

Chapter 10

Vector-Valued Functions

(向量值函數)

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Section 10.1

Vector-Valued Functions

(向量值函數)



Def. (向量值函數的定義)

- (1) A function of the form

$$r(t) = \vec{r}(t) = f(t)i + g(t)j = \langle f(t), g(t) \rangle \in \mathbb{R}^2$$

or

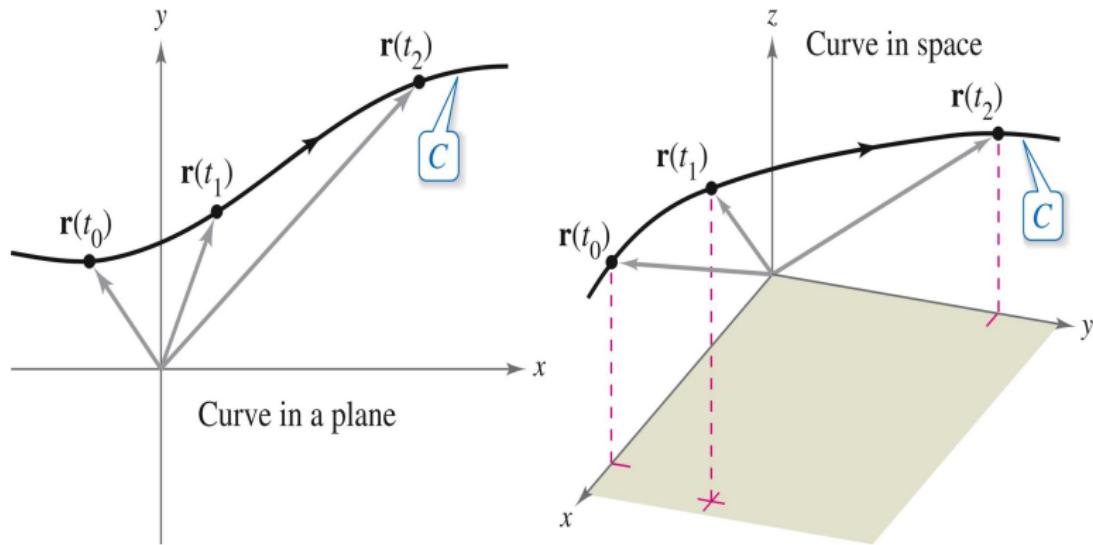
$$r(t) = \vec{r}(t) = f(t)i + g(t)j + h(t)k = \langle f(t), g(t), h(t) \rangle \in \mathbb{R}^3$$

is a vector-valued function, where f, g, h are real-valued functions of parameter t .

- (2) In this case, we say that f, g, h are the component functions (分量函數) of $r(t)$.



向量值函數的示意圖 (承上頁)

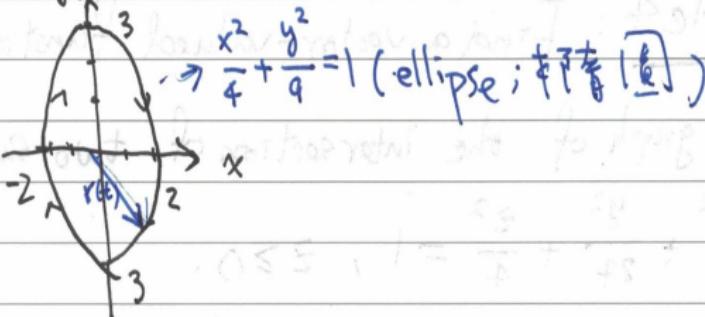


Example 1. (Plane Curve)

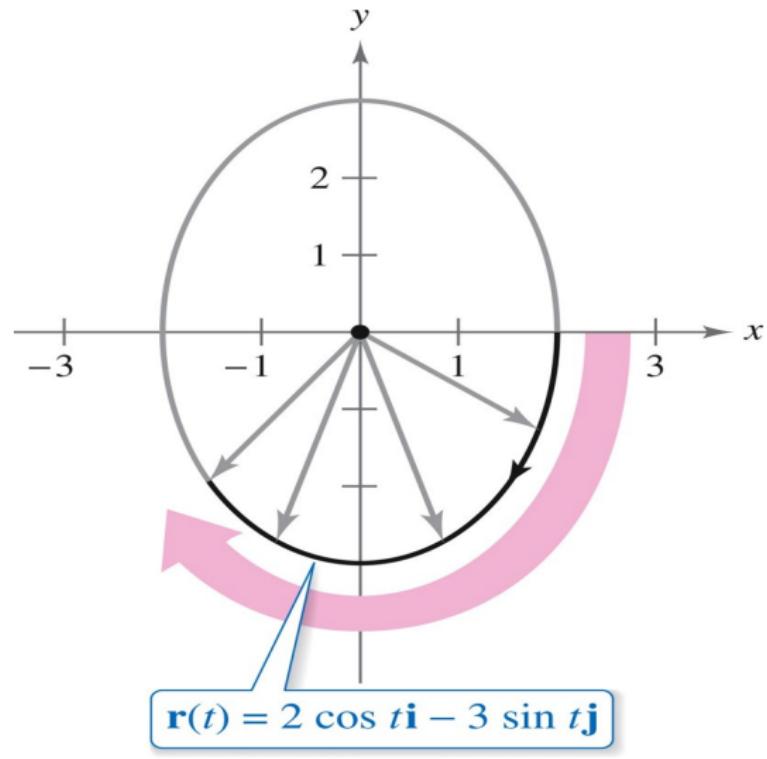
Sketch the graph of $\vec{r}(t) = 2\cos t \mathbf{i} - 3\sin t \mathbf{j}$ for $0 \leq t \leq 2\pi$.

Sol: Let $x = 2\cos t$ and $y = -3\sin t$. Then

$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{2}\right)^2 + \left(\frac{-y}{3}\right)^2 = \frac{x^2}{4} + \frac{y^2}{9}.$$



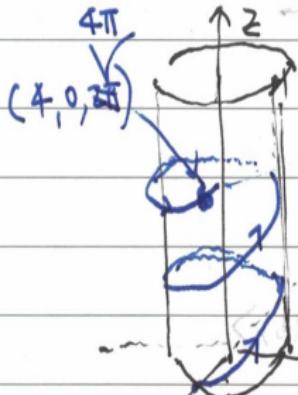
Example 1 的示意圖 (承上例)



Example 2 : (Space Curve)

Sketch the graph of a helix (螺旋旋轉曲線)

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t, t \rangle \text{ for } 0 \leq t \leq 4\pi$$

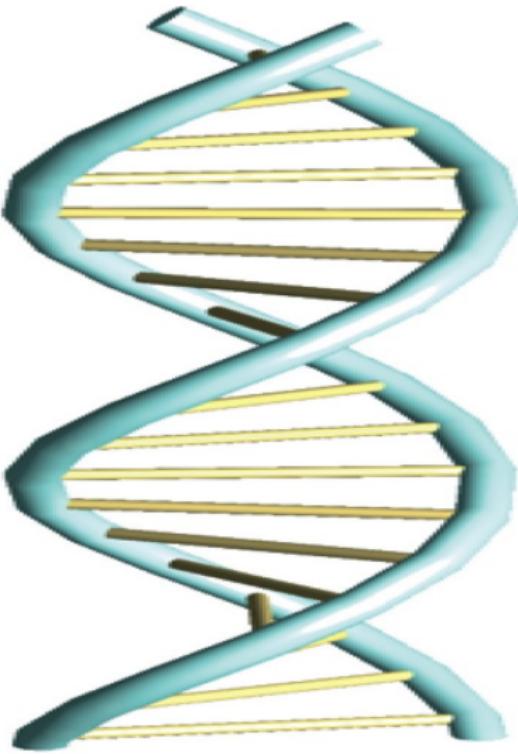
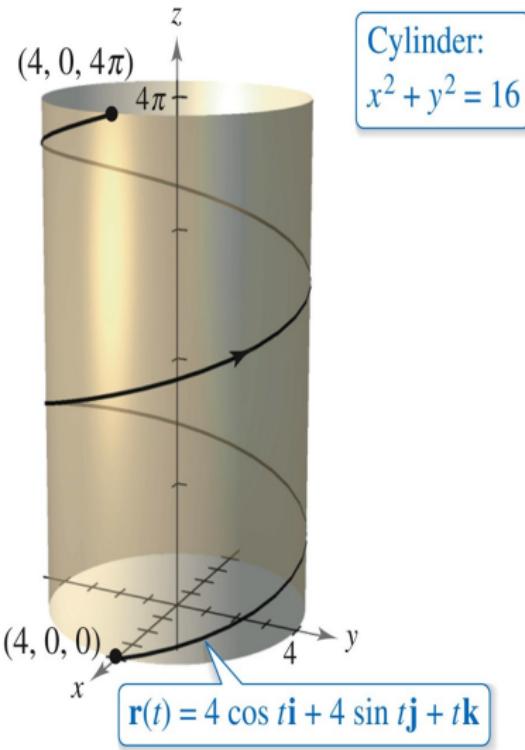


→ Two spirals on the helix

are traced out.



Example 2 的示意圖 (承上例)

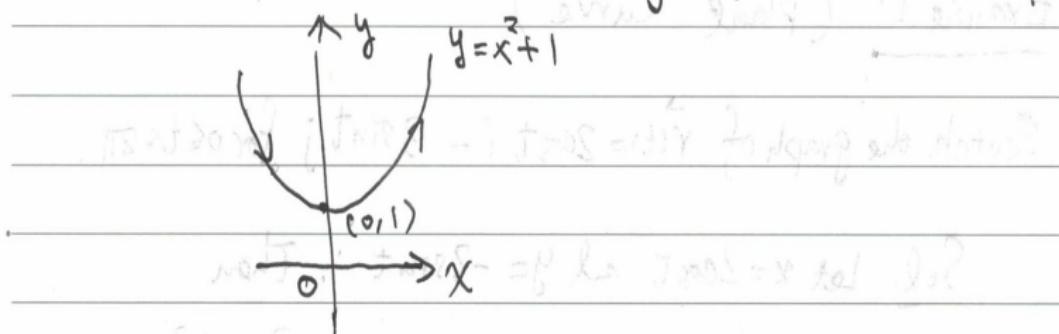


Example 3: Represent the parabola

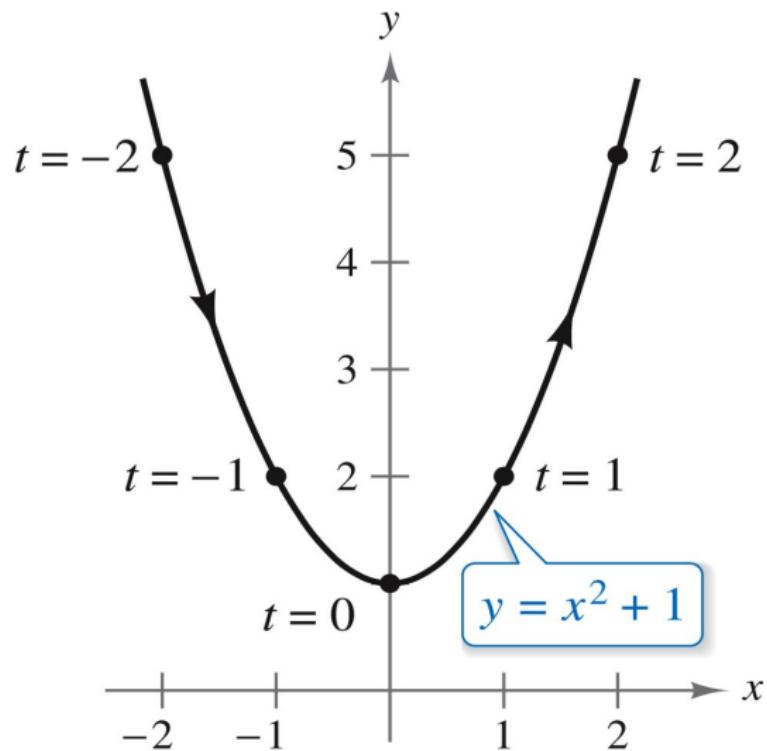
$y = x^2 + 1$ by a vector-valued function.

Sol: let $x = t$. Then $y = t^2 + 1 \quad \forall t \in \mathbb{R}$.

$$\Rightarrow \vec{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j} \quad \text{for } -\infty < t < \infty$$



Example 3 的示意圖 (承上例)



Example 4: Find a vector-valued function to represent the graph of the intersection of two surfaces

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, z \geq 0.$$

and $y = x^2$.

Sol.: Let $x = t$, Then $y = x^2 = t^2$ and hence



$$\text{such that } \frac{t^2}{12} + \frac{t^4}{24} + \frac{z^2}{4} = 1 \text{ and } z \geq 0.$$

$$\Rightarrow z = \sqrt{4\left(1 - \frac{t^2}{12} - \frac{t^4}{24}\right)} = \sqrt{\frac{-24 - 2t^2 - t^4}{6}}.$$

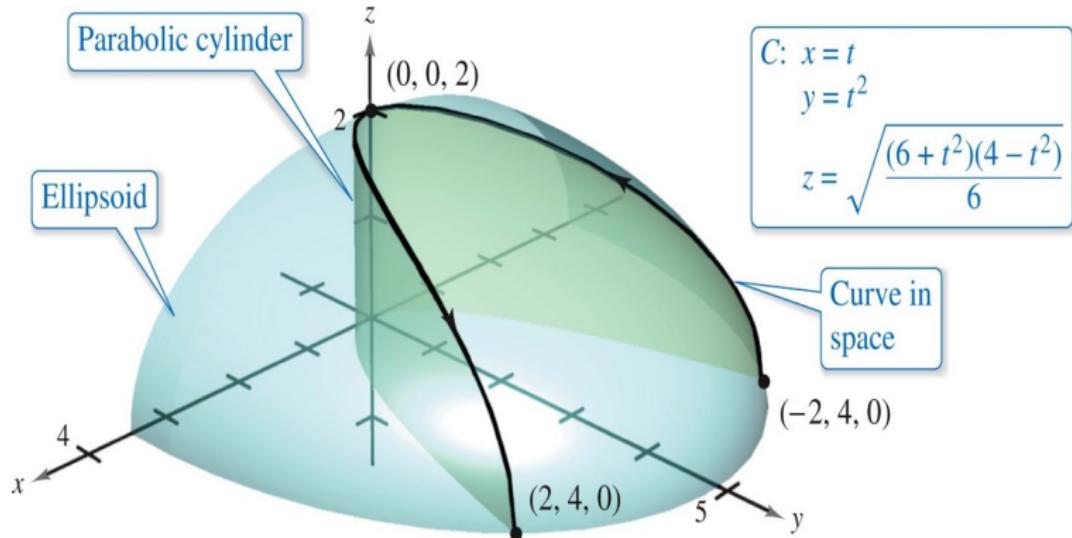
$$= \sqrt{\frac{(6+t^2)(4-t^2)}{6}} \geq 0 \text{ for } -2 \leq t \leq 2.$$

$$\Rightarrow \vec{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \sqrt{\frac{(6+t^2)(4-t^2)}{6}}\mathbf{k}$$

for $-2 \leq t \leq 2$.



Example 4 的示意圖 (承上例)



Def. (向量值函數的極限)

Suppose that $\lim_{t \rightarrow a} f(t)$, $\lim_{t \rightarrow a} g(t)$ and $\lim_{t \rightarrow a} h(t)$ both exist.

(1) For $r(t) = f(t)i + g(t)j$, define

$$\lim_{t \rightarrow a} r(t) = \left[\lim_{t \rightarrow a} f(t) \right] i + \left[\lim_{t \rightarrow a} g(t) \right] j.$$

(2) For $r(t) = f(t)i + g(t)j + h(t)k$, define

$$\lim_{t \rightarrow a} r(t) = \left[\lim_{t \rightarrow a} f(t) \right] i + \left[\lim_{t \rightarrow a} g(t) \right] j + \left[\lim_{t \rightarrow a} h(t) \right] k.$$



Def. (向量值函數的連續性)

Let $r(t)$ be a vector-valued function defined on the interval I containing a .

(1) $r(t)$ is conti. at $t = a$ if $\lim_{t \rightarrow a} r(t) = r(a)$.

(2) $r(t)$ is conti. on I if it is conti. at every parameter $t \in I$.



Thm (連續性的等價條件)

Let $f(t)$, $g(t)$ and $h(t)$ be real-valued functions defined on I .

$r(t) = f(t)i + g(t)j + h(t)k$ is conti. on I .

$\iff f(t), g(t)$ and $h(t)$ are conti. on I .



Example 5: For any $a \in \mathbb{R}$, the vector-valued function

$$\vec{r}(t) = t\vec{i} + a\vec{j} + (a^2 - t^2)\vec{k}$$

is conti. on $\tilde{I} = (-\infty, \infty) = \mathbb{R}$, since $\lim_{t \rightarrow c} \vec{r}(t) = \vec{r}(c), \forall c \in \mathbb{R}$.



Example 6: The vector-valued function

$$\vec{r}(t) = \vec{t}i + \sqrt{t+1} j + (t^2+1)k$$

is conti. on $\underline{I} = [-1, \infty)$, since $f(t) = t$, $g(t) = \sqrt{t+1}$ and

$$h(t) = t^2+1 \text{ both conti. on } \underline{I}.$$



Section 10.2

Differentiation and Integration of Vector-Valued Functions

(向量值函數的微分與積分)



Def. (向量值函數的微分)

Let $r(t)$ be a vector-valued function defined on the open interval I .

- (1) $r(t)$ is diff. at $t \in I$ if its derivative at t

$$r'(t) = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t} \quad \exists.$$

- (2) $r(t)$ is diff. on I if it is diff. at every parameter $t \in I$.



Thm 10.1 (微分公式)

Let f , g and h be diff. on an open interval I .

- ① If $r(t) = f(t)i + g(t)j \in \mathbb{R}^2$, then

$$r'(t) = f'(t)i + g'(t)j \quad \forall t \in I.$$

- ② If $r(t) = f(t)i + g(t)j + h(t)k \in \mathbb{R}^3$, then

$$r'(t) = f'(t)i + g'(t)j + h'(t)k \quad \forall t \in I.$$



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Example 2: ($\dot{\gamma}(t)$, $\ddot{\gamma}(t)$, $\gamma''(t)$)

For $\vec{\gamma}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k}$, find

(a) $\dot{\gamma}(t)$

(b) $\ddot{\gamma}(t)$

(c) $\dot{\gamma}(t) \cdot \ddot{\gamma}(t)$

(d) $\dot{\gamma}(t) \times \ddot{\gamma}(t)$



Sol: (a) $\vec{Y}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + 2 \mathbf{k}$.

(b) $\vec{Y}''(t) = -\cos t \mathbf{i} - \sin t \mathbf{j} + 0 \mathbf{k} = -\cos t \mathbf{i} - \sin t \mathbf{j}$.

(c) $\vec{Y}'(t) \cdot \vec{Y}''(t) = (-\sin t)(-\cos t) + (\cos t)(-\sin t) = 0$.



$$(d) \vec{Y}(t) \times \vec{Y}''(t) = \begin{vmatrix} i & j & k \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix} i - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix} j$$

(d) is I no benefit

$$+ \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} k = 2\sin t i - 2\cos t j + k.$$

for (d)



Thm 10.2 (Properties of $r'(t)$)

Let $r(t)$ and $u(t)$ be diff. vector-valued functions of t , and let $w(t)$ be a real-valued function of t .

$$① \frac{d}{dt} [r(t) \pm u(t)] = r'(t) \pm u'(t).$$

$$② \frac{d}{dt} [c \cdot r(t)] = c \cdot r'(t) \quad \forall c \in \mathbb{R}.$$

$$③ \frac{d}{dt} [w(t) \vec{r}(t)] = w'(t) \vec{r}(t) + w(t) \vec{r}'(t).$$

$$④ \frac{d}{dt} [r(t) \bullet u(t)] = r'(t) \bullet u(t) + r(t) \bullet u'(t).$$

$$⑤ \frac{d}{dt} [r(t) \times u(t)] = r'(t) \times u(t) + r(t) \times u'(t).$$

$$⑥ \text{Chain Rule: } \frac{d}{dt} [\vec{r}(w(t))] = \vec{r}'(w(t))w'(t).$$



Def. (空間中平滑曲線的定義)

A space curve \mathcal{C} represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth** on an open interval I if f, g, h are conti. on I , and
 $r'(t) \neq \vec{0} \quad \forall t \in I$.



Example 3: Find open intervals on which the epicycloid

$$\vec{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \leq t \leq 2\pi,$$

is smooth.

Sol: $\vec{r}'(t) = (-5\sin t + 5\sin 5t)\mathbf{i} + (5\cos t - 5\cos 5t)\mathbf{j}.$

$$\vec{r}'(t) = \vec{0} = 0\mathbf{i} + 0\mathbf{j} \Leftrightarrow \sin t = \sin 5t \text{ and } \cos t = \cos 5t.$$

for $0 \leq t \leq 2\pi$.

$$\Rightarrow t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$

So, the graph of $\vec{r}(t)$ is smooth on $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \pi)$, $(\pi, \frac{3\pi}{2})$

and $(\frac{3\pi}{2}, 2\pi)$, respectively.



Def. (向量值函數的積分)

(1) If $r(t) = f(t)i + g(t)j \in \mathbb{R}^2$, define

- $\int r(t) dt = \left[\int f(t) dt \right] i + \left[\int g(t) dt \right] j.$

- $\int_a^b r(t) dt = \left[\int_a^b f(t) dt \right] i + \left[\int_a^b g(t) dt \right] j,$ where f and g are integrable over $[a, b].$

(2) If $r(t) = f(t)i + g(t)j + h(t)k \in \mathbb{R}^3$, define

- $\int r(t) dt = \left[\int f(t) dt \right] i + \left[\int g(t) dt \right] j + \left[\int h(t) dt \right] k.$

- $\int_a^b r(t) dt = \left[\int_a^b f(t) dt \right] i + \left[\int_a^b g(t) dt \right] j + \left[\int_a^b h(t) dt \right] jk,$
where f , g and h are integrable over $[a, b].$



$\rightarrow \pi s^2 t^2 j \geq 0$, if $(s^2, t^2) = (t^2, s^2)$ then $s^2 = t^2$ or $s = \pm t$

Example 5: $\int (t^2 i + 3j) dt = \frac{1}{2} t^2 i + 3t j + C$

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Example 7: Find the antiderivative (or $\vec{r}(t)$) of

$$\vec{r}'(t) = \cos 2t i - 2 \sin t j + \frac{1}{1+t^2} k \quad \text{with } \vec{r}(0) = 3i - 2j + k.$$



Sol:

$$\vec{Y}(t) = \int \vec{r}'(t) dt = \left(\frac{1}{2} \sin 2t \right) \vec{i} + \left(2 \cos t \right) \vec{j} + \left(\tan t \right) \vec{k} + \vec{C}$$

$$\vec{r}(0) = 3\vec{i} - 2\vec{j} + \vec{k}$$

$$3\vec{i} - 2\vec{j} + \vec{k} = \vec{r}(0) = 0\vec{i} + 2\vec{j} + 0\vec{k} + \vec{C} \Rightarrow \vec{C} = 3\vec{i} - 4\vec{j} + \vec{k}$$

⇒ The antiderivative of $\vec{r}'(t)$ is

$$\vec{Y}(t) = \left(\frac{1}{2} \sin 2t + 3 \right) \vec{i} + \left(2 \cos t - 4 \right) \vec{j} + \left(\tan t + 1 \right) \vec{k}$$



Section 10.3

Velocity and Acceleration

(速度與加速度)



Def. (速度向量與加速度向量)

If a particle (質點) moves along a **smooth curve** \mathcal{C} represented by the **vector-valued function** $r = r(t) \quad \forall t \in I$, where I is some interval of time instants, we define

- (1) the velocity (vector) at time t : $v(t) \equiv r'(t)$.
- (2) the acceleration (vector) at time t : $a(t) \equiv v'(t) = r''(t)$.
- (3) the speed (速率) at time t : speed $\equiv \|v(t)\| = \|r'(t)\|$.



Example 1: (平面参数曲线的例子)

Find the velocity, speed and acceleration of a particle that moves along the curve

$$C: \vec{r}(t) = 2 \sin\left(\frac{t}{2}\right) \mathbf{i} + 2 \cos\left(\frac{t}{2}\right) \mathbf{j}.$$

Sol:

$$\vec{v}(t) = \vec{r}'(t) = \cos\left(\frac{t}{2}\right) \mathbf{i} - \sin\left(\frac{t}{2}\right) \mathbf{j}.$$

$$\text{Speed at time } t = \|\vec{v}(t)\| = \sqrt{\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)} = 1.$$

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = -\frac{1}{2} \sin\left(\frac{t}{2}\right) \mathbf{i} - \frac{1}{2} \cos\left(\frac{t}{2}\right) \mathbf{j}.$$



Example 3: (空間曲線的例子)

Find the velocity and acceleration vectors when $t=1$

$$\text{for } \vec{r}(t) = t\vec{i} + t^3\vec{j} + 3t\vec{k}, t \geq 0.$$

Sol: $\therefore \vec{v}(t) = \vec{r}'(t) = \vec{i} + 3t^2\vec{j} + 3\vec{k}$ and

$$\vec{a}(t) = \vec{v}'(t) = 0\vec{i} + 6t\vec{j} + 0\vec{k} = 6t\vec{j}$$

$\therefore \vec{v}(1) = \vec{i} + 3\vec{j} + 3\vec{k}$ and $\vec{a}(1) = 6\vec{j}$.



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Example 4: If an object starts from rest at $\underline{(1, 2, 0)}$

and moves with $\vec{a}(t) = \vec{j} + 2\vec{k}$, ($\|\vec{a}(t)\|$ in units ft/s^2)

find the location $\vec{r}(2)$ after 2 seconds.



Sol: We need to solve the 2nd order ODE:

$$\begin{cases} \ddot{y}(t) = \ddot{a}(t) = j + 2k \\ y(0) = 0 \text{ and } \dot{y}(0) = i + 2j. \end{cases}$$



$$\Rightarrow \vec{v}(t) = \int \vec{a}(t) dt = \int (j + 2k) dt = t j + 2t k + C_1$$

$$\Rightarrow \vec{v}(0) = \vec{v}(0) = C_1 \Rightarrow \vec{v}(t) = t j + 2t k.$$

Furthermore, $\vec{r}(t) = \int \vec{v}(t) dt = \int \vec{v}(t) dt = t^2 j + t^2 k + C_2$

$$\Rightarrow i + \cancel{j} + 2\cancel{j} + 0k = \vec{r}(0) = C_2 \Rightarrow \vec{r}(t) = i + \left(\frac{t^2}{2} + 2\right)j + t^2 k.$$

$$So, \vec{r}(2) = i + 4j + 4k = \langle 1, 4, 4 \rangle.$$



Section 10.4

Tangent Vectors and Normal Vectors

(切向量與法向量)



Def. (切向量的定義)

Let $\mathcal{C} : r = r(t)$ be a **smooth curve** defined on an open interval I .

- (1) The first derivative of r defined by

$$r'(t) = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

is called the tangent vector (切向量) to the curve \mathcal{C} at t .

- (2) The unit tangent vector (單位切向量) at t is defined by

$$T(t) = \vec{T}(t) \equiv \frac{r'(t)}{\|r'\|} = \frac{v(t)}{\|v(t)\|}.$$



Example 2: Consider the helix $\ell: \vec{r}(t) = 2\cos t \hat{i} + 2\sin t \hat{j} + t \hat{k}$.

(a) Find $\vec{T}(t)$.

(b) Use part (a) to find the equation of the tangent line

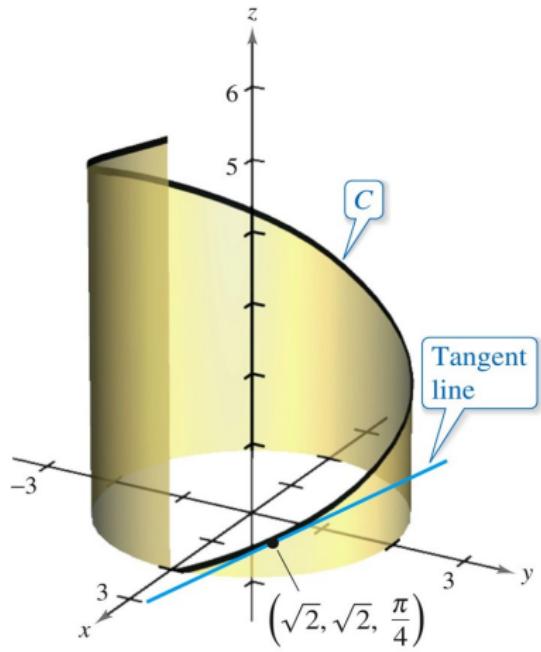
to ℓ at $(\sqrt{2}, \frac{\pi}{4})$.



Example 2 的示意圖

Curve:

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}$$



Sol: (a) $\vec{r}(t) = -2\sin t \vec{i} + 2\cos t \vec{j} + \vec{k}$.

$$\Rightarrow \|\vec{r}'(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + 1^2} = \sqrt{5}.$$

So, $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{1}{\sqrt{5}} (-2\sin t \vec{i} + 2\cos t \vec{j} + \vec{k})$.

(b) At the point $(\sqrt{2}, \sqrt{2}, \pi/4)$, we see that $t = \pi/4$.

$$\Rightarrow \vec{T}(\pi/4) = \frac{1}{\sqrt{5}} \left(-2 \cdot \frac{\sqrt{2}}{2} \vec{i} + 2 \cdot \frac{\sqrt{2}}{2} \vec{j} + \vec{k} \right) = \frac{1}{\sqrt{5}} (-\sqrt{2} \vec{i} + \sqrt{2} \vec{j} + \vec{k}).$$

So, the parametric equations of the tangent line to C



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at $(\sqrt{2}, \sqrt{2}, \frac{\pi}{4})$ is given by

$$x = \sqrt{2} - \sqrt{2}s, \quad y = \sqrt{2} + \sqrt{2}s, \quad z = \frac{\pi}{4} + s$$

for $-\infty < s < \infty$.



Recall (空間直線的參數方程式)

A straight line passing through $(x_0, y_0, z_0) \in \mathbb{R}^3$ in the direction $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ has the parametric equations of the form

$$x = x_0 + as, \quad y = y_0 + bs, \quad z = z_0 + cs,$$

for $-\infty < s < \infty$.



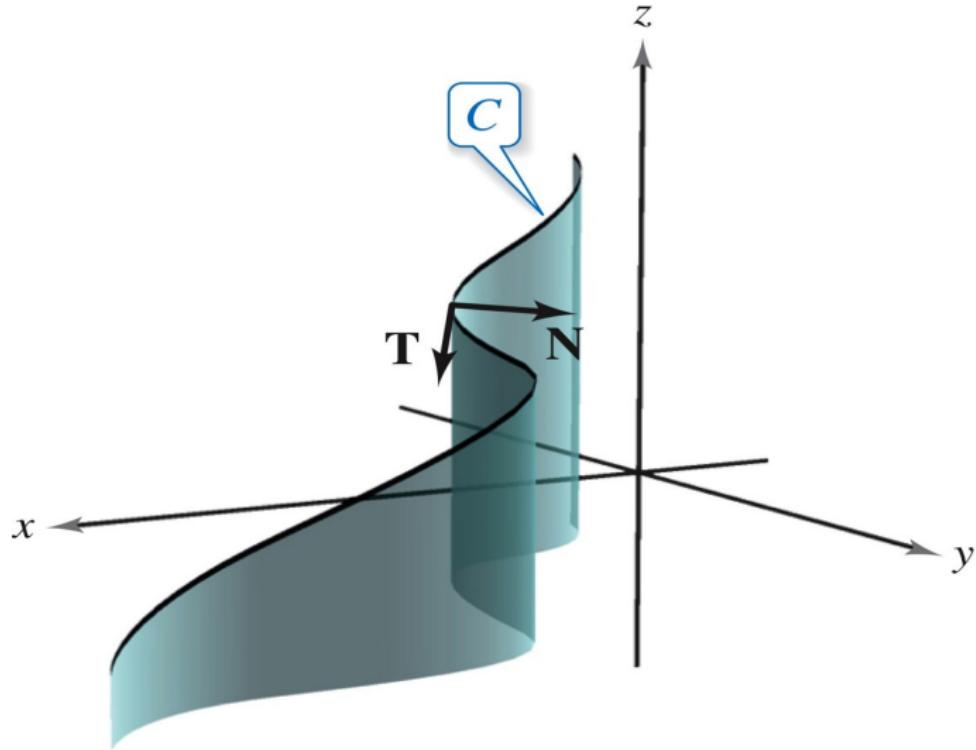
Def. (法向量的定義)

Let $\mathcal{C} : r = r(t)$ be a **smooth curve** defined on an open interval I .

If $T'(t) \neq \vec{0}$ $\forall t \in I$, the principal unit normal vector (主單位法向量) at t is defined by

$$N(t) = \vec{N}(t) \equiv \frac{T'(t)}{\|T'(t)\|}.$$





Example 4: Find $\vec{N}(t)$ for the helix $\vec{r}(t) = 2 \cos t \vec{i} + 2 \sin t \vec{j} + t \vec{k}$.

Sol: From Example 2, we see that

$$\vec{T}(t) = \frac{1}{\sqrt{5}} (-2 \sin t \vec{i} + 2 \cos t \vec{j} + \vec{k}).$$

$$\Rightarrow \vec{T}'(t) = \frac{1}{\sqrt{5}} (-2 \cos t \vec{i} - 2 \sin t \vec{j} + 0 \vec{k}) \text{ and hence}$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{5}} \sqrt{(-2 \cos t)^2 + (-2 \sin t)^2} = \frac{2}{\sqrt{5}}.$$

$$\text{So, } \vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{2} (-2 \cos t \vec{i} - 2 \sin t \vec{j})$$

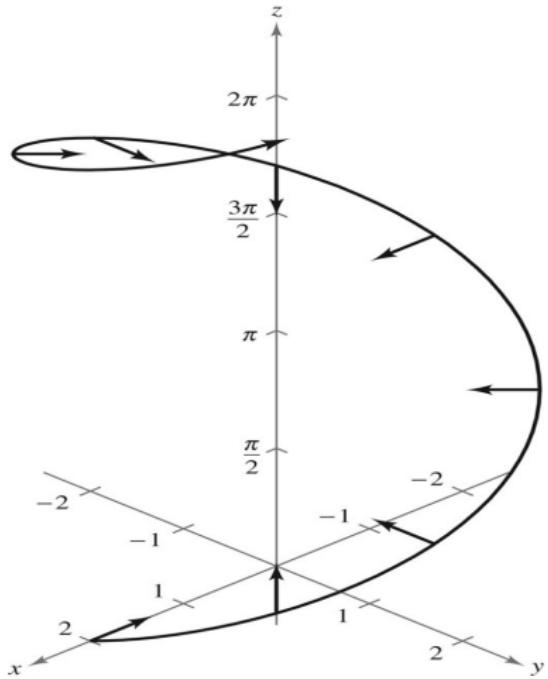
$$= -\cos t \vec{i} - \sin t \vec{j}.$$



Example 4 的示意圖

Helix:

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 2 \sin t \mathbf{j} + t \mathbf{k}$$



Section 10.5

Arc Length and Curvature

(弧長與曲率)



Thm 10.6 (空間中平滑曲線的弧長公式)

If \mathcal{C} is a **smooth curve** given by

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

on the interval $I = [a, b]$, then the arc length of \mathcal{C} on I is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} dt = \int_a^b \| \mathbf{r}'(t) \| dt.$$



Example : (Example 1 会使用到的结果)

$$\int \sqrt{x^2 - 3} dx = \frac{x\sqrt{x^2 - 3}}{2} - \frac{3}{2} \ln|x + \sqrt{x^2 - 3}| + C - (x)$$



Sol: let $x = \sqrt{3} \sec\theta$. Then $dx = \sqrt{3} \sec\theta \tan\theta d\theta$.

$$\Rightarrow \int \sqrt{x^2 - 3} dx = \int \sqrt{3(\sec^2\theta - 1)} (\sqrt{3} \sec\theta \tan\theta) d\theta = 3 \int \sec\theta \tan^2\theta d\theta$$

$$= 3 \int \sec\theta (\sec^2\theta - 1) d\theta = 3 \left(\int \sec^3\theta d\theta - \int \sec\theta d\theta \right).$$

$$= 3 \left(\frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| \right) + C$$

$$= \frac{3}{2} \sec\theta \tan\theta - \frac{3}{2} \ln|\sec\theta + \tan\theta| + C$$

$$= \frac{3}{2} \left(\frac{x}{\sqrt{3}} \right) \left(\frac{\sqrt{x^2 - 3}}{\sqrt{3}} \right) - \frac{3}{2} \ln \left| \frac{x}{\sqrt{3}} + \frac{\sqrt{x^2 - 3}}{\sqrt{3}} \right| + C$$

$$= \frac{1}{2} x \sqrt{x^2 - 3} - \frac{3}{2} \ln|x + \sqrt{x^2 - 3}| + C$$



Example 1: Find the arc length of the curve

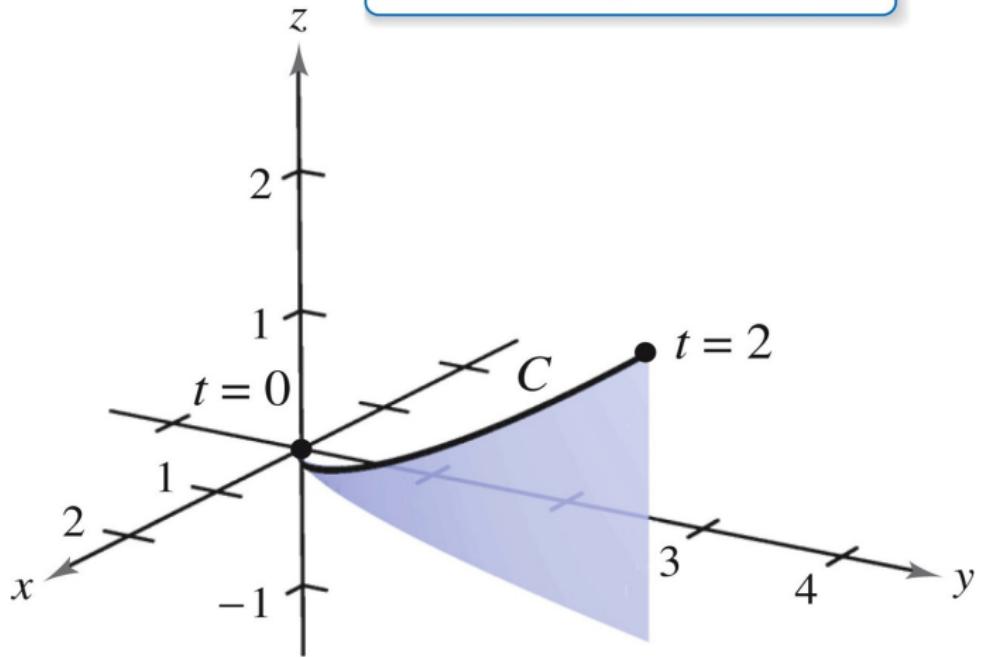
$$C: \vec{r}(t) = t \vec{i} + \frac{4}{3} t^{3/2} \vec{j} + \frac{1}{2} t^2 \vec{k}$$

on $I = [0, 2]$.



Example 1 的示意圖

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$



Sol. $\vec{r}(t) = \vec{i} + 2t^{1/2} \vec{j} + t^{\frac{3}{2}} \vec{k}$.

$$\text{So } s = \int_0^2 \|\vec{r}'(t)\| dt = \int_0^2 \sqrt{1+4t+t^2} dt$$

$$= \int_0^2 \sqrt{(t+2)^2 - 3} dt \quad (\text{Let } x=t+2 \text{ in (*)})$$

$$= \left[\frac{t+2}{2} \sqrt{(t+2)^2 - 3} - \frac{3}{2} \ln \left| (t+2) + \sqrt{(t+2)^2 - 3} \right| \right]_0^2$$

$$= 2\sqrt{13} - \frac{3}{2} \ln(4 + \sqrt{13}) - 1 + \frac{3}{2} \ln 3 \doteq 4.816$$



Example 2: Find the arc length of the helix

$$\vec{r}(t) = b \cos t \hat{i} + b \sin t \hat{j} + \sqrt{1-b^2} t \hat{k} \text{ with } |b| < 1$$

on $I = [0, 2\pi]$.

Sol: $\because \vec{r}'(t) = -b \sin t \hat{i} + b \cos t \hat{j} + \sqrt{1-b^2} \hat{k}$.

$$\therefore s = \int_0^{2\pi} \|\vec{r}'(t)\| dt = \int_0^{2\pi} \sqrt{(-b \sin t)^2 + (b \cos t)^2 + (1-b^2)} dt$$

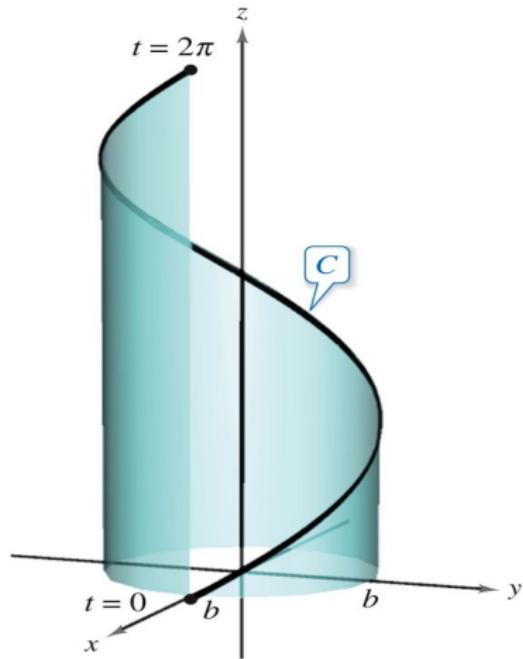
$$= \int_0^{2\pi} 1 dt = 2\pi$$



Example 2 的示意圖

Curve:

$$\mathbf{r}(t) = b \cos t \mathbf{i} + b \sin t \mathbf{j} + \sqrt{1 - b^2} t \mathbf{k}$$



Def. (弧長函數的定義)

Let $\mathcal{C} : r = r(t)$ be a **smooth curve** defined on the closed interval $I = [a, b]$. The arc length function (弧長函數) is defined by

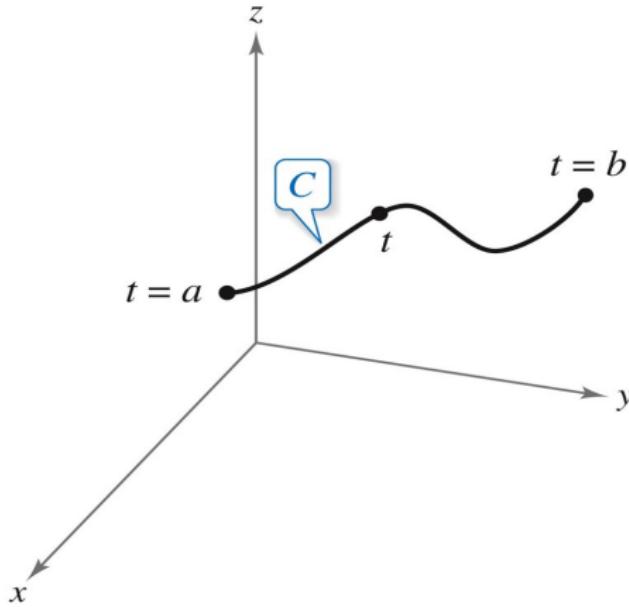
$$\textcolor{red}{s} = s(t) \equiv \int_a^t \|r'(u)\| du \quad \forall t \in I,$$

where s is called the **arc length parameter** (弧長參數).



弧長函數的示意圖

$$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2} du$$



Notes:

(1) $\frac{ds}{dt} = \frac{d}{dt} \int_a^t \|\vec{r}(u)\| du = \|\vec{r}'(t)\|$ by F.T.C.

(2) If $t=t(s)$ is obtained with an arc length parameter s ,

then $\vec{r}(t)$ can be rewritten as an arc length

parametrization $\vec{r}(s) = \vec{r}(t(s))$ for $a \leq s \leq \int_a^b \|\vec{r}'(t)\| dt$.

(弧長參式).



$$(3) \quad \left\| \vec{Y}(s) \right\| = \left\| \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} \right\| = \left\| \vec{r}'(t) \cdot \frac{1}{\|\vec{r}'(t)\|} \right\| = 1,$$

where s is the arc length parameter.

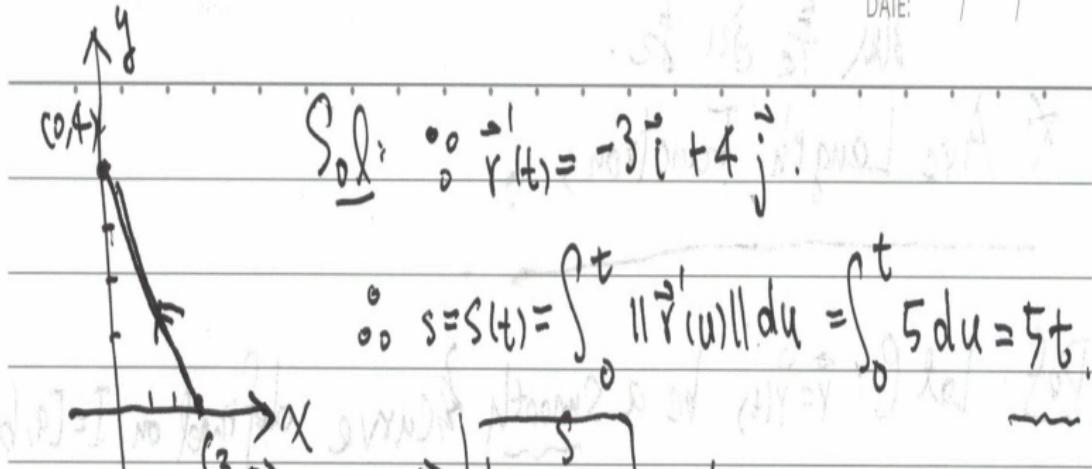
$$\Rightarrow \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{r}'(s) \cdot \frac{ds}{dt}}{\|\vec{r}'(s) \cdot \frac{ds}{dt}\|} = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} = \vec{r}'(s)$$

Example 4: (直線の弧長表示)

Find the arc length function for the line segment.

$$C: \vec{r}(t) = (3-3t) \vec{i} + 4t \vec{j}, \quad 0 \leq t \leq 1.$$





$$\therefore s = s(t) = \int_0^t \|\vec{r}'(u)\| du = \int_0^t 5 du = 5t$$

$$\Rightarrow t = \frac{s}{5}$$

with an arc length parameter

So, the line segment C can be written as

$$\vec{r}(s) = \vec{r}(t(s)) = \vec{r}\left(\frac{s}{5}\right) = \left(3 - \frac{3}{5}s\right)\vec{i} + \frac{4}{5}s\vec{j}, \text{ for } 0 \leq s \leq 5$$



*Curvature (曲率) :

Def. Let $C: \vec{r} = \vec{r}(s)$ be a smooth curve in \mathbb{R}^2 or \mathbb{R}^3 ,

where s is the arc length parameter. The curvature

$$K \text{ at } s \text{ is } K = \left\| \frac{d\vec{T}(s)}{ds} \right\| = \left\| \vec{r}'(s) \right\|.$$

Example 4: Show that the curvature of a circle

$$\vec{r}(t) = r \cos t \vec{i} + r \sin t \vec{j}, \quad 0 \leq t \leq 2\pi$$

of radius $r > 0$ is $K = \frac{1}{r}$.



$$\text{PF: } \textcircled{1} \quad S = S(t) = \int_0^t \|\vec{r}(u)\| du = \gamma t$$

$$\textcircled{2} \quad \boxed{t = \frac{s}{\gamma}}$$

$$\text{and hence } \vec{r}(s) = \sqrt{\cos(\frac{s}{\gamma})} \mathbf{i} + \sqrt{\sin(\frac{s}{\gamma})} \mathbf{j}.$$

$\Rightarrow \vec{T}(s) = \frac{d}{ds} \vec{r}(s) = -\sin(\frac{s}{\gamma}) \mathbf{i} + \cos(\frac{s}{\gamma}) \mathbf{j}$ is the unit tangent vector.

$$\Rightarrow \frac{d}{ds} \vec{T}(s) = -\frac{1}{\gamma} \cos(\frac{s}{\gamma}) \mathbf{i} - \frac{1}{\gamma} \sin(\frac{s}{\gamma}) \mathbf{j}$$

$$\Rightarrow K = \left\| \frac{d}{ds} \vec{T}(s) \right\| = \sqrt{\frac{1}{\gamma^2} \left[\cos^2(\frac{s}{\gamma}) + \sin^2(\frac{s}{\gamma}) \right]} = \frac{1}{\gamma}$$



* Two Useful Formulas for Curvature:

Thm 10.8 : (不用弧長參數式之曲率公式)

For any smooth curve \mathcal{C} , $\vec{r} = \vec{r}(t)$, the curvature

$$\text{of } \mathcal{C} \text{ at } t \text{ is } K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$



Example 4: Show that the curvature of a circle of radius r .

$$r > 0 \text{ so } K = \frac{1}{r}.$$

P: $\vec{r}(t) = r \cos t \vec{i} + r \sin t \vec{j}$ for $0 \leq t \leq 2\pi$.

$$\therefore \vec{r}'(t) = -r \sin t \vec{i} + r \cos t \vec{j} \text{ and}$$

$$\vec{r}''(t) = -r \cos t \vec{i} - r \sin t \vec{j}.$$

$$\Rightarrow \|\vec{r}'(t)\| = r \text{ and } \vec{r}'(t) \times \vec{r}''(t) = r^2 \vec{k}.$$

$$\Rightarrow K = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{r^2}{r^3} = \frac{1}{r} \text{ by Thm 10.8}$$



Thm 10.9 (直角座標下の曲線公式)

If C is the graph of a twice-differentiable function $y=f(x)$
then the curvature of C at $P(x,y)$ is

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$



Thank you for your attention!

