

# Chapter 7

# Infinite Series

## (無窮級數)

Hung-Yuan Fan (范洪源)

Department of Mathematics,  
National Taiwan Normal University, Taiwan

Spring 2019



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# Section 7.1

## Sequences

### (序列)



# Sequences (序列)

An infinite sequence of real numbers is denoted by

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \dots, a_n, \dots\},$$

where  $a_n$  is the  $n$ th term (第  $n$  項) of the sequence for  $n \in \mathbb{N}$ .



## Def. (序列的收斂性)

A sequence  $\{a_n\}$  converges to a limit  $L$ , denoted by

$$\lim_{n \rightarrow \infty} a_n = L,$$

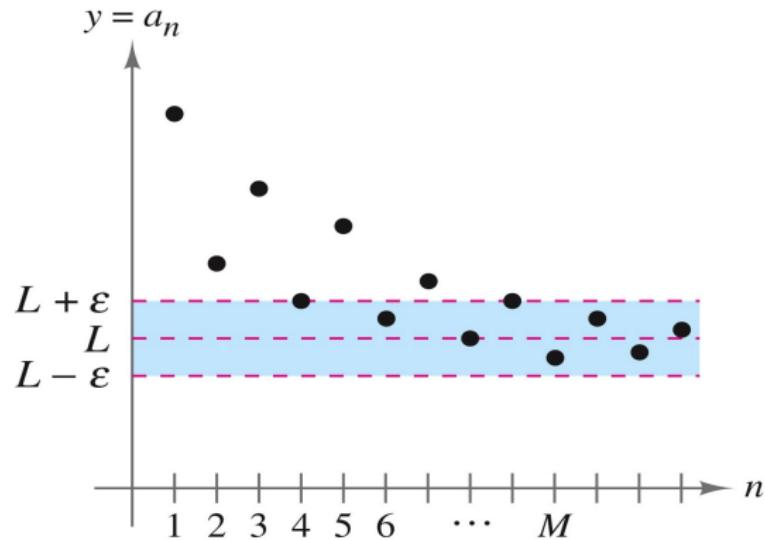
if  $\forall \varepsilon > 0$ ,  $\exists M > 0$  s.t.

$$n > M \implies |a_n - L| < \varepsilon.$$

Otherwise, we say that  $\{a_n\}$  diverges if the limit does not exist.



# 示意圖 (承上頁)



## Thm 7.1 (函數在 $\infty$ 處的極限 $\Rightarrow$ 序列的收斂性)

Let  $f$  be a real-valued function having the limit

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If  $a_n = f(n) \quad \forall n \in \mathbb{N}$ , then

$$\lim_{n \rightarrow \infty} a_n = L.$$



pf: Let  $\varepsilon > 0$  be given arbitrarily. Since  $\lim_{x \rightarrow \infty} f(x) = L$ ,  $\exists M > 0$  s.t.

$$x > M \implies |f(x) - L| < \varepsilon.$$

With  $a_n = f(n)$  for all  $n \in \mathbb{N}$ , if  $n > M$ , then

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

Thus, it follows from the Def. that

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f(n) = L.$$



## Example (Thm 7.1 的反敘述不成立)

Consider  $f(x) = \sin(\pi x)$  for all  $x > 0$  and let  $a_n = f(n)$  for  $n \in \mathbb{N}$ .  
Then

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} \sin(n\pi) = \lim_{n \rightarrow \infty} 0 = 0,$$

but we know that

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \sin(\pi x) \text{ does not exist!}$$



Example 2: Find the limit of  $a_n = \left(1 + \frac{1}{n}\right)^n$  when

Sol: Let  $f(x) = \left(1 + \frac{1}{x}\right)^x$  for  $x > 0$ .

$$\therefore \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

by L'Hôpital's Rule.

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} f(n) = e. \text{ by Thm 7.1}$$



## Thm 7.2 (Limit Laws for Sequences)

Suppose that  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = K$ . Then

- ①  $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K$ .
- ②  $\lim_{n \rightarrow \infty} (c \cdot a_n) = c \cdot L$  for all  $c \in \mathbb{R}$ .
- ③  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = L \cdot K$ .
- ④  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{K}$  if  $b_n \neq 0$   $\forall n \in \mathbb{N}$  and  $K \neq 0$ .



Example 3: Find the limit for  $\{a_n\}$ .

(a)  $\{a_n\} = \{3 + (-1)^n\} = \{2, 4, 2, 4, \dots\}$  diverges

because the  $n$ th term  $a_n$  oscillates between

2 and 4.

(b) If  $b_n = \frac{n}{1-2n} \quad \forall n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{n}{1-2n}$

$$= \lim_{n \rightarrow \infty} \frac{1}{(\frac{1}{n}) - 2} = \frac{-1}{2}$$



例 4：(要) 用 羅必達法則 求  $\{a_n\}$  的

Show that the sequence  $\{a_n\} = \left\{ \frac{n^2}{2^n - 1} \right\}$  converges.



Pf: Consider  $f(x) = \frac{x^2}{2^x - 1}$  for  $x > 0$ .

Applying the L'Hopital's Rule twice  $\Rightarrow$

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x^2}{2^x - 1} \stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2x}{(\ln 2)2^x}$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2}{(\ln 2)^2 2^x} = 0$$

$\therefore \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} f(n) = 0$  by Thm 7.1.



### Thm 7.3 (Squeeze Theorem for Sequences)

If  $\exists M > 0$  s.t.  $a_n \leq c_n \leq b_n \quad \forall n > M$ , and

$$\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n,$$

then the sequence  $\{c_n\}$  converges to the same limit  $L$ , i.e.,

$$\lim_{n \rightarrow \infty} c_n = L.$$



Example 5 : Show that  $\{c_n\} = \left\{ \frac{(-1)^n}{n!} \right\}$  converges.

PF:

$$\text{Since } n! = \cancel{n} \cdot \cancel{(n-1)} \cdot \cancel{(n-2)} \cdots \cancel{2} \cdot 1$$

Note that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{(n-4) \text{ terms}}$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = \underbrace{16 \cdot 2 \cdots 2}_{(n-4) \text{ terms}}$$

$$\Rightarrow \boxed{2^n < n! \text{ for } n \geq 4} \quad \rightarrow \text{理由}$$



## How to find the bounds for $\{c_n\}$ ?

Since  $n! \geq 2^n$  for all  $n \geq 4$ , it follows that

$$|c_n| = \left| \frac{(-1)^n}{n!} \right| = \frac{1}{n!} \leq \frac{1}{2^n},$$

and hence we immediately obtain

$$\frac{-1}{2^n} \leq c_n \leq \frac{1}{2^n}$$

for all  $n \geq 4$ .



$$\Rightarrow \frac{1}{2^n} \leq \frac{(-1)^n}{n!} < \frac{1}{2^n} \text{ for } n \geq 4.$$

$\circ\circ \lim_{n \rightarrow \infty} \frac{1}{2^n} = 0 = \lim_{n \rightarrow \infty} \frac{-1}{2^n}$

$\circ\circ \lim_{n \rightarrow \infty} \frac{(-1)^n}{n!} = 0 \text{ by Thm 7.3.}$



## Thm 7.4 (Absolute Value Thoerem)

Let  $\{a_n\}$  be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} |a_n| = 0 \iff \lim_{n \rightarrow \infty} a_n = 0.$$



If: ( $\Rightarrow$ ) Suppose that  $\lim_{n \rightarrow \infty} |a_n| = 0$ .

$$\therefore -|a_n| \leq a_n \leq |a_n| \quad \forall n \in \mathbb{N}.$$

$$\text{and } \lim_{n \rightarrow \infty} (-|a_n|) = 0 = \lim_{n \rightarrow \infty} |a_n|.$$

$$\therefore \lim_{n \rightarrow \infty} a_n = 0 \text{ by Squeeze Thm.}$$

( $\Leftarrow$ ) By the Def. directly !.



## Def. (單調序列的定義)

A sequence  $\{a_n\}$  is said to be monotonic (單調的) if its terms are nondecreasing (遞增的)

$$a_n \leq a_{n+1} \quad \forall n \in \mathbb{N},$$

or its terms are nonincreasing (遞減的)

$$a_n \geq a_{n+1} \quad \forall n \in \mathbb{N}.$$



## Def. (有界序列的定義)

- (1) The sequence  $\{a_n\}$  is bounded above (於上有界) if  $\exists M \in \mathbb{R}$  s.t.  $a_n \leq M \quad \forall n \in \mathbb{N}$ .
- (2) The sequence  $\{a_n\}$  is bounded below (於下有界) if  $\exists N \in \mathbb{R}$  s.t.  $a_n \geq N \quad \forall n \in \mathbb{N}$ .
- (3) The sequence  $\{a_n\}$  is bounded (有界的) if it is bounded above and bounded below, i.e.  $\exists M > 0$  s.t.  $|a_n| \leq M \quad \forall n \in \mathbb{N}$ .



## Thm 7.5 (Bounded Monotonic Sequences)

If the sequence  $\{a_n\}$  is **bounded** and **monotonic**, then it converges, i.e.,  $\exists! L \in \mathbb{R}$  s.t.  $\lim_{n \rightarrow \infty} a_n = L$ .



## Thm 7.5 (Bounded Monotonic Sequences)

If the sequence  $\{a_n\}$  is **bounded** and **monotonic**, then it converges, i.e.,  $\exists! L \in \mathbb{R}$  s.t.  $\lim_{n \rightarrow \infty} a_n = L$ .

### Example 9 (Thm 7.5 的例子)

- (a) The sequence  $\{a_n\} = \{1/n\}$  is bounded and nonincreasing, since

$$|a_n| \leq 1 \quad \text{and} \quad a_n = \frac{1}{n} \geq \frac{1}{n+1} = a_{n+1} \quad \forall n \in \mathbb{N}.$$

So, it must converge by Thm 7.5 with  $\lim_{n \rightarrow \infty} a_n = 0$ .



# Section 7.2

## Series and Convergence

### (級數與收斂)



## Def. (Partial Sums of a Series)

- (a) An infinite series (無窮級數) of real numbers is of the form

$$\sum a_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

- (b) For each  $n \in \mathbb{N}$ , the  $n$ th partial sum (第  $n$  個部分和) of  $\sum a_n$  is defined by

$$S_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n.$$



## Def. (Convergence of a Series)

- (a) We say that  $\sum a_n$  converges if the sequence  $\{S_n\}$  converges with  $\lim_{n \rightarrow \infty} S_n = S$ . In this case,  $S$  is called the sum of the series and write  $S = \sum a_n$ .
- (b) We say that  $\sum a_n$  diverges if the sequence  $\{S_n\}$  diverges.



## Two Questions

- ① Does a given series converge or diverge?
- ② What is the sum of a *convergent* series?

**Note:** these questions are not always easy to answer, especially  
the second one. (通常需要藉助數值方法求得近似和!)



# Example 1: (判斷級數的收斂或發散)

Determine the convergence for the series

$$(a) \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$(b) \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

$$(c) \sum_{n=1}^{\infty} 1$$

Sol: (a)  $\sum_{n=0}^{\infty} S_n = \sum_{i=1}^n \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = \frac{\frac{1}{2}(1 - (\frac{1}{2})^n)}{1 - \frac{1}{2}}$   
 $= \frac{2^n - 1}{2^n} \quad \forall n \in \mathbb{N}.$

∴  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{2^n - 1}{2^n} = 1 \text{ converges.}$



$$(b) \quad \text{if } S_n = \sum_{i=1}^n \left( \frac{1}{i} - \frac{1}{i+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1} \quad \forall n \in \mathbb{N}.$$

$$\therefore \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \rightarrow \infty} S_n = \frac{1}{\infty} \cdot \text{converges.}$$

$$(c) \quad \text{if } S_n = \sum_{i=1}^n 1 = n \quad \forall n \in \mathbb{N}.$$

$\therefore \sum_{n=1}^{\infty} 1$  diverges because  $\lim_{n \rightarrow \infty} S_n = \infty$ .

Remark: Note: ~~A~~ The series of the form (See Example 1(b))

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots$$

is called a telescoping series.



Example 4: Find the sum of the telescoping series

$$\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$$

Sol: May write  $\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)(2n+1)}$

$$= \sum_{n=1}^{\infty} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right).$$

$$\Rightarrow S_n = \sum_{i=1}^n \left( \frac{1}{2i-1} - \frac{1}{2i+1} \right) = \left( 1 - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{5} - \frac{1}{7} \right) + \dots + \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= 1 - \frac{1}{2n+1} \quad \text{The } n \in \mathbb{N}.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \lim_{n \rightarrow \infty} S_n = 1. \text{ converges. } *$$



# Geometric Series (幾何級數)

For  $a \neq 0$  and ratio (公比)  $r \neq 0$ , the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \dots .$$

## Thm 7.6 (幾何級數的收斂性)

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$



### Example 3 (Thm 7.6 的例子)

(a)  $\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{3}{1 - (1/2)} = 6$  by Thm 7.6.

(b)  $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$  diverges because the ratio  $|r| = |3/2| = 3/2 \geq 1$ .



## Thm 7.7 (Properties of Infinite Series)

Suppose that  $\sum a_n = A$  and  $\sum b_n = B$  are convergent series.

①  $\sum(c \cdot a_n) = c \cdot \left( \sum a_n \right) = c \cdot A$  for all  $c \in \mathbb{R}$ .

②  $\sum(a_n \pm b_n) = \left( \sum a_n \right) \pm \left( \sum b_n \right) = A \pm B$ .

③ In general, we know that

$$\sum(a_n b_n) \neq \left( \sum a_n \right) \left( \sum b_n \right), \quad \sum \left( \frac{a_n}{b_n} \right) \neq \frac{\sum a_n}{\sum b_n}.$$



## Thm 7.8

If  $\sum a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

(收斂級數第  $n$  項所形成的序列必定趨近 0!)

**pf:** If the series  $\sum a_n = S$  converges, then we know that

$$\lim_{n \rightarrow \infty} S_n = S = \lim_{n \rightarrow \infty} S_{n-1},$$

where  $S_n = \sum_{i=1}^n a_i$  is the  $n$ th partial sum of  $\sum a_n$ . So, we immediately obtain

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (S_n - S_{n-1}) = S - S = 0.$$



## Thm 7.9 (The $n$ th Term Test; 第 $n$ 項測試法)

If  $\lim_{n \rightarrow \infty} a_n \neq 0$ , the series  $\sum a_n$  diverges.

(若第  $n$  項不趨近到 0, 則級數必定發散!)

**Note:** 此定理為 Thm 7.8 的反敘述，常用於判斷級數的發散性。



## Example 5: (Thm 7.9 の 例)

(a)  $\sum_{n=0}^{\infty} 2^n$  diverges because  $\lim_{n \rightarrow \infty} 2^n = \infty$ .

(b)  $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$  diverges, since  $\lim_{n \rightarrow \infty} \frac{n!}{2(n!) + 1} = \frac{1}{2} \neq 0$ .

(c) For the series  $\sum_{n=1}^{\infty} \frac{1}{n}$ ,  $a_n = \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

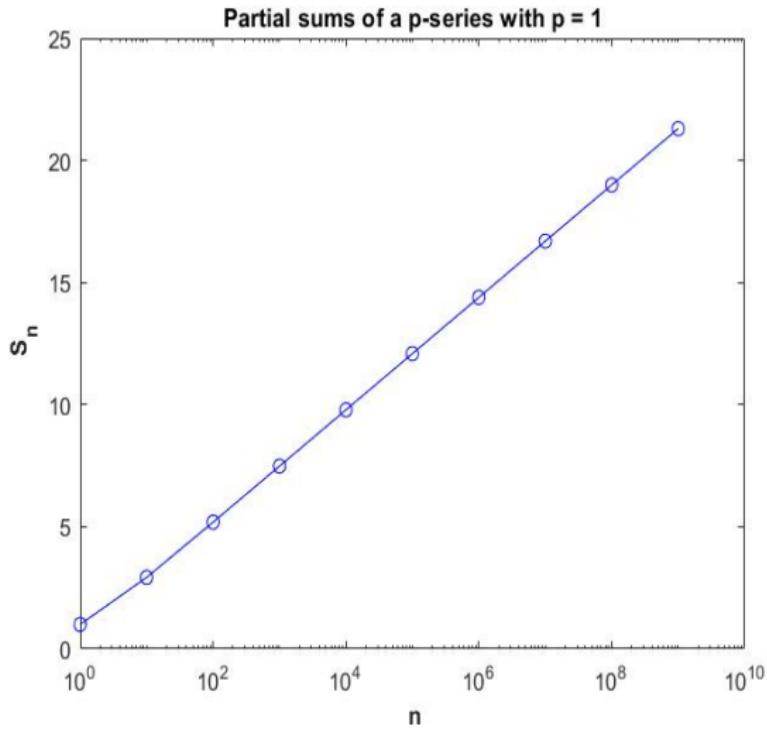
But  $\sum \frac{1}{n}$  diverges ! (See Sec. 7.3)

Note:  $\sum_{n=1}^{\infty} \frac{1}{n}$  is called a harmonic series.

(無窮和の反則)



# Divergence of Harmonic Series (示意圖)



# Section 7.3

## The Integral and Comparisons Tests (積分與比較測試法)



## Thm 7.10 (The Integral Test; 積分測試法)

If  $f$  is positive, conti. and  $\searrow$  on  $[1, \infty)$ , and  $a_n = f(n) \quad \forall n \in \mathbb{N}$ ,  
then  $\sum_{n=1}^{\infty} a_n$  and  $\int_1^{\infty} f(x) dx$  both converge or both diverge.

**Note:** 上述定理只說明級數與瑕積分是同收同發，但並沒有說明兩者相等喔！



## Example 2:

Apply Integral Test to  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ .

Sol: Let  $f(x) = \frac{1}{x^2+1}$  for  $x \geq 1$ .

$\Rightarrow f$  is positive, conti. and  $\downarrow$  on  $[1, \infty)$ .

$$\begin{aligned}\int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x^2+1} dx = \lim_{b \rightarrow \infty} \left( \tan^{-1} x \Big|_1^b \right) \\ &= \lim_{b \rightarrow \infty} \left( \tan^{-1} b - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}\end{aligned}$$

$\therefore \sum a_n = \sum f(n) = \sum \frac{1}{n^2+1}$  converges by ~~Integral~~  
by Integral Test.



## Def. ( $p$ -級數的定義)

(a)  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is called a  $p$ -series ( $p$ -級數) with  $p > 0$ .

(b) If  $p = 1$ , then  $\sum_{n=1}^{\infty} \frac{1}{n}$  is called a harmonic series (調和級數).



### Thm 7.11 ( $p$ -級數的收斂與發散)

The  $p$ -series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges for  $p > 1$ , and diverges for  $0 < p \leq 1$ .

**Note:** 此定理只探討  $p$ -級數的收斂性，但無法由此求出該級數的和！



If: let  $f(x) = \frac{1}{x^p} = x^{-p}$  for  $x \geq 1$ . Then

$$f'(x) = (-p)x^{-p-1} = \frac{-p}{x^{p+1}} < 0 \quad \forall x \in [1, \infty)$$

$\Rightarrow f$  is positive, conti. and  $\downarrow$  on  $[1, \infty)$ .

From Thm 6.5, we know that

$$\int_1^\infty \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 < p \leq 1. \end{cases}$$

So,  $\sum_{n=1}^{\infty} \frac{1}{n^p}$   $\begin{cases} \text{conv. if } p > 1 \\ \text{div. if } 0 < p \leq 1. \end{cases}$  by Integral Test



Example 3: (p-級數の例)

(a)  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges with  $p=1$

(b)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges with  $p=2 > 1$ .



## 計算 $p$ -級數的和 (檔名: p\_series.m)

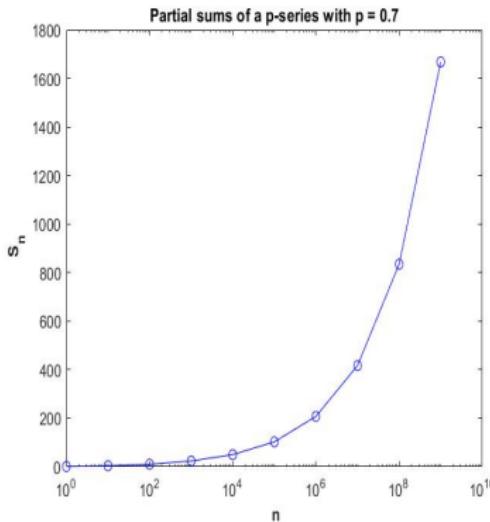
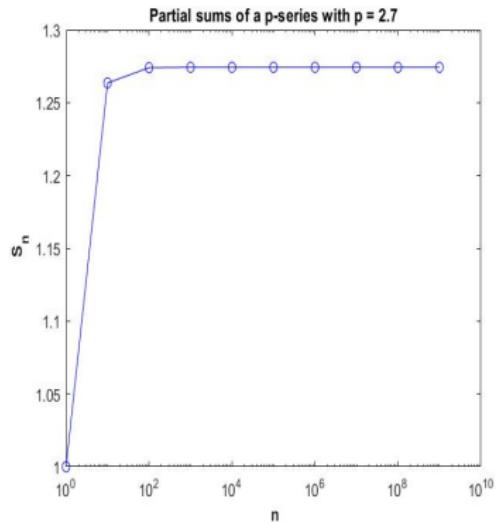
```
p = 2.7; data = [];
for k = 0:8
    N = 10^k;
    n = 1:N;
    S_N = sum(1./(n.^p));
    data = [data;N S_N];
end
semilogx(data(:,1),data(:,2),'bo-');
title('Partial sums of a p-series with p = 2.7');
xlabel('\bf n');
ylabel('\bf S_n');
```

**Note:**  $\sum_{n=1}^{\infty} \frac{1}{n^{2.7}} \approx 1.274265.$



# 程式執行結果 (承上例)

下列圖形顯示  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  的收斂與發散，其中  $p = 2.7$  和  $p = 0.7$ ：



80

Example 4 : Determine whether  $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

converges or diverges.

Sol: Let  $f(x) = \frac{1}{x \ln x}$  for  $x \geq 2$ .

$\Rightarrow f$  is positive and conti. on  $[2, \infty)$ .

$$\text{Since } f'(x) = \frac{d}{dx} \left( \frac{1}{x \ln x} \right) = \frac{-(\ln x + 1)}{(x \ln x)^2} < 0$$

for  $x \geq 2$ ,  $f$  is ↓ on  $[2, \infty)$ .



$$\therefore \int_2^{\infty} f(x) dx = \int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{b \rightarrow \infty} \left[ \ln(\ln x) \right]_2^b.$$

$$= \lim_{b \rightarrow \infty} [\ln(\ln b) - \ln(\ln 2)] = \infty.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges}$$

by Integral Test.  $\times$



## Thm 7.12 (Direct Comparison Test; 直接比較法)

If  $0 < a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then

- ①  $\sum b_n$  converges  $\implies \sum a_n$  converges, and
- ②  $\sum a_n$  diverges  $\implies \sum b_n$  diverges.

口訣:

- (1) 大的級數收斂保證小的級數也收斂!
- (2) 小的級數發散保證大的級數也發散!



Example 5: Determine the conv. or div. of

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Sol:  $\because 2+3^n \geq 3^n$

Then,

$$\therefore a_n = \frac{1}{2+3^n} \leq \frac{1}{3^n} = b_n \quad \text{Then } n \in \mathbb{N}.$$

Since  $\sum b_n = \sum \left(\frac{1}{3}\right)^n$  conv. with  $M = \frac{1}{3} < 1$

$\sum a_n = \sum \frac{1}{2+3^n}$  conv. by Direct Comparison Test.



Ex 6 Example 6: Determine the conv. or div.

of  $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ .

Sol:  $\because 2\sqrt{n} \leq n$  ~~when~~ for  $n \geq 4$ .

$$\therefore b_n = \frac{1}{2+\sqrt{n}} \geq \frac{1}{n} = a_n \text{ for } n \geq 4.$$

So,  $\sum b_n = \sum \frac{1}{2+\sqrt{n}}$  div. because  $\sum a_n = \sum \frac{1}{n}$  div.



## Thm 7.13 (Limit Comparison Test; 極限比較法)

Let  $a_n, b_n > 0 \quad \forall n \in \mathbb{N}$  with  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ .

- ① If  $0 < L < \infty$ , then  $\sum a_n$  and  $\sum b_n$  both converge or both diverge.
- ② If  $L = 0$ , then  $\sum b_n$  converges  $\Rightarrow \sum a_n$  converges, and  $\sum a_n$  diverges  $\Rightarrow \sum b_n$  diverges.
- ③ If  $L = \infty$ , then  $\sum a_n$  converges  $\Rightarrow \sum b_n$  converges, and  $\sum b_n$  diverges  $\Rightarrow \sum a_n$  diverges.



Example 8: Determine the conv. or div. of  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ .

Sol: Let  $a_n = \frac{\sqrt{n}}{n^2+1}$  and  $b_n = \frac{1}{n^{3/2}}$  then,

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{n}}{n^2+1} \right) n^{3/2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1 > 0$$

$\therefore a_n$  and  $\sum b_n = \sum \frac{1}{n^{3/2}}$  conv. with  $p = 3/2 > 1$ .

$\therefore \sum a_n$  conv. by Limit Comparison Test.



Example 9: Determine the conv. or div. of  $\sum_{n=1}^{\infty} \frac{n^2}{4n^3+1}$ .

Sol: Let  $a_n = \frac{n^2}{4n^3+1}$  and  $b_n = \frac{2^n}{n^2}$  then  $N$ .

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left( \frac{\frac{n^2}{4n^3+1}}{\frac{2^n}{n^2}} \right) = \frac{1}{4} > 0,$$

and  $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$  div. by the  $n$ th term test.

$\therefore \sum a_n = \sum \frac{n^2}{4n^3+1}$  div. by Limit Comparison



Test



# Section 7.4

## Other Convergence Tests

### (其他收斂測試法)



## Alternating Series (交錯級數)

An alternating series is of the form

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots,$$

where the  $n$ th term  $a_n > 0$  for all  $n \in \mathbb{N}$ .

## Questions

- Does the alternating series *always* converge?
- How to estimate the sum  $S$  of a *convergent* alternating series?



## Thm 7.14 (Alternating Series Test; 交錯級數測試法)

If the sequence of **positive** terms  $\{a_n\}$  satisfies

- ①  $\exists M > 0$  s.t.  $a_n \geq a_{n+1}$  for all  $n > M$ , and
- ②  $\lim_{n \rightarrow \infty} a_n = 0$ ,

then the alternating series  $S = \sum(-1)^{n+1}a_n$  converges with the property

$$|S_n - S| \leq a_{n+1} \quad \forall n \in \mathbb{N}.$$



Example 1: Determine the conv. or div. of  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ .

Sol. Let  $a_n = \frac{1}{n} > 0 \forall n \in \mathbb{N}$ . Since  $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$

then and  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , it follows

that  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$  conv. by Alternating Series

Test.



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Example 2: Determine the conv. or div. of

$$\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^{n-1}} = \sum_{n=3}^{\infty} (-1)^{n-1} \frac{n}{2^{n-1}}$$

Sol: Let  $a_n = \frac{n}{2^{n-1}} > 0 \forall n \in \mathbb{N}$  and consider  $f(x) = \frac{x}{2^{x-1}}$   
 for  $x \geq 3$ . Then  $a_n = f(n) \forall n \in \mathbb{N}$ .

$$\text{and } f'(x) = \frac{d}{dx} (x 2^{1-x}) = 2^{1-x} + x (-\ln 2) \cdot 2^{1-x},$$

$$= 2^{1-x} (1 - x \ln 2) < 0 \text{ for } x \geq 3.$$



$\Rightarrow f$  is  $\downarrow$  on  $[V, \infty)$  and hence

$$a_n \geq a_{n+1} \quad \text{for } n \geq 3 \quad \text{and hence} \quad - (*)$$

Moreover,  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x}{2^{x-1}} = \lim_{x \rightarrow \infty} \frac{1}{(\ln 2) 2^{x-1}} = 0$

by L'Hopital's Rule. So,  $\lim_{n \rightarrow \infty} a_n = 0$

by Thm 7.7 and  $a_n = f(n)$   $\forall n \in \mathbb{N}$ .

From (\*) and (\*\*)  $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2^{n-1}}$  Conv.

by Alternating Series Test.



## Def. (Types of Convergence for a Series)

- (1)  $\sum a_n$  is absolutely convergent (絕對收斂) if  $\sum |a_n|$  converges.
- (2)  $\sum a_n$  is conditionally convergent (條件收斂) if  $\sum a_n$  converges, but  $\sum |a_n|$  diverges.



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## Thm 7.16 (絕對收斂保證原級數收斂)

$\sum |a_n|$  converges  $\implies \sum a_n$  converges.



$\nexists$ :  $0 < -|a_n| \leq a_n \leq |a_n| \quad \forall n \in \mathbb{N}$ .

$$\therefore 0 \leq a_n + |a_n| \leq 2|a_n| \quad \forall n \in \mathbb{N}.$$

Since  $\sum a_n$  conv., we see that  $\sum(a_n + |a_n|)$  conv.

With  $a_n = (a_n + |a_n|) - |a_n| \quad \forall n \in \mathbb{N}$ , we further

conclude that  $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$  conv.



## Example (Thm 7.16 的反例)

- The series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges by the Alternating Series Test, but  $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$  is a **divergent  $p$ -series with  $p = 1$ !**
- In fact, any conditionally convergent series gives a counterexample of Thm 7.16.



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Example 6: Determine the conv. or div. of each series.

Which is the ~~absolut~~ absolutely or conditionally convergent?

$$(a) \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n}$$

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^{n(n+1)/2}}{3^n}$$

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$$



Sol: (a) Let  $a_n = \frac{(-1)^n n!}{2^n}$  for  $n=0,1,2,\dots$

$$\Rightarrow |a_n| = \frac{n!}{2^n} \geq \frac{2^n}{2^n} = \frac{1}{2} \quad \text{for } n \in \mathbb{N}.$$

$$\Rightarrow \lim_{n \rightarrow \infty} |a_n| \geq \frac{1}{2} \Rightarrow \lim_{n \rightarrow \infty} |a_n| \neq 0.$$

$\Rightarrow \lim_{n \rightarrow \infty} a_n \neq 0$  by Thm 7.4 (Absolute Value Thm)

$\Rightarrow \sum a_n$  diverges by the nth term Test.



(b) Let  $a_n = \frac{2(-1)^{n(n+1)/2}}{3^n}$  Then  $|a_n| = \frac{1}{3^n} \forall n \in \mathbb{N}$ .

$$\Rightarrow \sum |a_n| = \sum \left(\frac{1}{3}\right)^n \text{ conv. with ratio } r = \frac{1}{3}.$$

$\Rightarrow \sum a_n$  ~~is absolutely convergent~~  
is absolutely convergent.

~~∴~~  $\Rightarrow \sum a_n$  converges by Thm 7.16.



(c) Let  $a_n = \frac{(-1)^n}{\ln(n+1)}$  Then  $a_n > 0 \forall n \in \mathbb{N}$ ,

$a_{n+1} \in Q_n$  Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

So,  $\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$  conv. by Alternating Series Test.

Series Test.

But  $\sum |a_n| = \sum \frac{1}{\ln(n+1)}$  diverges because

$\frac{1}{\ln(n+1)} > \frac{1}{n}$  Then and  $\sum \frac{1}{n}$  diverges.

Hence,  $\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$  is conditionally convergent.



## Thm 7.17 (The Ratio Test; 比值法)

Let  $a_n \neq 0 \quad \forall n \in \mathbb{N}$  with  $\rho = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ .

①  $\rho < 1 \implies \sum a_n$  converges absolutely.

②  $\rho > 1$  or  $\rho = \infty \implies \sum a_n$  diverges.

③  $\rho = 1 \implies$  the test is inconclusive.



Example 8 : (Ratio Test of 13.1 3)

$$(a) \sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$$

Converges because by Ratio Test

$$\text{since } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^2 2^{n+2}}{3^{n+1}} \times \frac{3^n}{n^2 2^{n+1}} \right]$$

$$= \lim_{n \rightarrow \infty} \left[ \frac{2}{3} \frac{(n+1)^2}{n^2} \right] = \frac{2}{3} < 1.$$

$$(b) \sum_{n=1}^{\infty} \frac{n^n}{n!} \text{ diverges by Ratio Test, since}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right]$$

$$= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e > 1.$$



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### Example 9 (Ratio Test 失敗の例)

Determine the conv. or div. of  $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ .

Sol: Let  $a_n = \frac{\sqrt{n}}{n+1} \quad \forall n \in \mathbb{N}$ . Then  $a_n > 0 \quad \forall n \in \mathbb{N}$ .

$$\textcircled{1} \quad \lim_{n \rightarrow \infty} \frac{|(-1)^{n+1} a_{n+1}|}{|(-1)^n a_n|} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left( \frac{\sqrt{n+1}}{\sqrt{n}} \times \frac{n+1}{n} \right) = 1$$

∴ The Ratio Test fails.

\textcircled{2} Consider  $f(x) = \frac{\sqrt{x}}{x+1}$  for  $x > 1$ . Then  $a_n = f(n), \forall n \in \mathbb{N}$ .

Since  $f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2} < 0$  for  $x > 1$ ,  $a_n > a_{n+1}$  for  $n > 1$ .

Per-Duet



Moreover, we also have  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n+1} = 0$ .

$\Rightarrow \sum (-1)^n a_n$  converges by Alternating Series Test.

③  $|\sum (-1)^n a_n| = \sum a_n = \sum \frac{\sqrt{n}}{n+1}$  diverges because

$\frac{\sqrt{n}}{n+1} \geq \frac{1}{2\sqrt{n}}$  then and  $\sum \frac{1}{2\sqrt{n}}$  diverges with  $p = \frac{1}{2}$ .

From ② and ③, we know that  $\sum (-1)^n a_n$  is

converge conditionally convergent.



## Thm 7.18 (The Root Test; 根式法)

Let  $a_n \neq 0 \quad \forall n \in \mathbb{N}$  with  $\rho = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} (|a_n|)^{\frac{1}{n}}$ .

①  $\rho < 1 \implies \sum a_n$  converges absolutely.

②  $\rho > 1$  or  $\rho = \infty \implies \sum a_n$  diverges.

③  $\rho = 1 \implies$  the test is inconclusive.



## Example 10: (Root Test の 例)

$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$  converges absolutely by Root Test, since  $\lim_{n \rightarrow \infty} \left( \frac{e^{2n}}{n^n} \right)^{\frac{1}{n}}$

$$= \lim_{n \rightarrow \infty} \frac{e^2}{n} = 0 < 1.$$

$\Rightarrow \sum \frac{e^{2n}}{n^n}$  converges by Thm 7.16.



# Section 7.5

## Taylor Polynomials and Approximations

### (泰勒多項式與近似)



## Def. (Taylor and Maclaurin Polynomials)

Suppose that  $f$  has  $n$  derivatives at  $c \in \text{dom}(f)$ .

(1) A polynomial of the form

$$\begin{aligned}P_n(x) &= \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k \\&= f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n\end{aligned}$$

is called the  $n$ th Taylor poly. ( $n$  階泰勒多項式) for  $f$  at  $c$ .

(2) If  $c = 0$ , then  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$  is called the  $n$ th

Maclaurin poly. ( $n$  階馬克勞林多項式) for  $f$ .



Example 3: Find the  $n$ th MacLaurin Poly poly. for  $f(x) = e^x$ .

Sol:  $\stackrel{0}{\circ} f^{(k)}(x) = e^x \quad \forall k \in \mathbb{N}$

$\stackrel{0}{\circ} f^{(k)}(0) = e^0 = 1 \quad \forall k \in \mathbb{N}$  and hence the  $n$ th Mac

MacLaurin poly. is  $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^n$

$$= \sum_{k=0}^n \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



Example 4: Find Taylor poly.  $P_0, P_1, P_2, P_3$  and  $P_4$  for  $f(x) = \ln x$   
at  $c=1$ .

Sol:  $f'(x) = \frac{1}{x} = x^{-1}$ ,  $f''(x) = (-1)x^{-2}$ ,  $f'''(x) = (-1)2!x^{-3}$ ,  $f^{(4)}(x) = (-1)3!x^{-4}$ ,



$$\Rightarrow f(x) = (-1)^{k-1} (k-1)! x^{-k} \quad \forall k \in \mathbb{N}$$

$$\Rightarrow P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (k-1)!}{k!} (x-1)^k, \quad \left( \because f^{(0)}(1) = 0 \right)$$

$\sum_{k=1}^n \frac{(-1)^{k-1}}{k} (x-1)^k$  is the nth Taylor poly. for  $f$  at  $c=1$ .

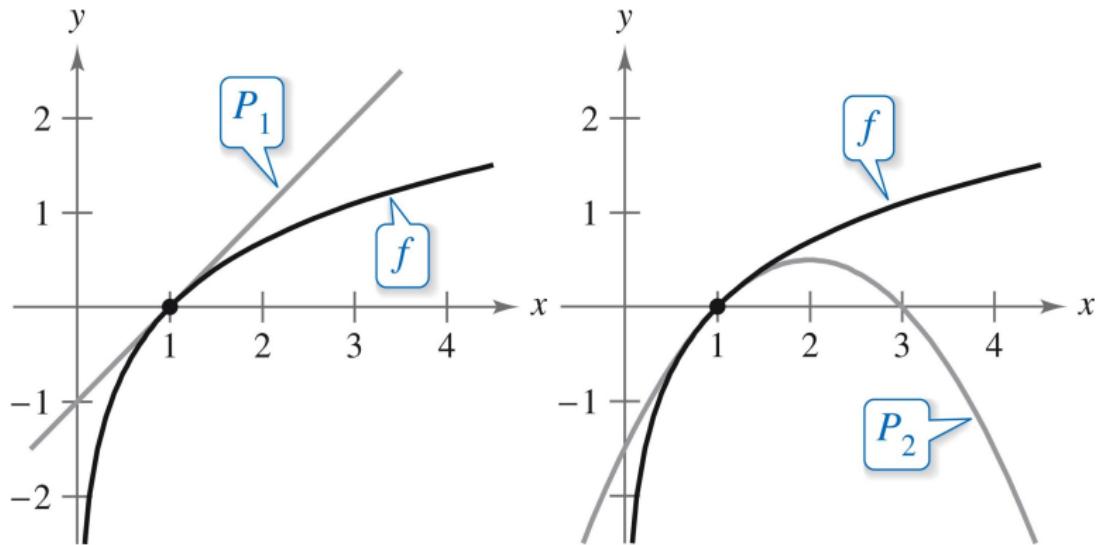
In particular,  $P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$ .

Note:  $\underline{P}_4(1.1) \approx 0.0953083$  and  $\underline{f}(1.1) = \ln(1.1) \approx 0.0953102$ .

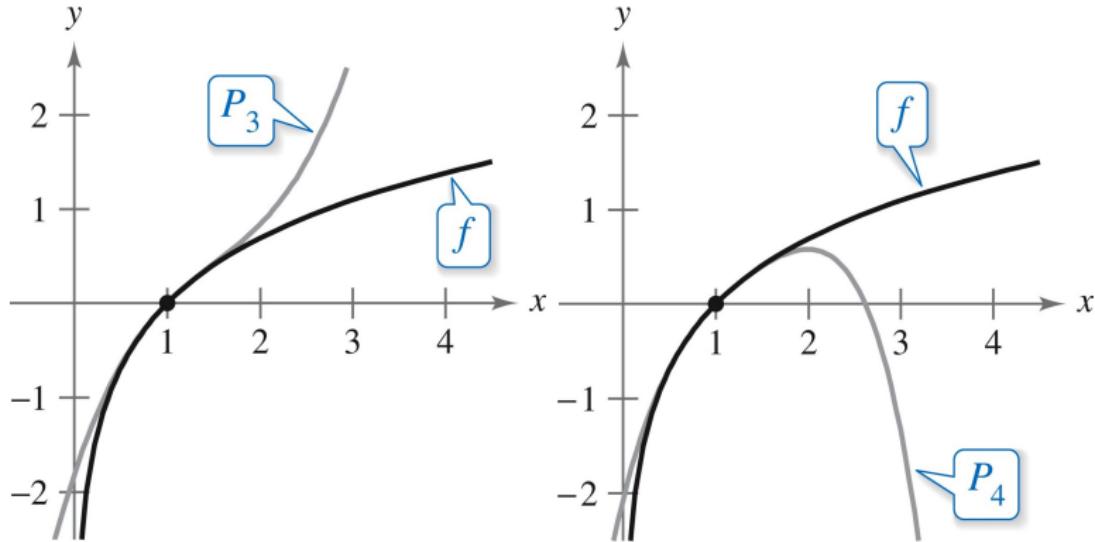
$\Rightarrow$  6 significant digits! (六個有效位數)



# 多項式 $P_1(x)$ 和 $P_2(x)$ 的示意圖 (承上例)



# 多項式 $P_3(x)$ 和 $P_4(x)$ 的示意圖 (承上例)



Example 5: Find the MacLaurin poly. for  $f(x) = \cos x$ , and use  $P_6(0, 1)$  to approximate  $\cos(0, 1)$ .

Sol:  $f'(x) = -\sin x$ ,  $f''(x) = -\cos x$ ,  $f'''(x) = \sin x$ ,  $f^{(4)}(x) = \cos x$ ,

$$\Rightarrow f(0) = 1, f''(0) = -1, f^{(4)}(0) = 1 \quad \cancel{\Rightarrow} \frac{f^{(2k)}}{2k+1} = 1$$

$$\Rightarrow f^{(2k)}(0) = (-1)^k \text{ for } k=0, 1, 2, \dots$$

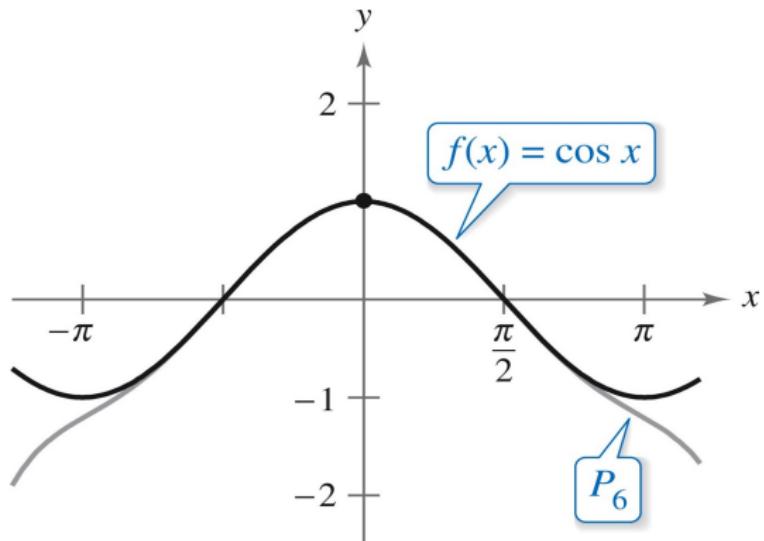
$\Rightarrow$  The MacLaurin Series for  $f$  is  $P_n(x) = \sum_{k=0}^n \frac{(-1)^k}{(2k)!} x^{2k}$ .

In particular,  $P_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$ .

$$\Rightarrow \cos(0, 1) \approx P_6(0, 1) = 0.99504165 \quad (\text{9个有效位数!})$$



# 多項式 $P_6(x)$ 的示意圖 (承上例)



Example 6: Find the Taylor Poly.  $P_3(x)$  for  $f(x) = \sin x$  at  $c = \frac{\pi}{6}$ .

Sol:  $f\left(\frac{\pi}{6}\right) = \frac{1}{2}, f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, f''\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}, f'''\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$

$\Rightarrow P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) - \frac{1}{2(2!)}(x - \frac{\pi}{6})^2 - \frac{\sqrt{3}}{2(3!)}(x - \frac{\pi}{6})^3.$



## Def. (The Remainder of $P_n(x)$ )

Let  $f$  have  $(n + 1)$  derivatives on an interval  $I$  containing  $c$ .

- (1)  $R_n(x) \equiv f(x) - P_n(x)$  is called the remainder (剩餘項) associated with  $P_n(x)$ .
- (2)  $|R_n(x)| = |f(x) - P_n(x)|$  is the error associated with  $P_n(x)$ .



## Thm 7.19 (Taylor's Theorem; 泰勒定理)

If  $f$  has  $(n+1)$  derivatives on an interval  $I$  containing  $c$ , then  
 $\forall x \in I$ ,  $\exists z$  between  $x$  and  $c$  s.t.

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k + R_n(x) = P_n(x) + R_n(x),$$

where the Lagrange form of the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(\textcolor{red}{z})}{(n+1)!} (x - c)^{n+1}.$$



# Section 7.6

## Power Series

### (幕級數)



## Def. (以 $c$ 點為中心的冪級數)

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots$$

is called a power series (冪級數) centered at  $c \in \mathbb{R}$ .



## Three Types of Convergence for Power Series

For a power series  $\sum_{n=0}^{\infty} a_n(x - c)^n$ , you will see that

- ① it converges only at  $x = c$  or
- ② it converges only for  $|x - c| < R$  with  $R > 0$  or
- ③ it converges for all  $x \in \mathbb{R}$ .



Denote

$R$  = Radius of Convergence (收斂半徑),  
 $I$  = Interval of Convergence (收斂區間).

## Type I of Convergence

$\sum_{n=0}^{\infty} a_n(x - c)^n$  converges only at  $x = c$ .

$$\implies R = 0 \quad \text{and} \quad I = \{c\}.$$



Example 2: Find the radius of conv. of  $\sum_{n=0}^{\infty} n! x^n$ .

Sol: Let  $a_n = n! x^n \quad \forall n \in \mathbb{N}$ .

① If  $x=0$ , then  $a_n=0 \quad \forall n \in \mathbb{N} \Rightarrow \sum a_n = 0$  conv.

② If  $x \neq 0$ , then  $r = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right|$

$$= \lim_{n \rightarrow \infty} [(n+1)|x|] = \infty \text{ for } x \neq 0.$$

$\Rightarrow \sum a_n$  div. for  $x \neq 0$  by Ratio Test.

From ① and ②, we know that  $\sum a_n = \sum n! x^n$

conv. only at  $x=0$ .

Per-Duet



## Type II of Convergence

$\sum_{n=0}^{\infty} a_n(x - c)^n$  converges absolutely for  $|x - c| < R$ , and

diverges for  $|x - c| > R$ .

$$\implies R > 0 \quad \text{and} \quad I = (c - R, c + R).$$



Example 3: Find the radius of conv. of

$$\sum_{n=0}^{\infty} 3(x-2)^n.$$

Sol: Let  $a_n = 3(x-2)^n \quad \forall n \in \mathbb{N}.$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{3(x-2)^{n+1}}{3(x-2)^n} \right| = \lim_{n \rightarrow \infty} |x-2|^n \\ = |x-2|.$$

$\Rightarrow \sum a_n$  conv. absolutely for  $|x-2| < 1$  and  
div. for  $|x-2| > 1$  by Ratio Test.

$\therefore S_0, R=1$  and  $I=(1, 3)$ .



## Type III of Convergence

$\sum_{n=0}^{\infty} a_n(x - c)^n$  converges for all  $x \in \mathbb{R}$ .

$\implies R = \infty \quad \text{and} \quad I = (-\infty, \infty)$ .

### Example 4 (Type III 的例子)

Find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$



Example 83 Sol: Let  $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \forall n \in \mathbb{N}$ .

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \times \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1 \quad \forall x \in \mathbb{R}.$$

$\Rightarrow$  ~~converges absolutely~~  $\sum a_n = \sum \frac{(-1)^n x^{2n+1}}{(2n+1)!}$  conv.

absolutely  $\forall x \in \mathbb{R}$ .

So,  $R = \infty$  and  $I = (-\infty, \infty)$ .  $\times$



## Note

In Section 7.8, we will further show that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \forall x \in \mathbb{R}.$$



## Endpoint Convergence (端點收斂性)

Besides the interval of convergence  $I = (c - R, c + R)$  in Type II,  
we also have

$$I = [c - R, c + R), \quad (c - R, c + R] \quad \text{or} \quad [c - R, c + R]$$

for different endpoint convergence of a power series.

## Example 5 (端點收斂的例子)

Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$



Sol: Let  $a_n = \frac{x^n}{n}$  Then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \times \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} |x| \right) = |x|$$

$\Rightarrow \sum a_n = \sum \frac{x^n}{n}$  conv. absolutely for  $|x| < 1$

and div. for  $|x| > 1$  by Ratio Test.

$\Rightarrow$  the radius of conv. is  $R = 1$ .



①

for  $x=1$ ,  $\sum a_n = \sum \frac{1}{n}$  div. with  $p=1$ .

②

For  $x=-1$ ,  $\sum a_n = \sum \frac{(-1)^n}{n}$  conv. by Alternating

Series Test.

From ① and ②  $\Rightarrow I = [-1, 1]$  is the interval

of convergence.



Example 7: Find the interval of conv.

of  $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ .

Sol: Let  $a_n = \frac{x^n}{n^2}$   $\forall n \in \mathbb{N}$ .

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2} \times \frac{n^2}{x^n} \right| = |x|$$

$\Rightarrow \sum a_n = \sum \frac{x^n}{n^2}$  conv. for  $|x| < 1$  and div. for  $|x| > 1$

by Ratio Test.

$\therefore R = 1$  is the radius of convergence.



① For  $x = -1$ ,  $\sum a_n = \sum \frac{(-1)^n}{n^2}$  conv. by Alternating Series Test.

② For  $x = 1$ ,  $\sum a_n = \sum \frac{1}{n^2}$  conv. with  $p = 2 > 1$ .

From ① and ②  $\Rightarrow I = [-1, 1]$  is the interval of convergence.



For the cases of Type II or Type III, we consider a real-valued function  $f$  defined by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \quad \forall x \in I,$$

where  $I = (c - R, c + R)$  with  $R > 0$ , or  $I = (-\infty, \infty)$ .



## Two Questions

- Is  $f$  differentiable and integrable on the open interval  $I$ ?
- If yes, what are  $f'(x)$  and  $\int f(x) dx$ ?



## Thm 7.21 (冪級數的微分與積分公式)

Let  $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$  be well-defined on  $I = (c - R, c + R)$ .

- ① Term-by-term differentiation (逐項微分):

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}[a_n(x - c)^n] = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1} \quad \forall x \in I.$$

- ② Term-by-term integration (逐項積分):

$$\begin{aligned} \int f(x) dx &= \sum_{n=0}^{\infty} \left[ \int a_n(x - c)^n dx \right] \\ &= C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - c)^{n+1} \quad \forall x \in I, \end{aligned}$$

where  $C$  is the constant of integration.



## Remarks

- (1) The radii of convergence of  $f'(x)$  and  $\int f(x) dx$  are the same as that of  $f(x) = \sum a_n(x - c)^n$ .
- (2) But, their intervals of convergence may differ from  $f$ .

(微分和積分後的收斂半徑與  $f$  相同，但收斂區間略有不同!)



8  
8

Example 8: Let  $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ .

Find the interval of conv. ~~for~~ each of the following.

(a)  $\int f(x) dx$  (b)  $f'(x)$  (c)  $f''(x)$ .

Sol: Let  $a_n = \frac{x^n}{n}$  then. Then  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{n} x \right|$

$\Rightarrow$  ~~By~~ Ratio Test.  $\sum a_n = \sum \frac{x^n}{n}$  conv. for  $|x| < 1$  and div. for  $|x| > 1$

by Ratio Test.

Applying Thm 7.21, we know that

$$f'(x) = \sum_{n=1}^{\infty} \left( \frac{x^n}{n} \right)' = \sum_{n=1}^{\infty} x^{n-1} \text{ and } \int f(x) dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$

both have the same radius of conv. of  $R=1$ .



(a)  $\int f(x) dx = \frac{1}{2}x^2 + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$  conv. at  $x=1$  and  $x=-1$ .

$\therefore$  its interval of conv. is  $I = [-1, 1]$ .

DATE: / /

(b)  $\int f(x) dx = \sum_{n=1}^{\infty} \frac{x^n}{n}$  conv. at  $x=-1$ , but div. at  $x=1$

$\therefore$  its interval of conv. is  $I = (-1, 1)$ .

(c)  $\int f(x) dx = \sum_{n=1}^{\infty} x^{n-1}$  div. at  $x=\pm 1$

$\therefore$  its interval of conv. is  $I = (-1, 1)$ . 



# Section 7.7

## Representation of Functions by Power Series

(以冪級數作為函數的表示式)



## Geometric Power Series (幾何冪級數)

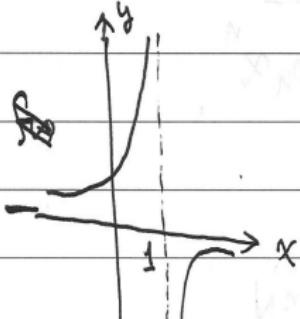
- The function  $f(x) = \frac{1}{1-x}$  is well-defined for  $x \neq 1$ .
- $f$  has a geometric power series centered at  $x = 0$ , i.e.,

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$$

for  $|x| < 1$ .

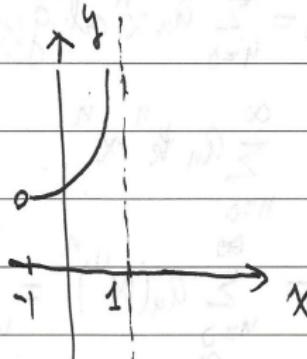


# 示意圖 (承上頁)



$$f(x) = \frac{1}{1-x}$$

$$\text{dom}(f) = (-\infty, 1) \cup (1, \infty)$$



$$\sum_{n=0}^{\infty} x^n \text{ conv. for } |x| < 1.$$



Example 1: Find a power series for  $f(x) = \frac{4}{2+x}$

Centered at  $x=0$ .

Sol:  $f(x) = \frac{4}{2+x} = \frac{2}{1+\frac{x}{2}} = \frac{2}{1-\left(-\frac{x}{2}\right)} = \sum_{n=0}^{\infty} 2\left(\frac{-x}{2}\right)^n$

for  $\left|\frac{-x}{2}\right| < 1$  or  $|x| < 2$

→ The interval of conv. is  $I = (-2, 2)$ .



Example 2: Find a power series for  $f(x) = \frac{1}{x}$  centered

at  $x=1$ .



$$\text{Sol: } f(x) = \frac{1}{x} = \frac{1}{1 - (-x+1)} = \sum_{n=0}^{\infty} (-x+1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \dots \quad \text{for } |x-1| < 1.$$

$\Rightarrow$  級數收斂域。The interval of conv. is  $I = (0, 2)$ .



## Operations with Power Series

Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} b_n x^n$  be well-defined.

① For any  $k \in \mathbb{R}$ ,  $f(kx) = \sum_{n=0}^{\infty} a_n (kx)^n = \sum_{n=0}^{\infty} (a_n k^n) x^n$ .

② For any  $N \in \mathbb{N}$ ,  $f(x^N) = \sum_{n=0}^{\infty} a_n (x^N)^n = \sum_{n=0}^{\infty} a_n x^{nN}$ .

③  $f(x) \pm g(x) = \left( \sum_{n=0}^{\infty} a_n x^n \right) \pm \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$ .



Example 3: Find a power series for  $f(x) = \frac{3x-1}{x^2-1}$

centered at  $\underline{x=0}$ .

Sol: Write the partial fraction decomposition for  $f$

$$\text{as } f(x) = \frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1} = \frac{2}{1-(-x)} - \frac{1}{1-x}.$$

$$= \sum_{n=0}^{\infty} 2(-x)^n - \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} [2(-1)^n - 1] x^n \text{ for } |x| < 1$$

$\Rightarrow$  The interval of conv. is  $I = (-1, 1)$ .



Example 4: Find a power series for  $f(x) = \ln x$

centered at  $\underline{x=1}$ .

Sol: From Example 2  $\Rightarrow \frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$ . for  $|x-1| < 1$ ,

$$\Rightarrow \ln x = \int \frac{1}{x} dx + C = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1} \text{ for } |x-1| < 1$$

by Thm 7.21.



For  $x \approx 1$ ,  $0 = \ln 1 = C + 0 \Rightarrow C = 0.$

$$\text{So, } \ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \dots \quad \text{for } |x-1| < 1.$$

*Fact:*  $\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^{n+1} \quad \text{for } 0 < x \leq 2.$



### Example 5\*: (補充題)

Find a power series for  $f(x) = \tan^{-1} x$ , centered at  $x=0$ .

Sol:  $\frac{1}{1+x^2} = \frac{1}{1-(x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$  for  $|x^2| < 1$  or  $|x| < 1$ .

$$\therefore \tan^{-1} x = \int \frac{dx}{1+x^2} + C = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } |x| < 1.$$

Note that  $0 = \tan^{-1} 0 = C + 0 \Rightarrow C = 0$ .

$$\text{So, } \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \text{ for } -1 \leq x \leq 1.$$



# Section 7.8

## Taylor and Maclaurin Series

### (泰勒級數與馬克勞林級數)



## Thm 7.22 (The Form of a Convergent Power Series)

If  $f$  is a real-valued function defined on  $I = (c - R, c + R)$  by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n \quad \forall x \in I,$$

then  $f$  has derivatives of all orders on  $I$ , and moreover, we have

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \forall n \in \mathbb{N},$$

with  $0! = 1$  and  $f^{(0)} = f$ .



$$\text{pf: } f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$$

$$\Rightarrow \underbrace{f(c)}_{=} = a_0 \quad \forall x \in I.$$

Applying Thm 7.21 again and again  $\Rightarrow$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$$

$$\Rightarrow \underbrace{f'(c)}_{=} = a_1 \quad \forall x \in I.$$

$$f''(x) = 2!a_2 + 3!(x-c) + 4 \cdot 3(x-c)^2 + \dots \quad \forall x \in I.$$

$$\Rightarrow \underbrace{f''(c)}_{=} = 2!a_2 \Rightarrow a_2 = \frac{f''(c)}{2!}.$$

⋮  
⋮

$$\text{By induction } \Rightarrow a_n(n!) = \underbrace{f^{(n)}(c)}_{\text{then N.}} \quad \forall n \in \mathbb{N}.$$

$$\Rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

then N.

Note:  $f$  has infinitely many derivatives on  $I$  !



## Def. (泰勒級數的定義)

Suppose that  $f$  has derivatives of all orders at  $c$ .

- (1) A power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots$$

is called the Taylor series (泰勒級數) for  $f$  at  $c$ .

- (2) If  $c = 0$ , a power series of the form  $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$  is called the Maclaurin series (馬克勞林級數) for  $f$ .



Example 1: Let  $f(x) = \sin x$ .

- (a) Find the Maclaurin series for  $f$ .  
(b) Determine the interval of convergence.



$$\underline{\text{Sof:}} \text{ (a)} \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad f^{(4)}(x) = \sin x, \quad \checkmark$$

$$\therefore f(0) = (-1)^k \text{ for } k=0,1,2,\dots$$

So, the MacLaurin Series for  $f(x) = \sin x$  is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$



(b)

$$\text{Let } a_n = \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \text{ Hung Yuan Fan}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1 \quad \forall x \in \mathbb{R}$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^n \text{ converges } \forall x \in \mathbb{R} \text{ by Ratio Test.}$$

$\therefore \Rightarrow I = (-\infty, \infty)$  is the interval of convergence.

Per-Duct



# Convergence of Taylor Series

Suppose that  $f$  has derivatives of all orders on an open interval  $I$  containing  $c$ . It follows from Taylor's Thm (Thm 7.19) that  
 $\forall x \in I, \exists z$  between  $x$  and  $c$  s.t.

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k + R_n(x) = P_n(x) + R_n(x),$$

where the Lagrange form of the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(\textcolor{red}{z})}{(n+1)!} (x - c)^{n+1}.$$



## Question

When does the Taylor series for  $f$  at  $c$  *always* converge to  $f$  on the open interval  $I$ ?



## Question

When does the Taylor series for  $f$  at  $c$  *always* converge to  $f$  on the open interval  $I$ ?

## Thm 7.23 (泰勒級數的收斂性)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n \text{ conv. on } I \iff \lim_{n \rightarrow \infty} R_n(x) = 0 \quad \forall x \in I.$$



Pf:  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$  converges on  $I$

$\Leftrightarrow \lim_{n \rightarrow \infty} P_n(x) = f(x)$ , since  $P_n(x)$  is the  $n$ th partial

Sum of the Taylor Series for  $f$  at  $c$ .

$$\Leftrightarrow \lim_{n \rightarrow \infty} [f(x) - R_n(x)] = f(x) \quad \text{by Taylor's Thm.}$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} P_n(x) = 0 \quad \forall x \in \mathbb{I}.$$



## Remark

In order to prove that

$$\lim_{n \rightarrow \infty} R_n(x) = \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(z)}{(n+1)!} (x - c)^{n+1} = 0 \quad \forall x \in I,$$

we often use the Squeeze Theorem and the following fact

$$\lim_{n \rightarrow \infty} \frac{|x - c|^{n+1}}{(n+1)!} = 0 \quad \forall x \in \mathbb{R}.$$



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$$\text{Thm: } \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \forall x \in \mathbb{R}$$

Pf: Let  $x \in \mathbb{R}$  be given arbitrarily.

① If  $|x| \leq 1$ , then  $0 \leq \frac{|x|^n}{n!} \leq \frac{1}{n!} \quad \forall n \in \mathbb{N}$ .

$\Rightarrow \lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0$  by Squeeze Thm and  $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$ .

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$$

② If  $|x| > 1$ , choose the smallest  $N \in \mathbb{N}$  s.t.  $|x| \leq N$ .

$$\Rightarrow 0 \leq \frac{|x|^n}{n!} = \frac{|x|}{n} \left[ \frac{|x|}{n-1} \cdot \frac{|x|}{n-2} \cdots \frac{|x|}{N} \right] \cdot \frac{|x|}{N-1} \cdots \frac{|x|}{2} \cdot |x|$$

$$\leq \frac{|x|}{n} \cdot \frac{|x|^{N-1}}{(N-1)!} = \frac{|x|^N}{n(N-1)!} \quad \text{for } n > N$$

Since  $\lim_{n \rightarrow \infty} \frac{|x|^N}{n(N-1)!} = 0$ , it follows from Squeeze Thm

$$\text{that } \lim_{n \rightarrow \infty} \frac{|x|^n}{n!} = 0 \Leftrightarrow \lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$$

From ① and ②, we conclude that  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0 \quad \forall x \in \mathbb{R}$ .



Example 2: Show that  $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$   $\forall x \in \mathbb{R}$ .

Sol:



② It follows from Taylor's Thm (Thm 7.19) that

$\forall x \in \mathbb{R}, \exists z$  between 0 and  $x$  s.t.

$$f(x) = \sin x = P_n(x) + R_n(x) \text{ with } R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} \cdot x^{n+1}$$

$$\Rightarrow |R_n(x)| \leq \frac{|f^{(n+1)}(z)| |x|^{n+1}}{(n+1)!} < \frac{|x|^{n+1}}{(n+1)!} \quad \forall x \in \mathbb{R}$$

Since  $f^{(n+1)}(x) = \pm \sin x$  or  $\pm \cos x$ ,



$\Rightarrow \lim_{n \rightarrow \infty} |R_n(x)| = 0 \quad \forall x \in \mathbb{R}$  by Squeeze Thm

$\Rightarrow \lim_{n \rightarrow \infty} R_n(x) = 0 \quad \forall x \in \mathbb{R}$ .

So, from (1), (2) and Thm 7.23  $\Rightarrow$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R}$$



## Useful Taylor or Maclaurin Series (1/2)

$$(1) \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } -1 < x < 1.$$

$$(2) \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ for } 0 < x \leq 2.$$

$$(3) e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ for } -\infty < x < \infty.$$

$$(4) \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ for } -\infty < x < \infty.$$



## Useful Taylor or Maclaurin Series (2/2)

$$(5) \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \text{ for } -\infty < x < \infty.$$

$$(6) \sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1} \text{ for } -1 \leq x \leq 1.$$

$$(7) \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \text{ for } -1 \leq x \leq 1.$$

(8) Binomial Series (二項級數) with  $k \in \mathbb{R}$ :

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n \quad \text{for } -1 < x < 1.$$



# Thank you for your attention!

