\int e^x\,dx=e^x$.

So, $\int xe^x\,dx=\int u\,dv=uv-\int v\,du$

$=xe^x-\int e^x\,dx=xe^x-e^x+C$. #

Sol: (第二種更簡潔的寫法)

$$\begin{aligned}\int x\, e^x dx = \int x\, d(e^x) &\equiv \int u\, dv \\ &= u\, v - \int v\, du \qquad \text{(使用 I.B.P. 公式)} \\ &= x\, e^x - \int e^x\, dx \\ &= xe^x - e^x + C,\end{aligned}$$

where C is a constant of integration.

Example 4 : Find $\int x^2 \sin x\,dx$ (使用兩次 I.B.P.)

Sol: Let $u = x^2$ and $dv = \sin x\,dx$.

$\Rightarrow du = 2x\,dx$ and $v = \int \sin x\,dx = -\cos x$.

$\Rightarrow \int x^2 \sin x\,dx = \int u\,dv = uv - \int v\,du = -x^2 \cos x - \int 2x(-\cos x)\,dx$

$= -x^2 \cos x + 2\int x \cos x\,dx$ — $(*)$.

Let $u = x$ and $dv = \cos x\,dx$

$\Rightarrow du = dx$ and $v = \int \cos x\,dx = \sin x$.

From $(*)$ $\Rightarrow \int x^2 \sin x\,dx = -x^2 \cos x + 2\int u\,dv$.

$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$ by I.B.P.

$= -x^2 \cos x + 2x\sin x + 2\cos x + C$. ※

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Example 7: (Tabular Method 表格法)

Find $\int x^2 \sin 4x \, dx$ using the tabular method.

Sol:

sign		
+	x^2	$\sin 4x \, dx$
−	$2x$	$-\frac{1}{4}\cos 4x$
+	2	$-\frac{1}{16}\sin 4x$
−	0	$\frac{1}{64}\cos 4x$

So, $\int x^2 \sin 4x \, dx = \frac{-x^2}{4}\cos 4x + \frac{x}{8}\sin 4x + \frac{1}{32}\cos 4x + C$

Type II of I.B.P.

The integrals of the form

$$\int x^n \ln x \, dx, \quad \int x^n \sin^{-1} x \, dx, \quad \int x^n \tan^{-1} x \, dx$$

with $n \neq -1$.

How to select u and dv for this case?

- $u = \ln x, \quad \sin^{-1} x, \quad \tan^{-1} x$.
- $dv = x^n dx$.

Example 2: Find $\int x^2 \ln x \, dx$.

Sol: Let $u = \ln x$ and $dv = x^2 dx$

$\Rightarrow \quad du = \frac{1}{x} dx$ and $v = \frac{1}{3} x^3$.

So, $\int x^2 \ln x \, dx = \int u \, dv = uv - \int v \, du$

$= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \left(\frac{1}{x}\right) dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx$

$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C$. #

Example 3: Find $\int_0^1 \sin^{-1} x \, dx$.

Sol: Let $u = \sin^{-1} x$ and $dv = dx$.

$\Rightarrow \quad du = \frac{1}{\sqrt{1-x^2}} dx$ and $v = x$.

$$\Rightarrow \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

$$= x \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx$$

$$= x \sin^{-1} x + \sqrt{1-x^2} + C$$

So, $\int_0^1 \sin^{-1} x \, dx = \left(x \sin^{-1} x + \sqrt{1-x^2} \right) \Big|_0^1 = \frac{\pi}{2} - 1$. #

Example 5 Find $\int \sec^3 x\,dx$ (重要公式，切記!!)

Sol: $\int \sec^3 x\,dx = \int \sec \cdot \sec^2 x\,dx = \int \sec x\, d(\tan x)$

$= \sec x \tan x - \int \tan x\, d(\sec x)$

$= \sec x \tan x - \int \sec x \tan^2 x\,dx.$

$= \sec x \tan x - \int \sec x(\sec^2 - 1)\,dx. \quad (\because \tan^2 x + 1 = \sec^2 x)$

$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$

So, $2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$

$\Rightarrow \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

Type III of I.B.P.

The integrals of the form

$$\int e^{ax} \sin(bx)\, dx, \quad \int e^{ax} \cos(bx)\, dx$$

with $a, \neq 0$ and $b, \neq 0$.

How to select u and dv for this case?

- $u = e^{ax}$ and $dv = \sin(bx)dx, \quad \cos(bx)dx$.
- $u = \sin(bx), \quad \cos(bx)$ and $dv = e^{ax}dx$.
- 通常使用兩次 I.B.P. 公式!

Example: Find $\int e^x \cos x\,dx$.

Sol: $\int e^x \cos x\,dx = \int e^x\,d(\sin x)$.

$= e^x \sin x - \int \sin x\,d(e^x) = e^x \sin x - \int e^x \sin x\,dx$

$= e^x \sin x + \int e^x\,d(\cos x) = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$.

So, $2\int e^x \cos x\,dx = e^x(\sin x + \cos x)$

$\Rightarrow \int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C$.

Another Solution

Applying the tabular method, we immediately obtain

$+$	$\cos x$	$e^x\,dx$
$-$	$-\sin x$	e^x
$+$	$-\cos x$	e^x

Thus the original integral satisfies

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x + \int e^x(-\cos x)\,dx,$$

and hence the indefinite integral is given by

$$\int e^x \cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + C,$$

where C is a constant of integration.

Section 6.2
Trigonometric Integrals
(三角積分)

Main Goals

Try to find the integrals of the form

(INT-1) : $\displaystyle\int \sin^m x \cos^n x \, dx$, where $m, n \in \mathbb{Q}$.

(INT-2) : $\displaystyle\int \sec^m x \tan^n x \, dx$, where $m, n \in \mathbb{Q}$.

(INT-3) : $\displaystyle\int \sin\alpha\cos\beta \, dx, \quad \int \sin\alpha\sin\beta \, dx, \quad \int \cos\alpha\cos\beta \, dx,$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are functions of x.

Type I of (INT-1)

If $m = 2k + 1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \cdot \sin x \, dx$$

$$= - \int (1 - \cos^2 x)^k \cos^n x \, d(\cos x) = - \int (1 - u^2)^k u^n \, du,$$

where we let $u = \cos x$.

Example 1 (Type I a 例子) Find $\int \sin^3 x \cos^4 x \, dx$.

Sol: $\int \sin^3 x \cos^4 x \, dx = \int \sin^2 x \cos^4 x \, \underline{\sin x \, dx}$

$= -\int (1-\cos^2 x)\cos^4 x \, d(\cos x) = \int (\cos^6 x - \cos^4 x) \, d(\cos x)$.

$= \dfrac{\cos^7 x}{7} - \dfrac{\cos^5 x}{5} + C$. ※

Type II of (INT-1)

If $n = 2k + 1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cdot \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \, d(\sin x) = \int u^m (1 - u^2)^k \, du,$$

where we let $u = \sin x$.

Example 2: Find $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-\frac{1}{2}} \cos^3 x\,dx.$

Sol: $\displaystyle\int_{\pi/6}^{\pi/3} (\sin x)^{-1/2} \cos^2 x \cdot \cos x\,dx = \int_{\pi/6}^{\pi/3} (\sin x)^{-1/2} (1-\sin^2 x)\, d(\sin x)$

$\displaystyle = \int_{\pi/6}^{\pi/3} \left[(\sin x)^{-1/2} - (\sin x)^{3/2} \right] d(\sin x).$

$\displaystyle = \left[2(\sin x)^{1/2} - \frac{2}{5}(\sin x)^{5/2} \right]_{\pi/6}^{\pi/3}$

$\displaystyle = 2\left(\frac{\sqrt{3}}{2}\right)^{1/2} - \frac{2}{5}\left(\frac{\sqrt{3}}{2}\right)^{5/2} - \sqrt{2}$ ~~$+\frac{\sqrt{2}}{20}$~~ $\displaystyle + \frac{\sqrt{2}}{20}$ //

Type III of (INT-1)

If m and n are even and nonnegative, try to use the identities

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$

(當 m, n 為偶數或零，試用倍角公式!)

Example 3: Find $\int \cos^4 x\,dx$ with $m=0$ and $n=4$.

Sol: $\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1+2\cos 2x+\cos^2(2x))\,dx$

$= \frac{1}{4}\int \left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}\int \left(\frac{3}{2}+2\cos 2x+\frac{1}{2}\cos 4x\right)dx.$

$= \int \left(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx$

$= \frac{3x}{8}+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C.$

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Type I of (INT-2)

If $m = 2k$ is even for some $k \in \mathbb{N}$, then

$$\int \sec^{2k} x \tan^n x \, dx = \int (\sec^2 x)^{k-1} \tan^n x \cdot \sec^2 x \, dx$$
$$= \int (1 + \tan^2 x)^{k-1} \tan^n x \, d(\tan x) = \int (1 + u^2)^{k-1} u^n \, du,$$

where we let $u = \tan x$.

Example 5: Find $\int \sec^4(3x)\tan^3(3x)\,dx$.

Sol: ~~$\int \sec^2(3x)\tan^3(3x)\cdot\sec^2(3x)\,dx$~~ Let $u = \tan(3x)$. Then

$$du = 3\sec^2(3x)\,dx \Rightarrow \sec^2(3x)\,dx = \frac{1}{3}\,du.$$

So, $\int \sec^4(3x)\tan^3(3x)\,dx = \int \sec^2(3x)\tan^3(3x)\cdot\sec^2(3x)\,dx$

$$= \int [1+\tan^2(3x)]\tan^3(3x)\cdot\sec^2(3x)\,dx = \frac{1}{3}\int (1+u^2)u^3\,du$$

$$= \frac{1}{3}\int (u^3+u^5)\,du = \frac{\tan^4(3x)}{12} + \frac{\tan^6(3x)}{18} + C$$

Type II of (INT-2)

If $n = 2k+1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sec^m x \tan^{2k+1} x\, dx = \int \sec^{m-1} x (\tan^2)^k \cdot \sec x \tan x\, dx$$

$$= \int \sec^{m-1} x (\sec^2 - 1)^k\, d(\sec x) = \int u^{m-1}(u^2 - 1)^k\, du,$$

where we let $u = \sec x$.

Example 4: Find $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int (\sec x)^{-1/2} \tan^3 x\,dx.$

Sol: $\int (\sec x)^{-1/2} \tan^3 x\,dx = \int (\sec x)^{-3/2} \tan^2 x\,(\sec x \tan x)\,dx$

$= \int (\sec x)^{-3/2} (\sec^2 x - 1)\,d(\sec x) = \int \left[(\sec x)^{1/2} - (\sec x)^{-3/2}\right] d(\sec x)$

$= \frac{2}{3}(\sec x)^{3/2} + 2(\sec x)^{-1/2} + C.$ #

Type III of (INT-2)

If *m* is odd or *n* is even, try to use the identity

$$tan^2 x = \sec^2 x - 1.$$

(當 m 是奇數或 n 是偶數時，試用上述等式!)

Example 6: Find $\int_0^{\pi/4} \tan^4 x \, dx$.

Sol: $\int_0^{\pi/4} \tan^2 x \cdot \tan^2 x \, dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan^2 x \, dx$

$$= \int_0^{\pi/4} \sec^2 x \tan^2 x \, dx - \int_0^{\pi/4} (\sec^2 x - 1) \, dx$$

$$= \left(\frac{\tan^3 x}{3} \Big|_0^{\pi/4} - \tan x + x \right) \Big|_0^{\pi/4} = \frac{-2}{3} + \frac{\pi}{4} \doteq 0.119$$

※

Integrals of Sine-Cosine Products

If $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are conti. functions of x, how to evaluate

$$\text{(INT-3)}: \int \sin\alpha \cos\beta \, dx, \quad \int \sin\alpha \sin\beta \, dx, \quad \int \cos\alpha \cos\beta \, dx$$

using the product-to-sum identities (積化和差等式)?

Product-to-Sum Identities

(1) $\sin\alpha\sin\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big].$

(2) $\cos\alpha\cos\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big].$

(3) $\sin\alpha\cos\beta = \frac{1}{2}\Big[\sin(\alpha-\beta) + \sin(\alpha+\beta)\Big].$

Example 8*: (積化和差の例子)

Find $\int \sin(5x)\cos(4x)\,dx$ with $\alpha = 5x$ and $\beta = 4x$.

Sol:
$$\int \sin(5x)\cos(4x)\,dx = \frac{1}{2}\int \left[\sin(5x-4x) + \sin(5x+4x)\right]dx$$
$$= \frac{1}{2}\int (\sin x + \sin 9x)\,dx = \frac{-\cos x}{2} - \frac{\cos(9x)}{18} + C.$$

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Section 6.3
Trigonometric Substitution
(三角代換)

Main Goal

- To deal with the integrals involving

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}$$

  with $a > 0$.
- More precisely, how to evaluate the three types of aforementioned integrals using the technique of trigonometric substitution?

Type I of Trigonometric Substitution

For integrals involving $\sqrt{a^2 - x^2}$ with $a > 0$, let

$$x = a \sin\theta.$$

Then $dx = a\cos\theta\, d\theta$ and we see that

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Example 1: Find $\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$.

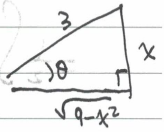
Sol: Let $x = 3\sin\theta$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

$$dx = 3\cos\theta\,d\theta \text{ and } \sqrt{9-x^2} = 3\cos\theta.$$

So,

$$\int \frac{dx}{x^2\sqrt{9-x^2}} = \int \frac{3\cos\theta}{(9\sin^2\theta)(3\cos\theta)}\,d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\,d\theta = \frac{-1}{9}\cot\theta + C$$



$$= \frac{-1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right) + C = \frac{-\sqrt{9-x^2}}{9x} + C$$

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Type II of Trigonometric Substitution

For integrals involving $\sqrt{a^2 + x^2}$ with $a > 0$, let

$$x = a \tan \theta.$$

Then $dx = a \sec^2 \theta \, d\theta$ and we see that

$$\sqrt{a^2 + x^2} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = a \sec \theta$$

for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Example 2: Find $\int \frac{dx}{\sqrt{4x^2+1}}$.

Sol: Note that $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+\frac{1}{4}}}$.

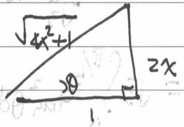
Let $x = \frac{1}{2}\tan\theta$. Then $dx = \frac{1}{2}\sec^2\theta\, d\theta$ and

$$\sqrt{x^2+\frac{1}{4}} = \sqrt{\frac{1}{4}\sec^2\theta} = \frac{1}{2}\sec\theta \text{ for } -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

So, $\displaystyle\int \frac{\frac{1}{2}\,dx}{\sqrt{x^2+\frac{1}{4}}} = \int \frac{(\frac{1}{2})(\frac{1}{2})\sec^2\theta}{(\frac{1}{2})\sec\theta}\,d\theta$

$\displaystyle = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

$\displaystyle = \frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right| + C$. #



$(\tan\theta = 2x)$

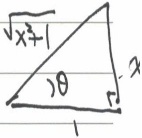
Example 3: § Find $\displaystyle\int \frac{dx}{\cdots}$

Example 3: Find $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$ $\quad(\tan\theta = 2x)$

Sol: Let $x = \tan\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{(x^2+1)^{3/2}} = \int \frac{\sec^2\theta \, d\theta}{(\tan^2\theta+1)^{3/2}} = \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta$$

$$= \int \frac{d\theta}{\sec\theta} = \int \cos\theta \, d\theta = \sin\theta + C = \frac{x}{\sqrt{x^2+1}} + C.$$



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$(\tan\theta = x)$

Type III:

Type III of Trigonometric Substitution

For integrals involving $\sqrt{x^2 - a^2}$ with $a > 0$, let

$$x = a \sec \theta.$$

Then $dx = a \sec \theta \tan \theta \, d\theta$ and we see that

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$$

for $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$.

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Example 4: Find $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$.

Sol: Let $x = \sqrt{3}\sec\theta$ for $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

Then $dx = \sqrt{3}\sec\theta\tan\theta\,d\theta$, ~~$\sqrt{x^2-3} = \sqrt{3}\tan\theta$~~

x	$\sqrt{3}$	2
θ	0	$\pi/6$

and $\sqrt{x^2-3} = \sqrt{3(\sec^2\theta - 1)} = \sqrt{3\tan^2\theta} = \sqrt{3}\tan\theta$ for $0 \le \theta \le \pi/6$.

Hence, $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx = \int_0^{\pi/6} \frac{(\sqrt{3}\tan\theta)(\sqrt{3}\sec\theta\tan\theta)}{\sqrt{3}\sec\theta}\,d\theta$.

$\displaystyle = \sqrt{3}\int_0^{\pi/6} \tan^2\theta\,d\theta = \sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta$.

$\displaystyle = \sqrt{3}(\tan\theta - \theta)\Big|_0^{\pi/6} = \sqrt{3}\left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = 1 - \frac{\sqrt{3}\pi}{6}$. #

Section 6.4
Partial Fractions
(部分分式)

Long Division (長除法)

If $N(x)$ and $D(x)$ are polynomials with $\deg(N) \geq \deg(D)$, then

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N_1(x)}{D(x)},$$

where $Q(x)$ and $N_1(x)$ are polynomials with $\deg(N_1) < \deg(D)$. In this case, $\frac{N_1(x)}{D(x)}$ is called a proper rational function of x (真分式).

Note: integrating w.r.t. $x \Longrightarrow$

$$\int \frac{N(x)}{D(x)}\,dx = \int Q(x)\,dx + \int \frac{N_1(x)}{D(x)}\,dx.$$

Remarks

- $\int Q(x)\,dx$ can evaluated easily by the power rule.
- But, how to evaluate the integral $\int \frac{N_1(x)}{D(x)}\,dx$ using the technique of partial fraction decomposition?

Partial Fraction Decomposition

If $D(x) = (px+q)^m(ax^2+bx+c)^n$ with $p \neq 0$ and $b^2-4ac<0$, may write

$$\frac{N_1(x)}{D(x)} = \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m} + \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n},$$

where A_i $(i=1,\ldots,m)$ and B_j, C_j $(j=1,\ldots,n)$ are unknowns.

Type I: Distinct or Repeated Linear Factors

The denominator (分母) $D(x)$ contains

- distinct linear factors (相異的一次因式)

$$(p_1x + q_1)(p_2x + q_2)\cdots(p_mx + q_m)$$

- repeated linear factors (重複的一次因式)

$$(px + q)^m,$$

where $p_j \neq 0$ $(j = 1, 2, \ldots, m)$ and $p \neq 0$ for some $m \in \mathbb{N}$.

Example 1: (相異の一次因式)

Write the partial fraction decomposition for $\dfrac{1}{x^2-5x+6}$.

Sol:

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}.$$

$$\Rightarrow 1 = A(x-2) + B(x-3) = (A+B)x + (-2A-3B).$$

$$\Rightarrow \begin{cases} A+B=0 \\ -2A-3B=1 \end{cases} \Rightarrow B=-1 \text{ and } A=1.$$

So,

$$\frac{1}{x^2-5x+6} = \frac{1}{x-3} - \frac{1}{x-2}. \quad \#$$

Example 2: (重複的一次因式)

Find $\displaystyle\int \frac{5x^3+20x+6}{x^3+2x^2+x}\,dx$.

Sol: Let $f(x) = \dfrac{5x^2+20x+6}{x^3+2x+x}$. Since $x^3+2x^2+x = x(x+1)^2$,

may write $f(x) = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$.

$$\Rightarrow 5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$$

$$= (A+B)x^2 + (2A+B+C)x + A.$$

$\Rightarrow A = 6,\ 2A + B + C = 20$ and $A + B = 5$,

$\Rightarrow A = 6,\ B = -1,\ C = 9. \Rightarrow f(x) = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$.

So, $\int f(x)\,dx = \int \left(\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right) dx = 6\ln|x| - \ln|x+1| - \frac{9}{x+1} + C$

$= \ln\left|\frac{x^6}{x+1}\right| - \frac{9}{x+1} + C$ #

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Type II: Distinct or Repeated Quadratic Factors (二次因式)

The denominator $D(x)$ contains

- distinct quadratic factors

$$(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_nx^2 + b_nx + c_n)$$

- repeated quadratic factors

$$(ax^2 + bx + c)^n,$$

where $b_j^2 - 4a_jc_j < 0$ $(j = 1, 2, \ldots, n)$ and $b^2 - 4ac < 0$ for some $n \in \mathbb{N}$.

Note: these quadratic factors are called irreducible (不可既約)!

Example 3: (相異的一次與二次因式)

Find $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$.

Sol: Let $f(x) = \dfrac{2x^3-4x-8}{(x^2-x)(x^2+4)}$. Since $(x^2-x)(x^2+4) = x(x-1)(x^2+4)$,

write $f(x) = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+4}$;

$$\Rightarrow 2x^3-4x-8 = A(x-1)(x^2+4) + Bx(x^2+4) + (Cx+D)(x^2-x) \quad (*)$$

Take $x=0$ in $(*)$ $\Rightarrow$ $-8 = A(-1)(4)$ $\Rightarrow$ $A=2$.

Take $x=1$ in $(*)$ $\Rightarrow$ $2-4-8 = 5B$ $\Rightarrow$ $B=-2$.

So, $2x^3-4x-8 = 2(x-1)(x^2+4) + (-2)x(x^2+4) + (Cx+D)(x^2-x)$.

Take $x=-1$ $\Rightarrow$ $2 = -C+D$

Take $x=2$ $\Rightarrow$ $8 = 2C+D$

$\Rightarrow C=2$ and $D=4$.

$$\text{So, } \int f(x)\,dx = \int \left(\frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}\right) dx$$

$$= \int \frac{2}{x}\,dx - \int \frac{2}{x-1}\,dx + \int \frac{2x}{x^2+4}\,dx + \int \frac{4}{x^2+4}\,dx$$

$$= 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

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Example 4: (重複の二次因式)

Find $\displaystyle\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$.

Sol: Write $f(x) = \dfrac{8x^3+13x}{(x^2+2)^2} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{(x^2+2)^2}$.

$$\Rightarrow 8x^3+13x = (Ax+B)(x^2+2) + (Cx+D).$$

$$= Ax^3 + Bx^2 + (2A+C)x + (2B+D).$$

$\Rightarrow A=8,\ B=0,\ 2A+C=13$ and $2B+D=0$.

$\Rightarrow A=8,\ B=0,\ C=-3,\ D=0$.

So, $\displaystyle\int f(x)\,dx=\int\left(\frac{8x}{x^2+2}+\frac{-3x}{(x^2+2)^2}\right)dx$.

$\displaystyle = 4\int\frac{2x}{x^2+2}\,dx-\frac{3}{2}\int\frac{2x}{(x^2+2)^2}\,dx$.

$\displaystyle = 4\ln(x^2+2)+\frac{3}{2(x^2+2)}+C$. #

Section 6.7
Improper Integrals
(瑕積分)

Two Types of Improper Integrals

Type I : Infinite Limits of Integration (無窮積分上下限), e.g.,

$$\int_a^\infty f(x)\,dx \quad \text{or} \quad \int_{-\infty}^b f(x)\,dx.$$

Type II : Infinite Discontinuities (無窮不連續點), e.g., we say that $\displaystyle\int_a^b f(x)\,dx$ is an improper integral of Type II if

$$\lim_{x\to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to c^-} f(x) = \pm\infty$$

for some $c \in [a, b]$.

Type I: Infinite Limits of Integration (1/2)

(1) If f is conti. on $[a, \infty)$, the improper integral

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\left[\int_a^b f(x)\,dx\right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

(2) If f is conti. on $(-\infty, b]$, the improper integral

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\left[\int_a^b f(x)\,dx\right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

Type I: Infinite Limits of Integration (2/2)

(3) If f is conti. on $(-\infty, \infty)$, the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

converges (收斂) whenever the improper integrals on the RHS both converge for all $c \in \mathbb{R}$.

Note: we say that the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ diverges if either of the improper integrals on the RHS diverges.

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Example 2: Evaluate the improper integral.

(a) $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\left[\int_0^b e^{-x}dx\right] = \lim_{b\to\infty}\left(-e^{-x}\Big|_0^b\right)$

$= \lim_{b\to\infty}(-e^{-b}+1) = 1$. #

(b) $\int_0^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty}\left(\int_0^b \frac{dx}{x^2+1}\right) = \lim_{b\to\infty}\left(\tan^{-1}x\Big|_0^b\right)$

$= \lim_{b\to\infty}(\tan^{-1}b - 0) = \frac{\pi}{2}$. #

Example 4: Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$.

Sol: $\because \int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+(e^x)^2}\,d(e^x) = \tan^{-1}(e^x) + C$.

$$\therefore \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx.$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}(e^x)\Big|_a^0\right] + \lim_{b\to\infty} \left[\tan^{-1}(e^x)\Big|_0^b\right].$$

$$= \lim_{a\to-\infty} \left[\frac{\pi}{4} - \tan^{-1}(e^a)\right] + \lim_{b\to\infty} \left[\tan^{-1}(e^b) - \frac{\pi}{4}\right]$$

$$= \left(\frac{\pi}{4} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}. \quad \#$$

Type II: Infinite Discontinuities (1/2)

(1) If f is conti. on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, the integral

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \left[\int_a^c f(x)\,dx \right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

(2) If f is conti. on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$, the integral

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \left[\int_c^b f(x)\,dx \right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

Type II: Infinite Discontinuities (2/2)

(3) If f is conti. on $[a, b]$ and has an infinite discontinuity at $c \in (a, b)$, the improper integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

converges (收斂) whenever the improper integrals on the RHS both converge.

Note: we say that the improper integral $\int_a^b f(x)\,dx$ diverges if either of the improper integrals on the RHS diverges.

Example 5: Evaluate $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \int_0^1 x^{-1/3}\,dx$. #

Sol: $\because \lim_{x\to 0^+} \frac{1}{\sqrt[3]{x}}$ ~~$\left(\frac{1}{+0}\right)$~~ $= \infty$ and $f(x) = \frac{1}{\sqrt[3]{x}}$ is conti. on $(0,1]$.

$\therefore \int_0^1 x^{-1/3}\,dx$ is an improper integral of Type II.

So, $\int_0^1 x^{-1/3}\,dx = \lim_{a\to 0^+}\left(\int_a^1 x^{-1/3}\,dx\right) = \lim_{a\to 0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_a^1\right) = \frac{3}{2}$. #

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Example 7: Evaluate $\int_{-1}^{2} \frac{1}{x^3}\,dx$.

Sol: Note that $\int_{-1}^{2} x^{-3}\,dx = \frac{-1}{2}x^{-2}\Big|_{-1}^{2} = \cdots$ (X)

$\because \lim_{x\to 0^+} \frac{1}{x^3} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x^3} = -\infty$.

$\therefore \int_{-1}^{2} x^{-3}\,dx$ is an improper integral of Type II.

$$\Rightarrow \int_{-1}^{2} x^{-3}\,dx = \int_{-1}^{0} x^{-3}\,dx + \int_{0}^{2} x^{-3}\,dx.$$

$$= \lim_{b\to 0^-} \left(\frac{-1}{2} x^{-2} \Big|_{-1}^{b} \right) + \lim_{a\to 0^+} \left(\frac{-1}{2} x^{-2} \Big|_{a}^{2} \right)$$

$$= \lim_{b\to 0^-} \left(\frac{-1}{2b^2} + \frac{1}{2} \right) + \lim_{a\to 0^+} \left(\frac{-1}{8} + \frac{1}{2a^2} \right) \text{ diverges,}$$

Since $\int_{-1}^{0} \frac{1}{x^3}\,dx = \lim_{b\to 0^-} \left(\frac{-1}{2b^2} + \frac{1}{2} \right) = -\infty$ and $\int_{0}^{2} \frac{1}{x^3}\,dx = \infty$. #

Example 8 (Type I 和 Type II 的瑕積分同時出現)

Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

Sol: Let $u=\sqrt{x}$ for $x\geq 0$. Then $du=\frac{1}{2\sqrt{x}}\,dx$ or $\frac{dx}{\sqrt{x}}=2\,du$.

$$\Rightarrow \int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2\,du}{u^2+1} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C \quad \text{--- } (*)$$

$$\because \lim_{x\to 0^+} \frac{1}{\sqrt{x}(x+1)} = \infty$$

$$\therefore \int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{a\to 0^+}\left(2\tan^{-1}1 - 2\tan^{-1}\sqrt{a}\right)$$

$$+ \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{b} - 2\tan^{-1}1\right) = 2\left(\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \pi. \quad \#$$

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Thm 6.7 (A Special Type of Improper Integral)

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \le 1. \end{cases}$$

Note: Thm 6.7 will be used in the Integral Test (積分測試法) for determining the convergence of an infinite series (無窮級數的收斂性), see Section 7.3 for details.

pf: ① If $p=1$, then $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left(\int_1^b \frac{1}{x}\,dx\right) = \lim_{b\to\infty}(\ln|b| - \ln 1)$

$= \infty$. (diverges.)

② If $p \neq 1$, then $\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\left(\frac{1}{-p+1} x^{-p+1}\Big|_1^b\right)$

$= \lim_{b\to\infty}\left(\frac{1}{-p+1} b^{-p+1} + \frac{1}{p-1}\right) = \begin{cases} \frac{1}{p-1}, & p>1 \\ \infty, & p<1. \end{cases}$

So, from ① and ② $\Rightarrow$ $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \text{diverges} & \text{if } p \le 1. \end{cases}$

Thm (瑕積分的比較定理)

Suppose that f and g are conti. on $[a,\infty)$ with $f(x) \leq g(x) \quad \forall\, x \in [a,\infty)$.

(1) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

(2) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example (補充題)

Does the integral $\int_1^\infty e^{-x^2}\tan^{-1}x\,dx$ converge or diverge?

Sol: Since $x^2 \geq x$ for all $x \geq 1$, it follows that $-x^2 \leq -x$ for all $x \geq 1$ and hence we know that

$$f(x) \equiv e^{-x^2}\tan^{-1}x \leq \frac{\pi}{2}e^{-x} \equiv g(x) \qquad \forall\, x \in [1,\infty).$$

In addition, since the improper integral

$$\int_1^\infty g(x)\,dx = \frac{\pi}{2}\int_1^\infty e^{-x}\,dx = \frac{\pi}{2}\lim_{b\to\infty}(-e^{-x})\Big|_1^b = \frac{\pi}{2e},$$

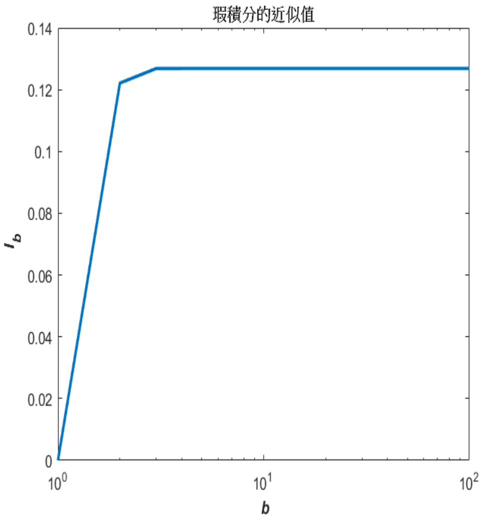
it follows from the Comparison Thm that the improper integral $\int_1^\infty f(x)\,dx = \int_1^\infty e^{-x^2}\tan^{-1}x\,dx$ converges.

Numerical Results (1/2)

- What is the true value of $I = \int_1^{\infty} e^{-x^2} \tan^{-1} x \, dx$?
- Although it is usually difficult to get the true value of I, we may apply the technique of numerical integration (數值積分) to compute its approximate value.
- If $I_b \equiv \int_1^{b} e^{-x^2} \tan^{-1} x \, dx$ for $b \geq 1$, we see that

$$I = \lim_{b \to \infty} I_b \approx 0.1268694335685518.$$

Numerical Results (2/2)



Thank you for your attention!