

Chapter 8

Conics, Parametric Equations, and Polar Coordinates

(圓錐曲線、參數方程式與極坐標)

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Section 8.1

Plane Curves and Parametric Equations

(平面曲線與參數方程式)

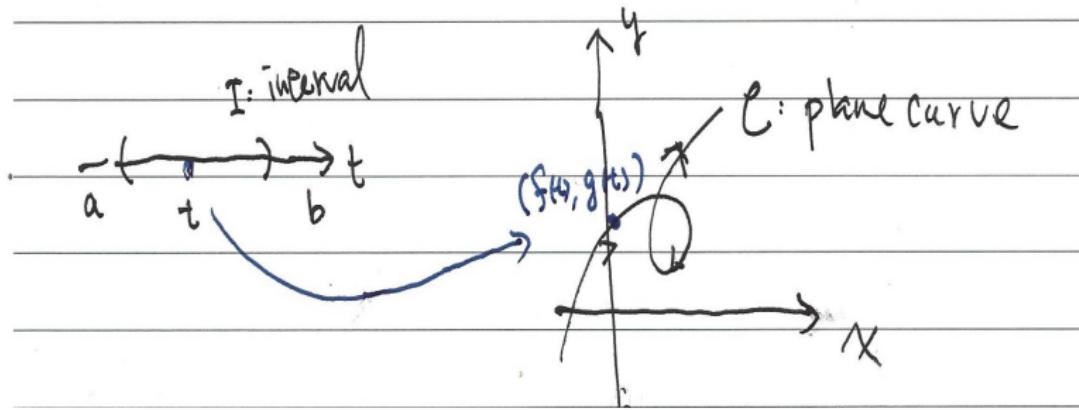


Def. (參數曲線的定義)

Let I be an interval. A plane curve \mathcal{C} is often defined by the graph of the parametric equations (參數方程式)

$$x = f(t) \quad \text{and} \quad y = g(t) \quad \forall t \in I,$$

where f and g are conti. functions of t , and t is a parameter (參數).



Example 1: Sketch the curve described by

$$x = f(t) = t^2 - 4 \quad \text{and} \quad y = g(t) = t/2$$

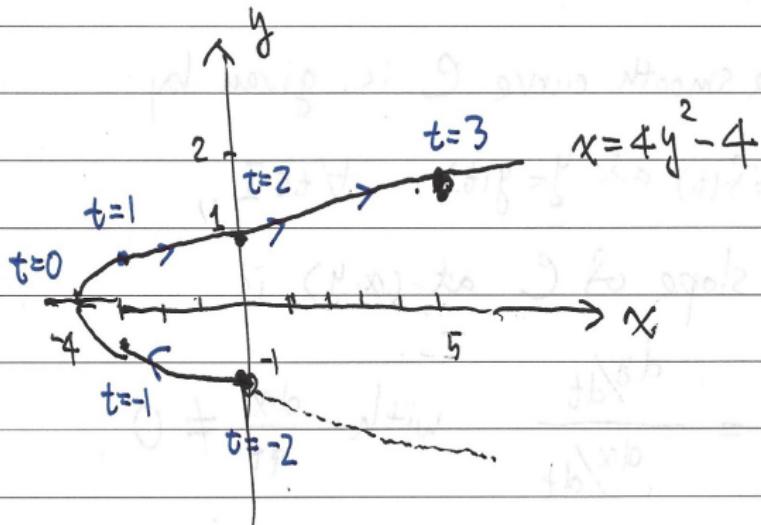
for $-2 \leq t \leq 3$.

Sol: $\because x = t^2 - 4$ and $y = t/2$

$$\therefore x = t^2 - 4 = (2y)^2 - 4 = 4y^2 - 4.$$



<u>t</u>	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	-1/2	0	1/2	1	3/2



Example 4: Sketch the curve represented by

$$x = f(\theta) = 3\cos\theta \text{ and } y = g(\theta) = 4\sin\theta$$

for $0 \leq \theta \leq 2\pi$.



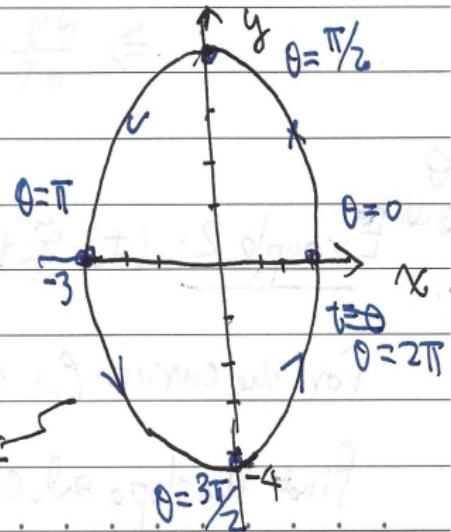
Sol: $\cos\theta = \frac{x}{3}$ and $\sin\theta = \frac{y}{4}$

$$\therefore 1 = \cos^2\theta + \sin^2\theta = \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \frac{x^2}{9} + \frac{y^2}{16}$$

is the rectangular equation of the given curve.

(直角坐标方程式)

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
x	3	0	-3	0	3
y	0	4	0	-4	0



$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$



Section 8.2

Parametric Equations and Calculus

(參數方程式及其微積分)



Thm 8.1 (參數曲線的微分)

If \mathcal{C} is a smooth curve defined by

$$x = f(t) \quad \text{and} \quad y = g(t) \quad \forall t \in I,$$

with $dx/dt = f'(t) \neq 0 \quad \forall t \in I$, then

- (1) the slope of \mathcal{C} at the point (x, y) is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \equiv m(t) \quad \forall t \in I.$$

- (2) the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d(dy/dt)/dt}{dx/dt} = \frac{m'(t)}{f'(t)} \quad \forall t \in I.$$



Example 1. Find $\frac{dy}{dx}$ for the curve given

by $x = \sin t$ and $y = \cos t$.

Sol: Applying Thm 8.1 $\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{-\cos t}$

$\Rightarrow \frac{dy}{dx} = -\tan t$ is a function of t .



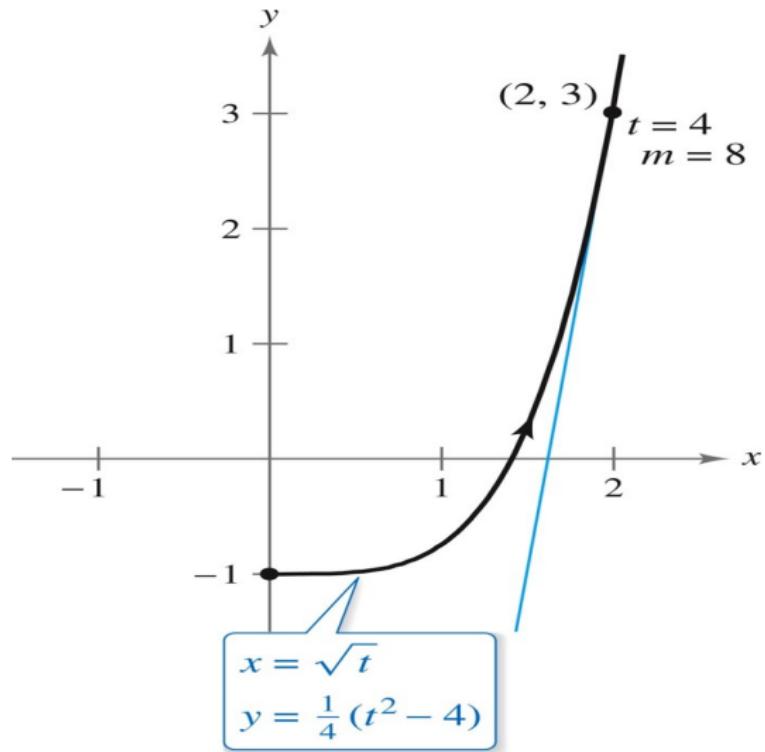
例題 2：(求參數曲線的斜率與凹凸性)

For the curve ℓ : $x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$ $\forall t \geq 0$,

find the slope and concavity at $(x, y) = (2, 3)$,



Example 2 的示意圖



Sol: Note that when $t=4$, $x=2$ and $y=3$.

① The slope at $(x,y) = (2,3)$ is

$$m = \frac{dy}{dx} \Big|_{t=4} = \frac{\frac{d}{dt}(\frac{1}{4}t^2 - 4)}{\frac{d}{dt}(t)} \Big|_{t=4} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} \Big|_{t=4}$$

$$= t^{3/2} \Big|_{t=4} = (4)^{3/2} = 8.$$

② The second derivative at $(2,3)$ is

$$\frac{d^2y}{dx^2} \Big|_{t=4} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \Big|_{t=4} = \frac{\frac{d}{dt}(t^{3/2})}{\frac{d}{dt}(t^{1/2})} \Big|_{t=4}$$

$$= \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} \Big|_{t=4} = 3t \Big|_{t=4} = 12 > 0.$$

So, the graph is concave upward (c.u.) at $(2,3)$.



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Example 3 (參看曲線在同一處有兩條切線)

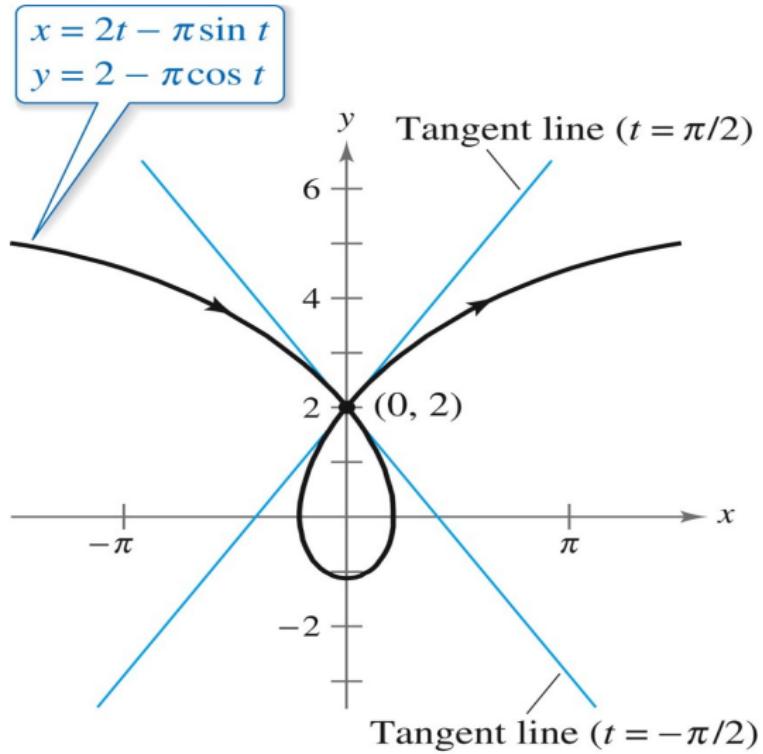
For the curve given by

$$C: x = 2t - \pi \sin t \text{ and } y = 2 - \pi \cos t,$$

find the equations of tangent lines at $(x,y) = (0,2)$,

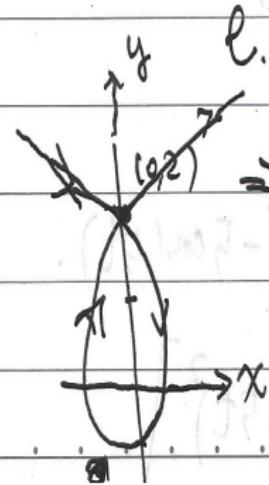


Example 3 的示意圖



Sol:

At $(x,y) = (0,2)$, $2 = y = 2 - \pi \cos t$ and $\cancel{2 = 2\sin t}$



$$0 = x = 2t - \pi \sin t$$

$$\begin{cases} 2t - \pi \sin t = 0 \\ \cos t = 0 \end{cases} \Rightarrow t = \frac{-\pi}{2} \text{ or } \frac{\pi}{2}.$$

The slope at $(x,y) \in C$ is



$$m = \frac{dy}{dx} = \frac{\frac{d}{dt}(2 - \pi \cos t)}{\frac{d}{dt}(2t - \pi \sin t)} = \frac{\pi \sin t}{2 - \pi \cos t}$$

When $t = -\frac{\pi}{2}$, $m = \left. \frac{dy}{dx} \right|_{t=-\frac{\pi}{2}} = \frac{-\pi}{2}$ and hence

the equation of tangent line at $(0, 2)$ is $y = \underline{\underline{\left(-\frac{\pi}{2} \right)x + 2}}$.

When $t = \frac{\pi}{2}$, $m = \frac{\pi}{2}$ and the equation of

the tangent line at $(0, 2)$ is $y = \underline{\underline{\frac{\pi}{2}x + 2}}$



Thm 8.2 (Arc Length in Parametric Form)

Let \mathcal{C} be a smooth curve defined by

$$x = f(t) \quad \text{and} \quad y = g(t) \quad \forall t \in I = [a, b].$$

If \mathcal{C} does not intersect itself on I , then the arc length of \mathcal{C} on I is given by

$$\begin{aligned}s &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + \left[\frac{y'(t)}{x'(t)}\right]^2} [x'(t)dt] \\&= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.\end{aligned}$$



Example 4: Find the arc length of the

epicycloid (外環線; 圓外旋轉線)

$$x = 5\cos t - \cos(5t) \text{ and } y = 5\sin t - \sin(5t)$$

for $0 \leq t \leq 2\pi$.

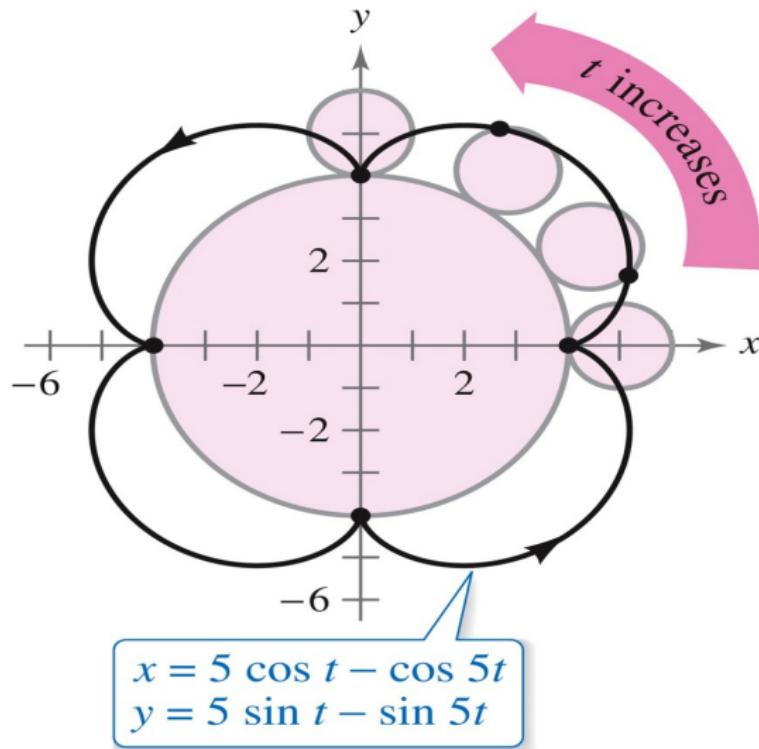
Sol: $\frac{dx}{dt} = -5\sin t + 5\sin(5t)$ and $\frac{dy}{dt} = 5\cos t - 5\cos(5t)$.

$$\frac{d}{dt} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] = (25) \left[(-\sin t + \sin 5t)^2 + (\cos t - \cos 5t)^2 \right].$$

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Example 4 的示意圖



$$= b\bar{y} \left(2 - 2 \sin t \sin 5t - 2 \cos t \cos 5t \right)$$

$$= (50) (1 - \cos 4t) = (50) (2 \sin^2 2t)$$

$$\Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(10) \sin^2 2t} = (10) \sin 2t \quad \text{for } 0 \leq t \leq \frac{\pi}{2}.$$

So, the arc length of the epicycloid is

$$s = 4 \int_0^{\pi/2} (10) \sin 2t dt = (40) \left(-\frac{1}{2} \cos 2t\right) \Big|_0^{\pi/2}$$

$$= (40) \left(-\frac{1}{2}\right) (-2) = 40$$



Useful Formulas

Recall the following identities for the sine and cosine functions:

$$① \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$② \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$③ \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$④ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$



Section 8.3

Polar Coordinates and Polar Graphs

(極坐標與極坐標圖形)



Def. (極坐標的定義)

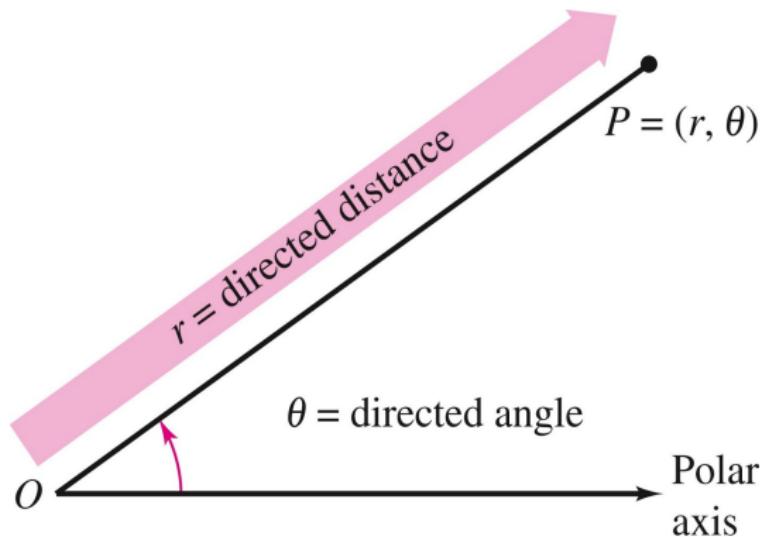
The polar coordinates (r, θ) of a point $P(x, y) \in \mathbb{R}^2$ is defined by

r = directed distance from the pole (極點) O to P .

θ = directed angle, counterclockwise from the polar axis (極軸)
to the line \overline{OP} .

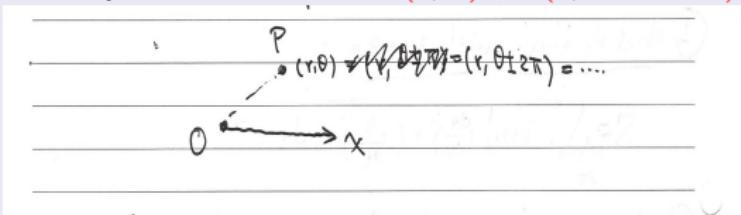


極坐標的示意圖 (承上頁)



Notes

- (1) The polar coordinates of the pole is $O = (0, \theta)$ for any $\theta \in \mathbb{R}$.
- (2) The polar coordinates (r, θ) and $(r, \theta + 2n\pi)$ represent the same point in \mathbb{R}^2 , i.e., $(r, \theta) = (r, \theta + 2n\pi) \quad \forall n \in \mathbb{Z}$.



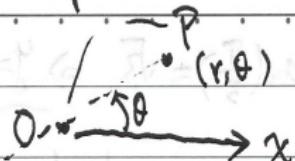
- (3) If $r > 0$, then $(-r, \theta) = (r, \theta + (2n+1)\pi) \quad \forall n \in \mathbb{Z}$.



示意圖 (承上頁)

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$$r > 0$$



$$(-r, \theta) = \cancel{(r, \theta)} (r, \theta \pm \pi) = (r, \theta \pm 3\pi) = \dots$$

($(-r, \theta)$ 和 (r, θ) 反對稱於極點 O)

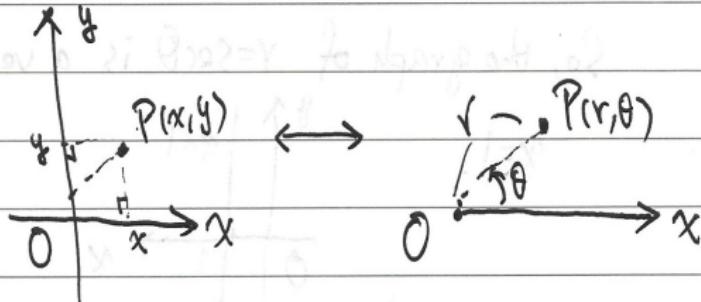
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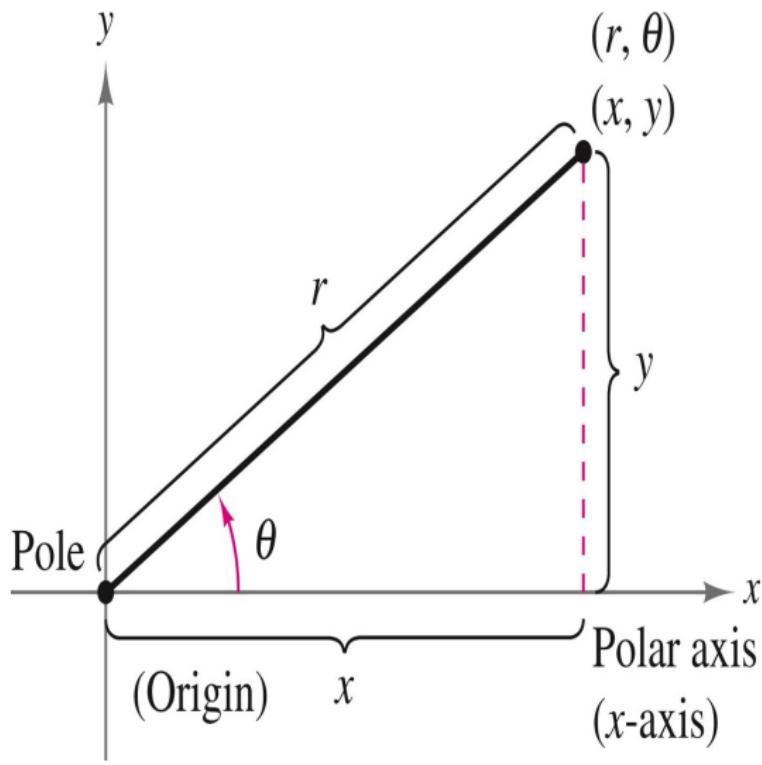
* Rectangular Coordinates vs. Polar Coordinates:

① Polar to Rectangular: $x = r \cos \theta$, $y = r \sin \theta$.

② Rectangular to Polar: $r^2 = x^2 + y^2$, $\tan \theta = \frac{y}{x}$.



直角坐標和極坐標的關係



* Polar Equations: (極坐標方程式)

- $r = f(\theta)$

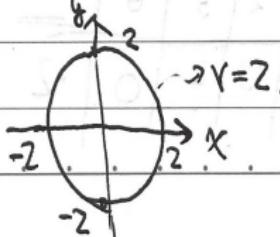
- $F(r, \theta) = 0$

Example 3: Sketch the graph of each polar equation.

(a) $r = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 2^2 = 4.$

\Rightarrow The graph of $r=2$ is a circle of radius 2 centered

at $(0,0)$.

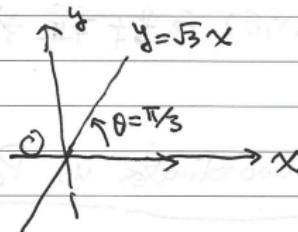


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$$(b) \theta = \frac{\pi}{3} \Rightarrow \frac{y}{x} = \tan \theta = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \Rightarrow y = \sqrt{3}x.$$

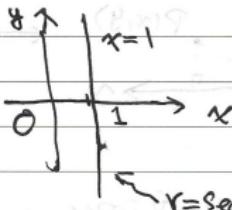
So, the graph of $\theta = \frac{\pi}{3}$ is a straight line of slope $\sqrt{3}$ passing through $(0, 0)$.



$$(c) r = \sec \theta \Rightarrow r = \frac{1}{\cos \theta} \Rightarrow r \cos \theta = 1 \Rightarrow x = 1, \text{ since } x = r \cos \theta \text{ and } y = r \sin \theta.$$

So, the graph of $r = \sec \theta$ is a vertical line

$$x = 1.$$



* Rose Curves: (玫瑰線)

$r = a \cosh(\theta)$ or $r = a \sin(\theta)$, with $a > 0$ and $n \in \mathbb{N}$.

Example: (三葉玫瑰線)

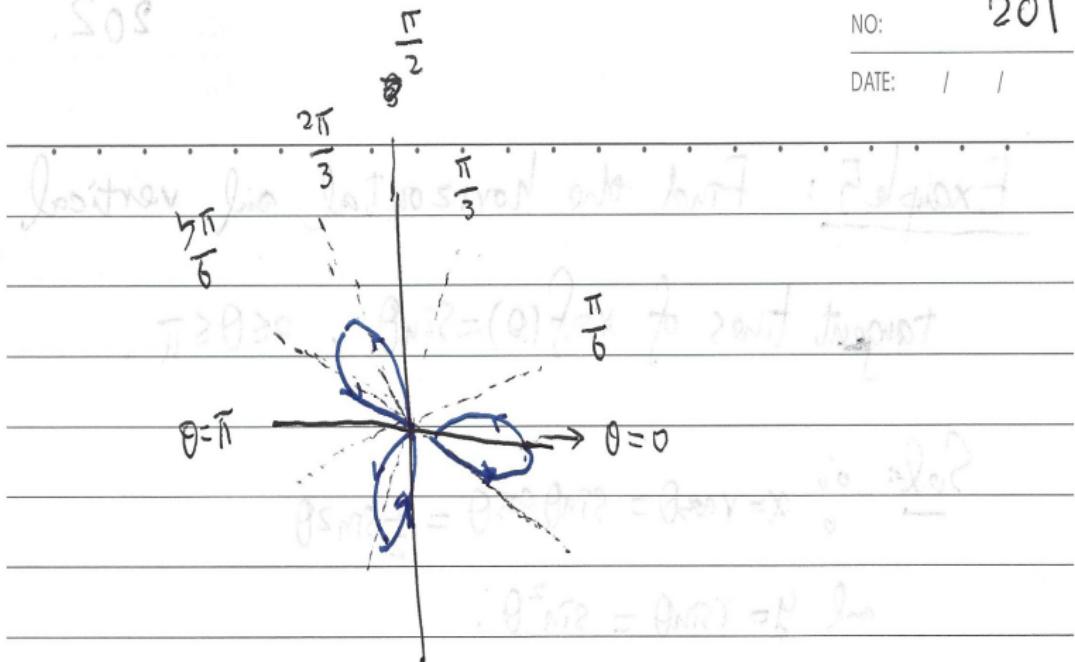
Sketch the graph of $r = 2 \cos 3\theta$ for $0 \leq \theta \leq \pi$.



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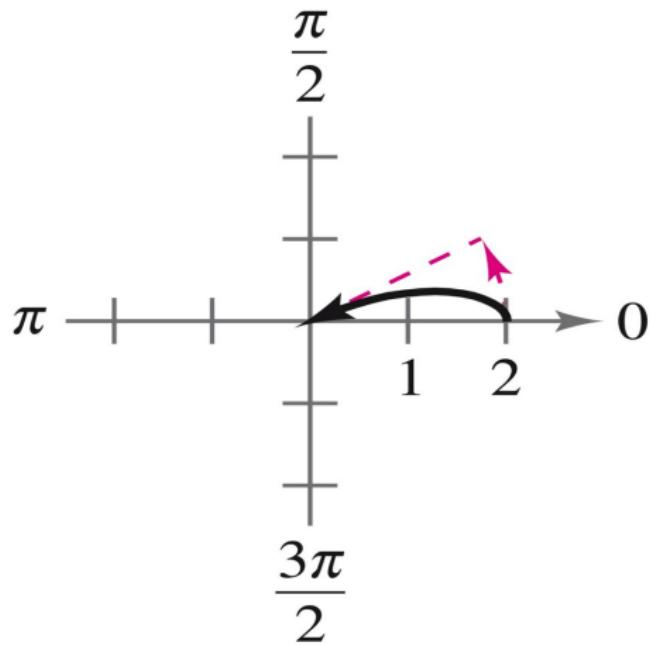
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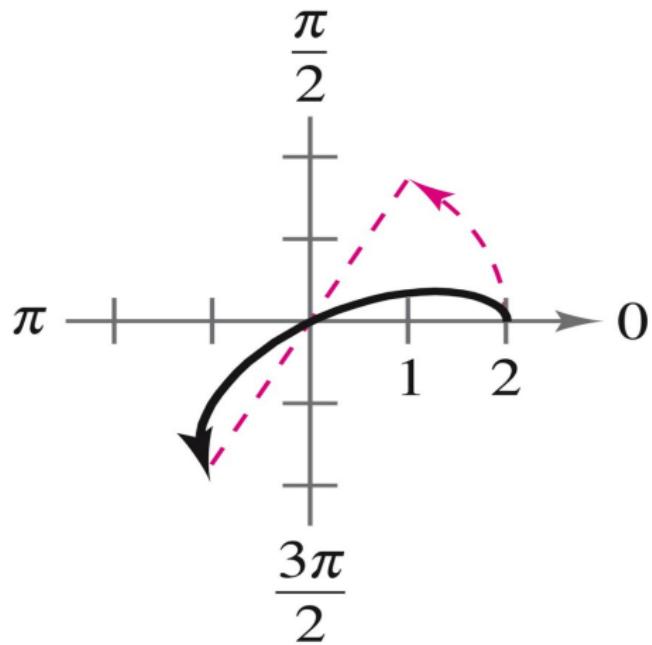
$$r = f(\theta) = 2 \cos 3\theta \text{ for } 0 \leq \theta \leq \pi$$



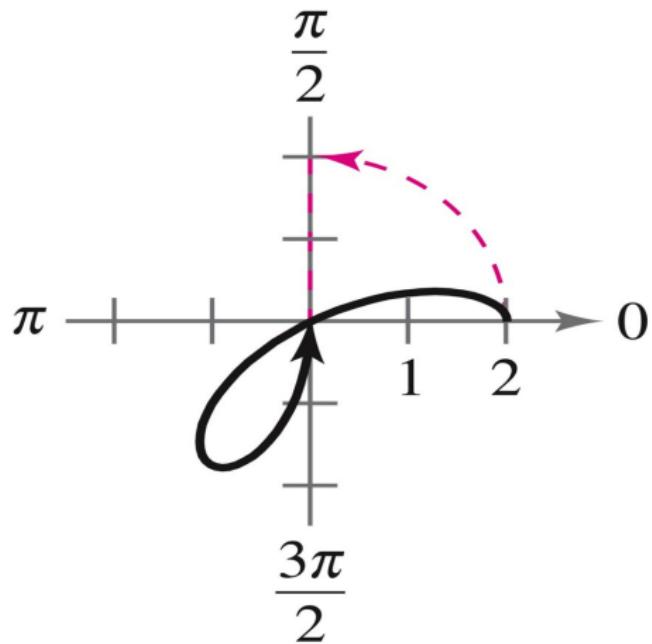
三瓣玫瑰線的動態示意圖



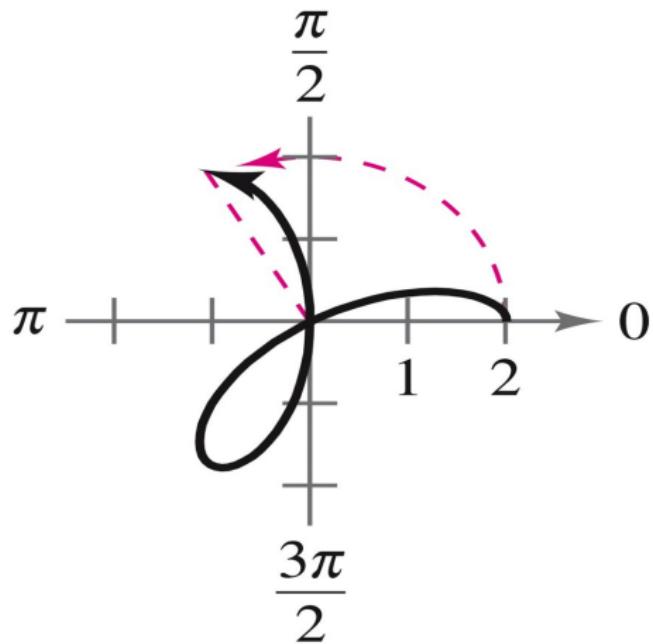
三瓣玫瑰線的動態示意圖



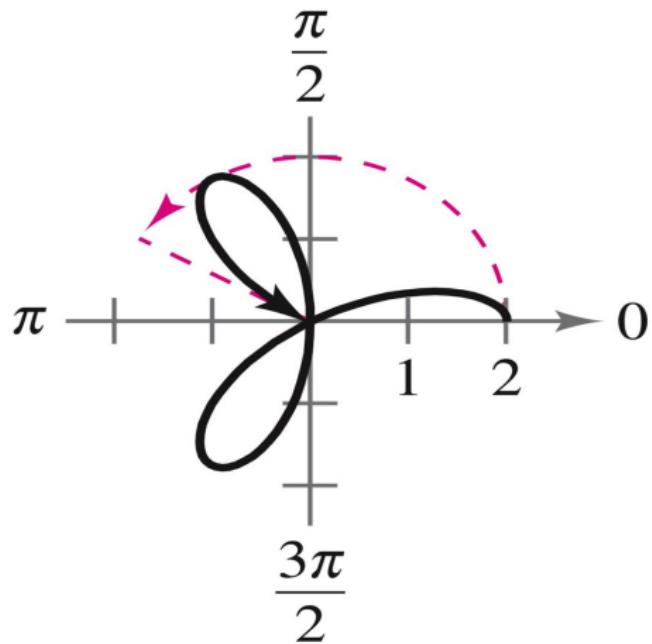
三瓣玫瑰線的動態示意圖



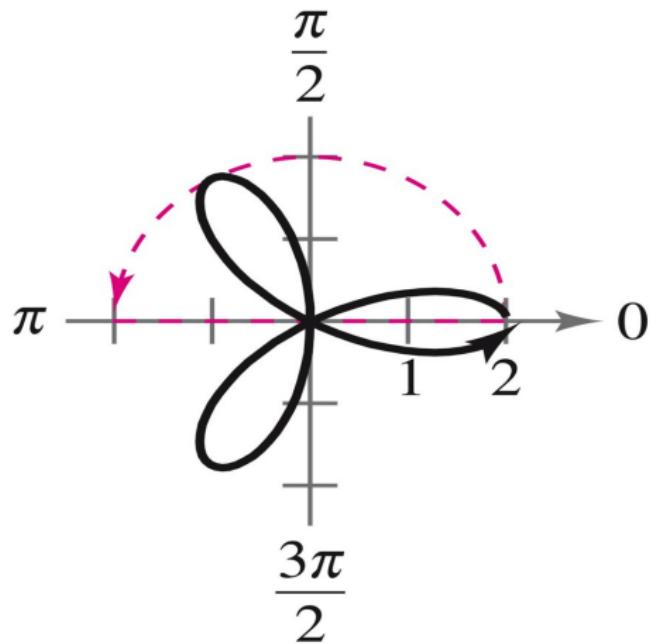
三瓣玫瑰線的動態示意圖



三瓣玫瑰線的動態示意圖



三瓣玫瑰線的動態示意圖



Thm 8.5 (Slope in Polar Form)

If f is a diff. function of θ , then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}.$$



If: use the identities:

$$x = r \cos \theta = f(\theta) \cos \theta \quad \text{and}$$

~~$$y = r \sin \theta = f(\theta) \sin \theta$$~~

$$\Rightarrow \frac{dx}{d\theta} = -f(\theta) \sin \theta + f'(\theta) \cos \theta$$

$$\frac{dy}{d\theta} = f(\theta) \cos \theta + f'(\theta) \sin \theta$$



Example 5: Find the horizontal and vertical

tangent lines of $r = f(\theta) = \sin\theta$, $0 \leq \theta \leq \pi$.



Sol:

$$x = r \cos \theta = \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$$

and $y = r \sin \theta = \sin^2 \theta$.

$$\therefore \frac{dx}{d\theta} = \cos 2\theta = 0 \Leftrightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \Leftrightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

and $\frac{dy}{d\theta} = 2\sin \theta \cos \theta = \sin 2\theta = 0 \Leftrightarrow 2\theta = 0, \pi$

$$\Leftrightarrow \theta = 0, \frac{\pi}{2}$$



So, the graph has horizontal tangent lines at

$$r(\theta) = (0, 0) \text{ and } (1, \frac{\pi}{2}),$$

and it has vertical tangent lines at

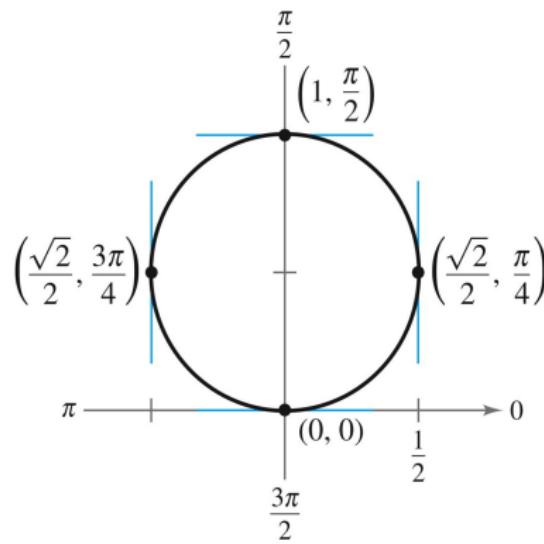
$$(r, \theta) = (\frac{1}{\sqrt{2}}, \frac{\pi}{4}) \text{ and } (\frac{1}{\sqrt{2}}, \frac{3\pi}{4}).$$



函數 $r = \sin \theta$ 的示意圖 (承上例)

The rectangular equation for $r = f(\theta) = \sin \theta$ is given by

$$\begin{aligned} r = \sin \theta &\Rightarrow r^2 = r \sin \theta \Rightarrow x^2 + y^2 = y \\ &\Rightarrow x^2 + (y - \frac{1}{2})^2 = (\frac{1}{2})^2. \end{aligned}$$



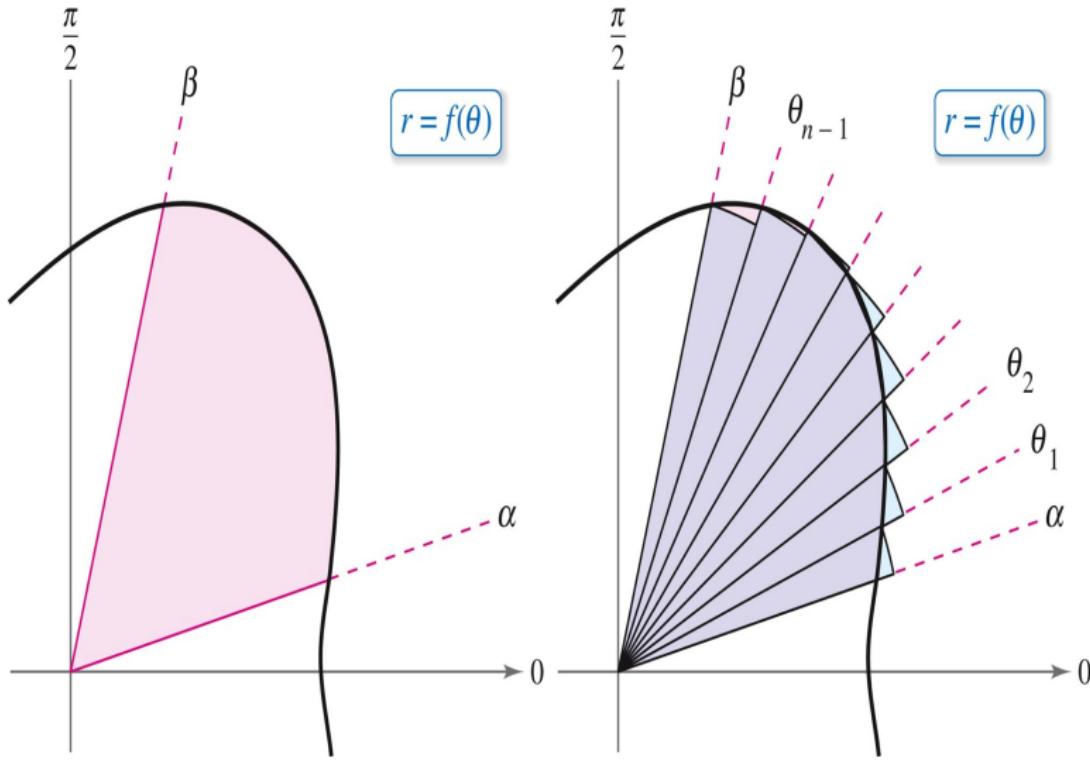
Section 8.4

Area and Arc Length in Polar Coordinates

(極坐標上的圖形面積與弧長)



示意圖



Consider a polar region given by

$$\mathcal{R} = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, 0 \leq r \leq f(\theta)\}$$

with $0 < \beta - \alpha \leq 2\pi$. If the region \mathcal{R} is partitioned into n polar sectors (極坐標扇形) by the rays $r = \theta_i$ ($i = 1, 2, \dots, n$) with

$$\alpha \equiv \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n \equiv \beta,$$

then the area of \mathcal{R} should be

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{2} [f(\theta_i)]^2 \Delta\theta_i,$$

where $\Delta\theta_i \equiv \theta_i - \theta_{i-1}$ for $i = 1, 2, \dots, n$.



Thm 8.7 (Area in Polar Coordinates)

If $f(\theta)$ is conti. on $[\alpha, \beta]$ with $0 < \beta - \alpha \leq 2\pi$, then the area of the polar region

$$\mathcal{R} = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, 0 \leq r \leq f(\theta)\}$$

is given by

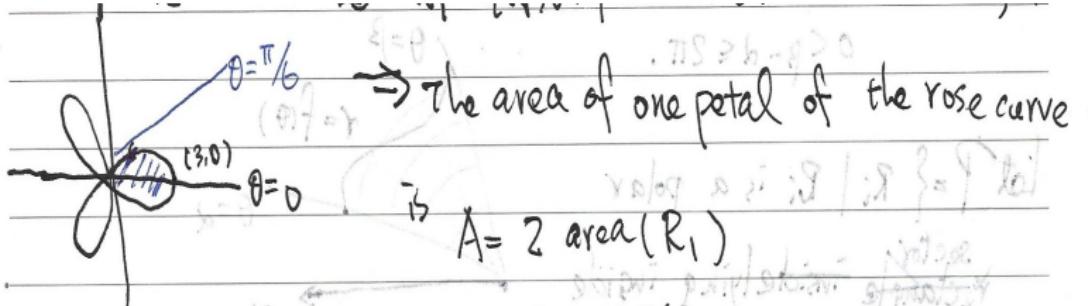
$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta.$$



Example 1: Find the area of one petal of the
rose curve $r = f(\theta) = 3 \cos(3\theta)$.

Sol:
 $\theta = \pi/6$
 Let $R_1 = \{(r, \theta) \mid 0 \leq \theta \leq \pi/6, 0 \leq r \leq 3 \cos(3\theta)\}$.





$$r = 3\cos(3\theta)$$

$$0 \leq \theta \leq \pi.$$

$$A = 2 \text{ area}(R_1)$$

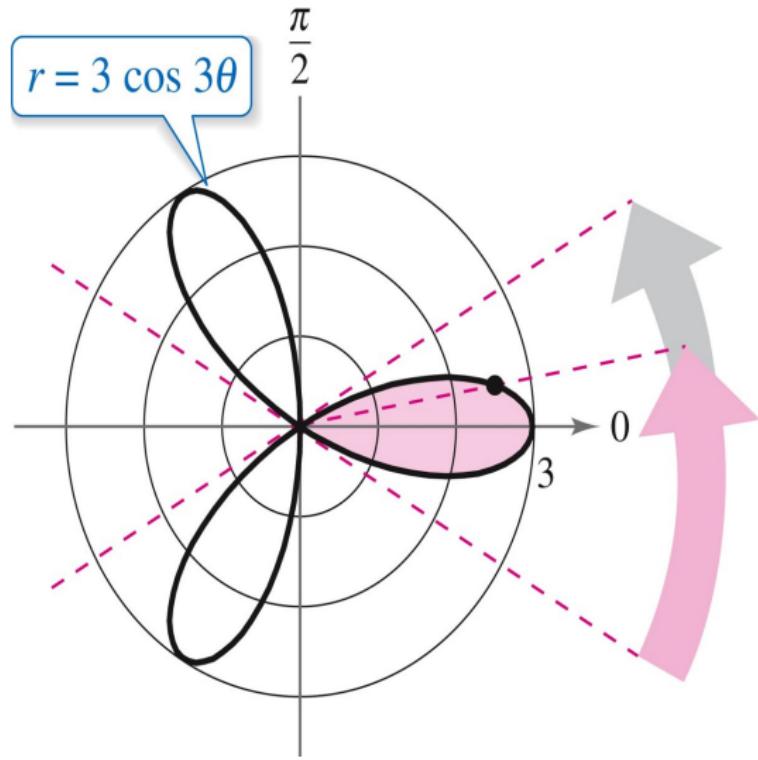
$$= 2 \cdot \frac{1}{2} \int_0^{\pi/6} [f(\theta)]^2 d\theta = \int_0^{\pi/6} (3\cos(3\theta))^2 d\theta$$

$$= 9 \int_0^{\pi/6} \cos^2(3\theta) d\theta = 9 \int_0^{\pi/6} \frac{1 + \cos(6\theta)}{2} d\theta$$

$$= \frac{9}{2} \left(\theta + \frac{1}{6} \sin(6\theta) \right) \Big|_0^{\pi/6} = \frac{9}{2} \left(\frac{\pi}{6} \right) = \frac{3}{4}\pi$$



示意圖 (承上例)



Thm 8.8 (Arc Length of a Polar Curve)

If $f(\theta)$ and $f'(\theta)$ are conti. on $[\alpha, \beta]$ with $0 < \beta - \alpha \leq 2\pi$, then the arc length of a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$s = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta.$$



pf: From the proof of Thm 8.5, we know that

$$\frac{dx}{d\theta} = -f(\theta) \sin \theta + f'(\theta) \cos \theta \quad \text{and} \quad \frac{dy}{d\theta} = f(\theta) \cos \theta + f'(\theta) \sin \theta.$$

Then we immediately get

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left[-f(\theta) \sin \theta + f'(\theta) \cos \theta\right]^2 \\ &\quad + \left[f(\theta) \cos \theta + f'(\theta) \sin \theta\right]^2 \\ &= [f(\theta)]^2 + [f'(\theta)]^2.\end{aligned}$$

So, the arc length of the polar curve $r = f(\theta)$ is given by

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta.$$

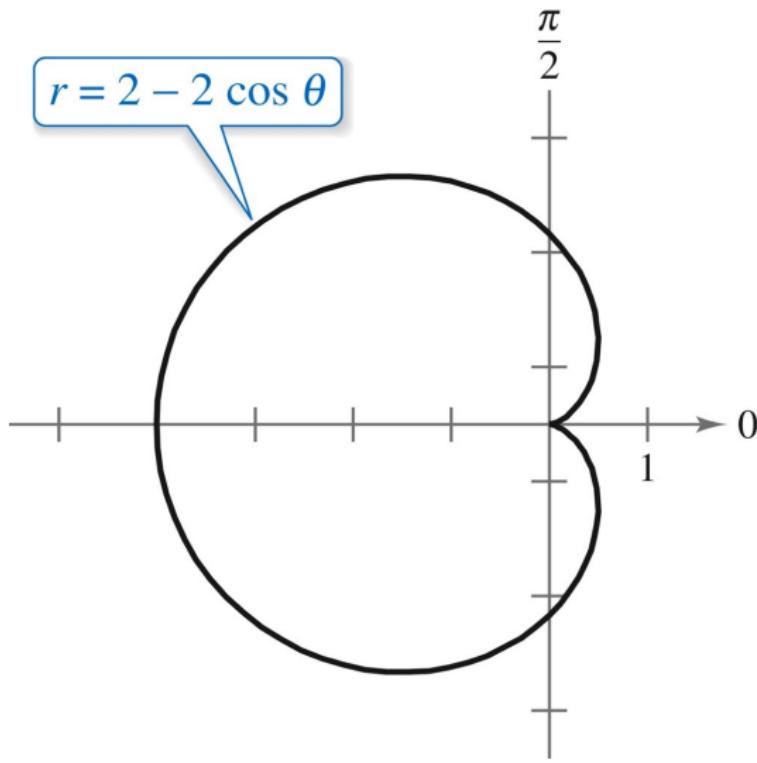


Example 4: Find the arc length of the cardioid
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$$r = f(\theta) = 2 - 2 \cos \theta \text{ from } \theta=0 \text{ to } \theta=2\pi,$$



心臟線的示意圖 (承上例)



Sol:

$$s = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + [f'(\theta)]^2} d\theta$$

$$= \int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2} d\theta .$$

$$= 2 \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta} d\theta = 2 \int_0^{2\pi} \sqrt{2 - 2\cos\theta} d\theta$$

$$= 2\sqrt{2} \int_0^{2\pi} \sqrt{1-\cos\theta} d\theta = 2\sqrt{2} \int_0^{2\pi} \sqrt{\frac{1}{2}(1-\cos\theta)} d\theta .$$

$$= 4 \int_0^{2\pi} \sin \frac{\theta}{2} d\theta \quad \left(\because \sin \frac{\theta}{2} \geq 0 \text{ for } 0 \leq \theta \leq 2\pi \right)$$

$$= -8 \cos \frac{\theta}{2} \Big|_0^{2\pi} = (-8)(-1 - 1) = 16$$



Thank you for your attention!

