

Chapter 12

Multiple Integration

(多變量積分)

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Section 12.1

Iterated Integrals and Area in the Plane

(疊積分與平面上的面積)



Iterated Integrals of $f(x, y)$

Type 1 : $\int \left[\int f(x, y) dx \right] dy = \int G(y) dy.$

Type 2 : $\int \left[\int f(x, y) dy \right] dx = \int F(x) dx.$

[Q]: How to evaluate the functions $G(y)$ and $F(x)$?

- $G(y) = \int f(x, y) dx$ = 將 y 視為常數且對 x 積分.
- $F(x) = \int f(x, y) dy$ = 將 x 視為常數且對 y 積分.



Example 2 (Type 2 9 例子)

Evaluate $\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y) dy \right] dx$.

Sol: $\int_1^2 \left[\int_1^x (2x^2y^{-2} + 2y) dy \right] dx = \int_1^2 (-2x^2y^{-1} + y^2) \Big|_{y=1}^{y=x} dx$

$$= \int_1^2 (-2x + x^2 + 2x - 1) dx = \int_1^2 (x^2 - 2x - 1) dx = (x^3 - x^2 - x) \Big|_1^2 = 2 - (-1)$$

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Type I: 第一型平面區域面積

If $g_1(x), g_2(x)$ are conti. on $[a, b]$ and let

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

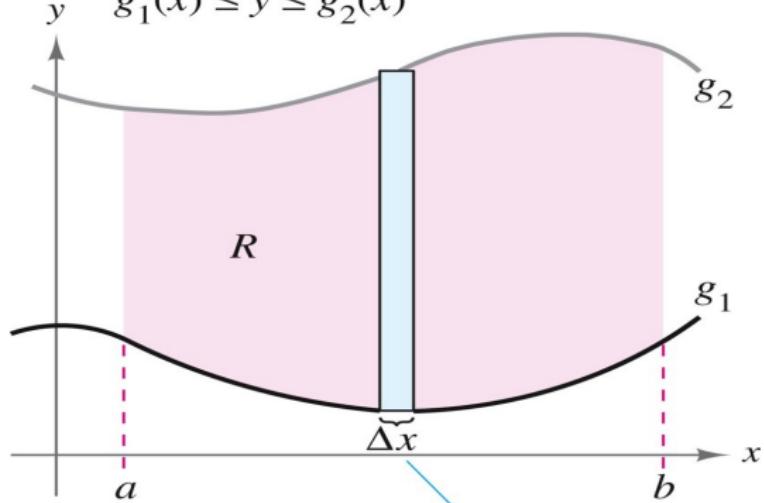
then the area of the plane region \mathcal{R} is given by

$$\begin{aligned} A &= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy \right) dx \\ &= \int_a^b [g_2(x) - g_1(x)] dx. \quad (\text{上學期學過的結果!}) \end{aligned}$$



Type I 的示意圖

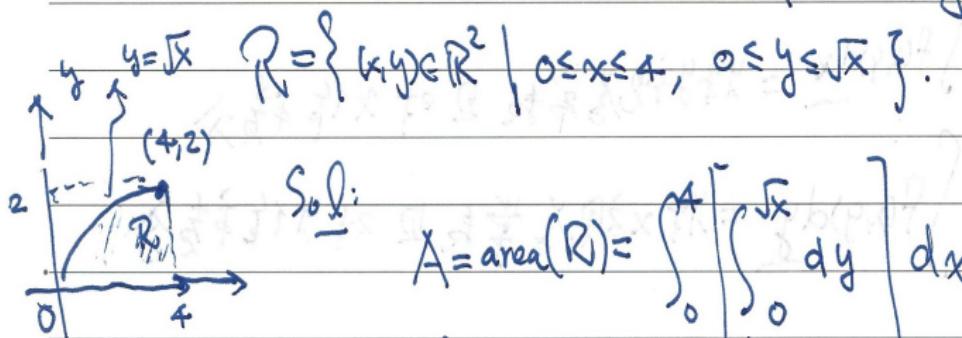
Region is bounded by
 $a \leq x \leq b$ and
 $g_1(x) \leq y \leq g_2(x)$



$$\text{Area} = \int_a^b \int_{g_1(x)}^{g_2(x)} dy \quad (dx)$$



Example 5: Find the area of the plane region



Sol:

$$\begin{aligned} A &= \text{area}(R) = \int_0^4 \left[\int_0^{\sqrt{x}} dy \right] dx \\ &= \int_0^4 x^{1/2} dx = \frac{2}{3} x^{3/2} \Big|_0^4 = \frac{16}{3}. \end{aligned}$$



Type II: 第二型平面區域面積

If $h_1(y), h_2(y)$ are conti. on $[c, d]$ and let

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\},$$

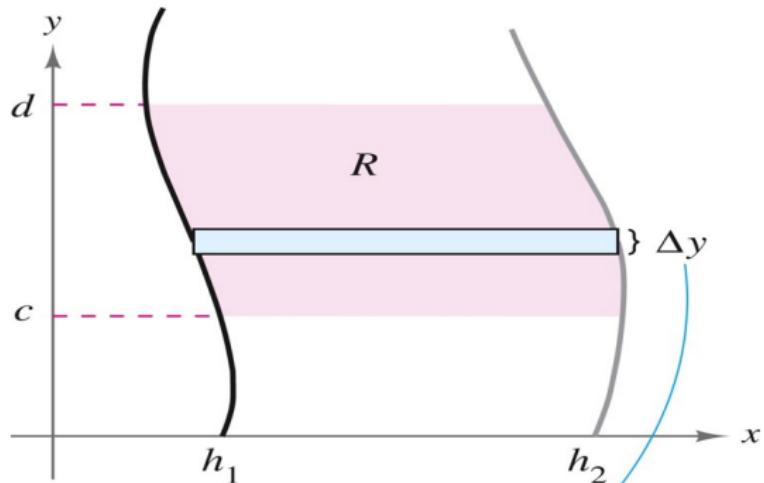
then the area of the plane region \mathcal{R} is given by

$$\begin{aligned} A &= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} dx \right) dy \\ &= \int_c^d [h_2(y) - h_1(y)] dy. \quad (\text{上學期學過的結果!}) \end{aligned}$$



Type II 的示意圖

Region is bounded by
 $c \leq y \leq d$ and
 $h_1(y) \leq x \leq h_2(y)$

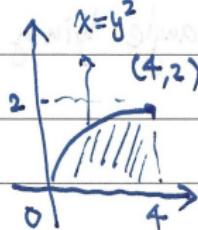


$$\text{Area} = \int_c^d \int_{h_1(y)}^{h_2(y)} dx \, dy$$



Example 5: Find the area of the plane

$$\text{Region } R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2, y^2 \leq x \leq 4\}.$$



Sol: $A = \text{area}(R) = \int_0^2 \left[\int_{y^2}^4 dx \right] dy.$

$$= \int_0^2 (4 - y^2) dy \quad \left[4y - \frac{1}{3}y^3 \right] \Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}.$$



Section 12.2

Double Integrals and Volume

(雙重積分與體積)



Double Integrals of $f(x, y)$

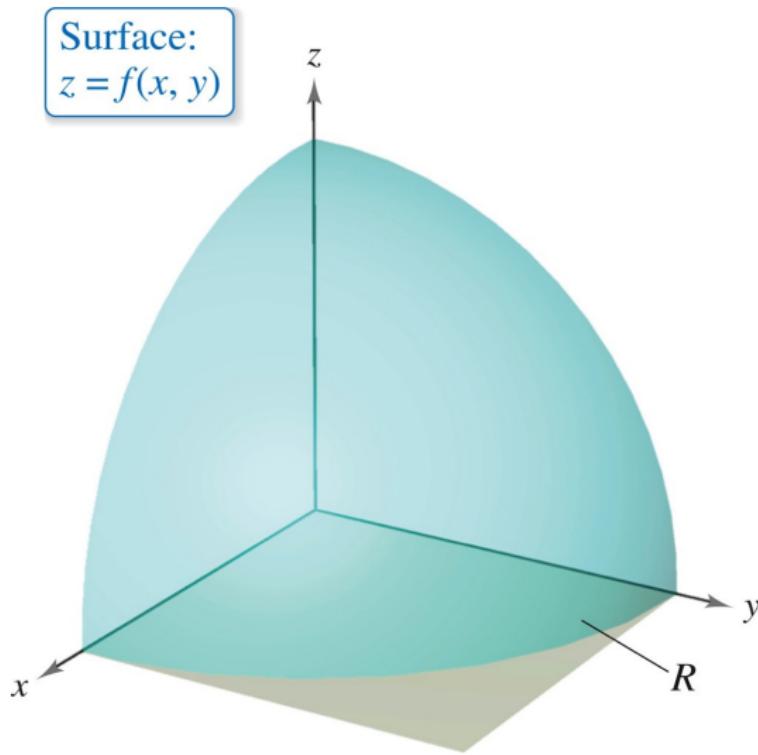
Let $f(x, y)$ be defined on a **closed and bounded** plane region $\mathcal{R} \subseteq \mathbb{R}^2$. Choose an **inner** partition of \mathcal{R} as

$$\Delta = \{R_i \mid R_i \text{ is a small rectangle lying inside } \mathcal{R}, 1 \leq i \leq n\}.$$

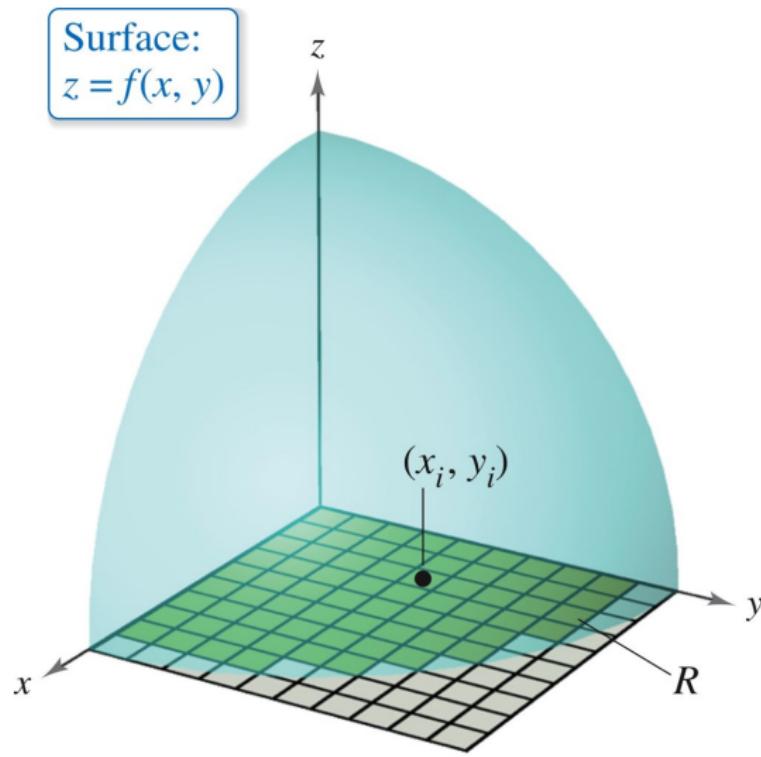
- If $d_i = \text{length of the diagonal of } R_i$ for $i = 1, 2, \dots, n$, the norm of partition Δ is defined by $\|\Delta\| = \max_{1 \leq i \leq n} d_i$.
- The Riemann sum of f associated with Δ is $\sum_{i=1}^n f(x_i, y_i) \Delta A_i$, where $(x_i, y_i) \in R_i$ and $\Delta A_i = \text{area}(R_i)$ for $i = 1, 2, \dots, n$.



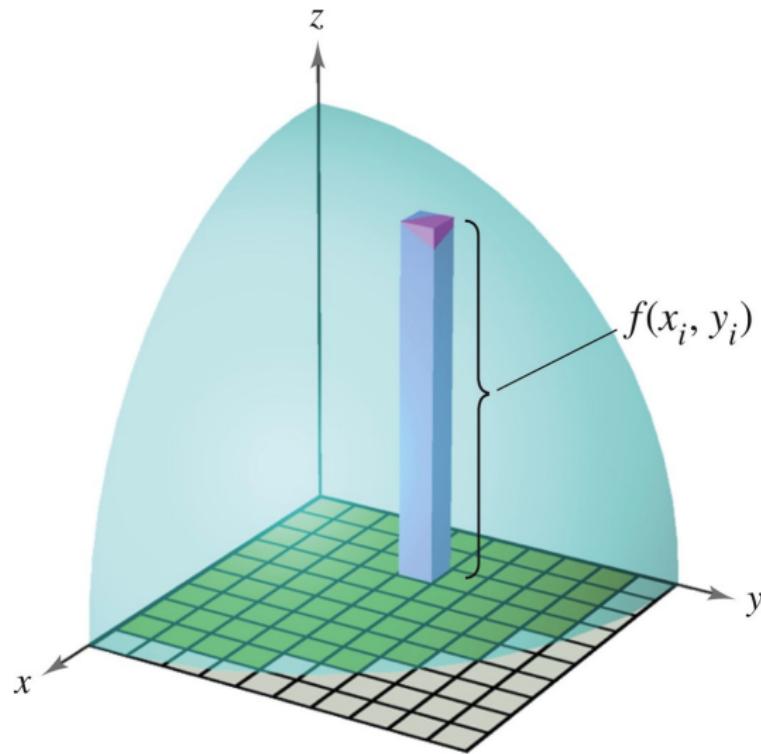
雙重積分的示意圖 (1/4)



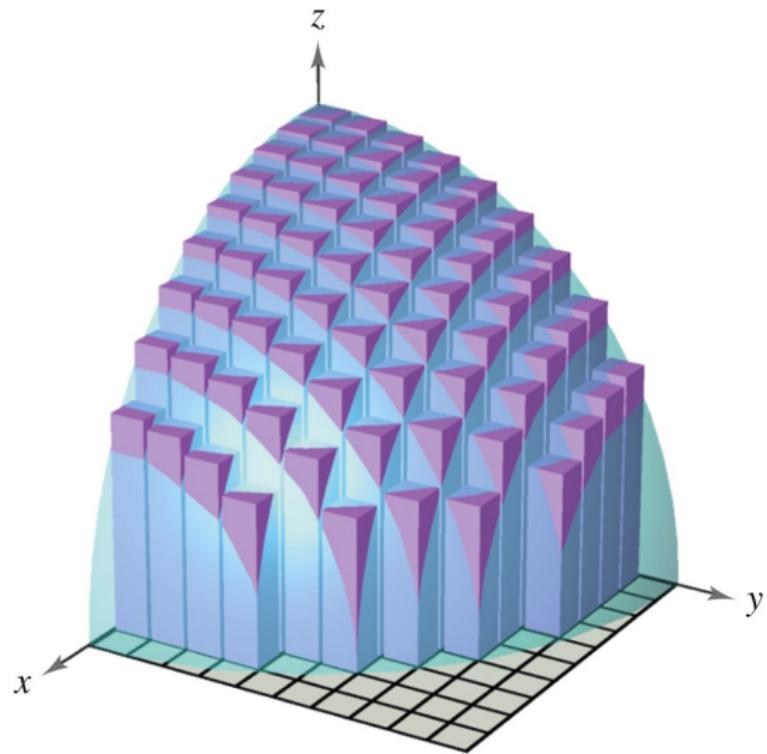
雙重積分的示意圖 (2/4)



雙重積分的示意圖 (3/4)



雙重積分的示意圖 (4/4)



Double Integrals (雙重積分)

Def. (雙重積分的定義)

Let \mathcal{R} be a closed and bounded region in \mathbb{R}^2 .

- (1) The double integral of $f(x, y)$ over the plane region \mathcal{R} is

$$\iint_{\mathcal{R}} f(x, y) dA = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i) \Delta A_i$$

for any inner partition Δ of \mathcal{R} .

- (2) We say that f is integrable over \mathcal{R} if $\iint_{\mathcal{R}} f(x, y) dA$ \exists .



Thm (雙重積分存在性的充分條件)

If $f(x, y)$ is **conti.** on a closed and bounded plane region $\mathcal{R} \subseteq \mathbb{R}^2$,
then f is **integrable** over \mathcal{R} , i.e., the double integral

$$\iint_{\mathcal{R}} f(x, y) dA \quad \exists.$$



Thm 12.1 (Properties of Double Integrals; 1/2)

Suppose f and g are **integrable** over a closed, bounded region \mathcal{R} .

$$(1) \iint_{\mathcal{R}} [c \cdot f(x, y)] dA = c \cdot \left[\iint_{\mathcal{R}} f(x, y) dA \right] \quad \forall c \in \mathbb{R}.$$

$$(2) \iint_{\mathcal{R}} [f(x, y) \pm g(x, y)] dA = \left[\iint_{\mathcal{R}} f(x, y) dA \right] \pm \left[\iint_{\mathcal{R}} g(x, y) dA \right].$$

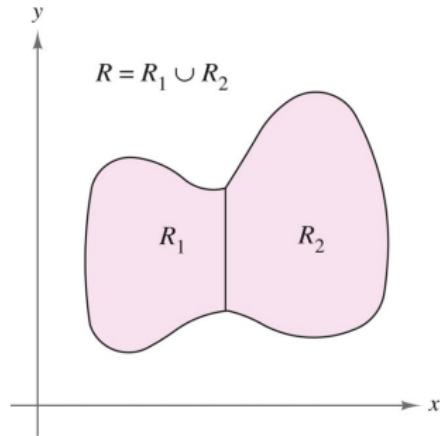
$$(3) \iint_{\mathcal{R}} f(x, y) dA \geq 0 \text{ if } f(x, y) \geq 0 \quad \forall (x, y) \in \mathcal{R}.$$

$$(4) \iint_{\mathcal{R}} f(x, y) dA \geq \iint_{\mathcal{R}} g(x, y) dA \text{ if } f(x, y) \geq g(x, y) \quad \forall (x, y) \in \mathcal{R}.$$



Thm 12.1 (Properties of Double Integrals; 2/2)

(5) $\iint_{\mathcal{R}} f(x, y) dA = \iint_{\mathcal{R}_1} f(x, y) dA + \iint_{\mathcal{R}_2} f(x, y) dA$, where \mathcal{R}_1 and \mathcal{R}_2 are nonoverlapping subregions (非重疊子區域) of \mathcal{R} with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.



Some Questions

- What is the relationship between double integrals and iterated integrals?
- Are these two integrals *always* equal for each function $f(x, y)$?
- How to evaluate its double integral quickly when f is integrable over \mathcal{R} ?
- Fubini's theorem will answer above questions partly!



Thm 12.2 (Fubini's Theorem; 1/2)

Let $f(x, y)$ be conti. on a plane region $\mathcal{R} \subseteq \mathbb{R}^2$.

- (1) If $g_1(x), g_2(x)$ are conti. on $[a, b]$ and the region is defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

then the double integral of f over \mathcal{R} is given by

$$\iint_{\mathcal{R}} f(x, y) dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx.$$



Thm 12.2 (Fubini's Theorem; 2/2)

(2) If $h_1(y), h_2(y)$ are conti. on $[c, d]$ and the region is defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\},$$

then the double integral of f over \mathcal{R} is given by

$$\iint_{\mathcal{R}} f(x, y) dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy.$$



Example 2: Evaluate $\iint_R (1 - \frac{1}{2}x^2 - \frac{1}{2}y^2) dA$, where

the region $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Sol:

Method 1: $\iint_R (1 - \frac{1}{2}x^2 - \frac{1}{2}y^2) dA = \int_0^1 \int_0^1 (1 - \frac{x^2}{2} - \frac{y^2}{2}) dy dx = \frac{2}{3}$.

Method 2: $\iint_R (1 - \frac{x^2}{2} - \frac{y^2}{2}) dA = \int_0^1 \int_0^1 (1 - \frac{x^2}{2} - \frac{y^2}{2}) dx dy$



$$= \int_0^1 \left[\left(1 - \frac{y^2}{2}\right)x - \frac{x^3}{6} \right]_{x=0}^{x=1} dy = \int_0^1 \left(\frac{5}{6} - \frac{y^2}{2}\right) dy$$

$$= \left(\frac{5}{6}y - \frac{1}{6}y^3 \right) \Big|_0^1 = \frac{4}{6} = \underline{\underline{\frac{2}{3}}} \quad \text{※}$$



Def. (實心區域的體積)

If $f(x, y)$ is integrable over a closed and bounded region \mathcal{R} , and $f(x, y) \geq 0 \quad \forall (x, y) \in \mathcal{R}$, then the volume of the solid region that lies over \mathcal{R} and below the graph of f is

$$V = \iint_{\mathcal{R}} f(x, y) dA \geq 0.$$



Example 3: Find the volume of the solid region

bounded by $z = f(x, y) = 4 - x^2 - 2y^2$ and the xy -plane.

Sol: Note that $f(x, y) \geq 0$ is conti over the plane.

$$\text{region } R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 2y^2 \leq 4\}$$

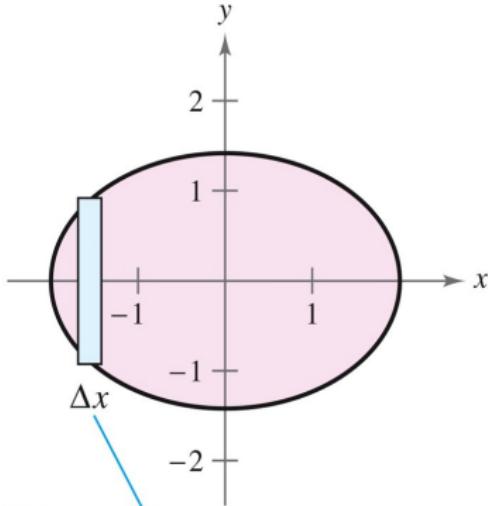
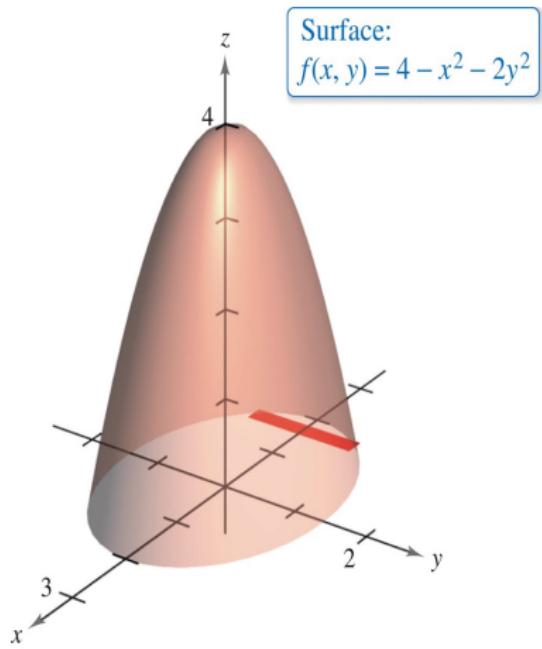
Per-Dust



Example 3 的示意圖

Base: $-2 \leq x \leq 2$

$$-\sqrt{(4-x^2)/2} \leq y \leq \sqrt{(4-x^2)/2}$$



Volume:

$$\int_{-2}^2 \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (4 - x^2 - 2y^2) dy dx$$



If we write $R = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 2, -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}}\}$,

then $V = \iint_R f(x, y) dA = \int_0^2 \int_0^{\sqrt{\frac{4-x^2}{2}}} (4-x^2-y^2) dy dx$.

$$= \int_0^2 \left[(4-x^2)y - \frac{2}{3}y^3 \right]_{y=0}^{y=\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \frac{8}{3\sqrt{2}} \int_0^2 (4-x^2)^{3/2} dx.$$



Let $x = 2\sin\theta$. Then $dx = 2\cos\theta d\theta$.

$$S_0, V = \frac{8}{3\sqrt{2}} \int_0^{\pi/2} (8\cos^3\theta)(2\cos\theta) d\theta$$

$$= \frac{128}{3\sqrt{2}} \int_0^{\pi/2} \cos^4\theta d\theta \stackrel{!}{=} \frac{128}{3\sqrt{2}} \left(\frac{3\pi}{16} \right) = 4\sqrt{2}\pi.$$



Homework

Evaluate the following definite integral

$$\int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta = \frac{3\pi}{16}.$$

(此結果會被應用於 Example 3 和 Example 5!)

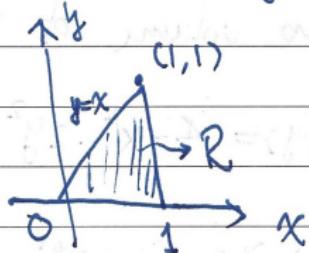




Example 4 (比較豊積分の積分順序の例子).

Find the volume of the solid region bounded by

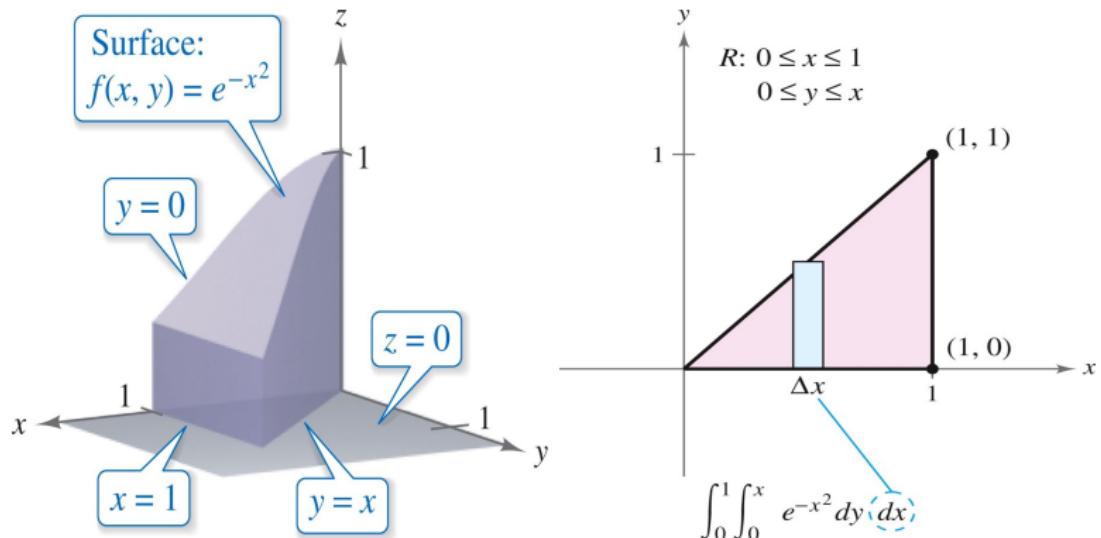
$$z = f(x,y) = e^{-x^2} \text{ over the plane region } R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq x\}.$$



$$\text{Sol: } V = \iiint_R e^{-x^2} dA = \int_0^1 \int_0^x e^{-y^2} dy dx \text{ by Fubini's Thm}$$



Example 4 的示意圖



$$= \int_0^1 (e^{-x^2} y) \Big|_{y=0}^{y=x} dx = \int_0^1 x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} \Big|_0^1$$

$$= \frac{-1}{20} + \frac{1}{2} = \frac{e-1}{20} \approx 0.316$$

Remark: If we write $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1, y \leq x \leq 1\}$,

$$\text{then } V = \iint_R e^x dA = \int_0^1 \int_{y+1}^1 e^x dx dy \quad (\times)$$



\$\text{Q}\$

Example 5: (夾在兩曲面間的區域體積.)

Find the volume of the solid region bounded above

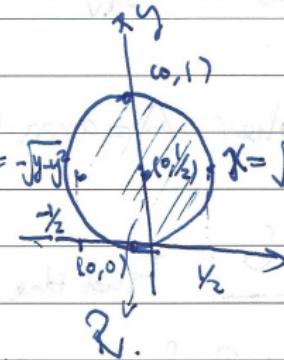
by $z = f(x,y) = 1 - x^2 - y^2$ and below by $z = g(x,y) = 1 - y$.

Sol: Note that $1 - y = z = 1 - x^2 - y^2 \Leftrightarrow x^2 + y^2 - y = 0$.

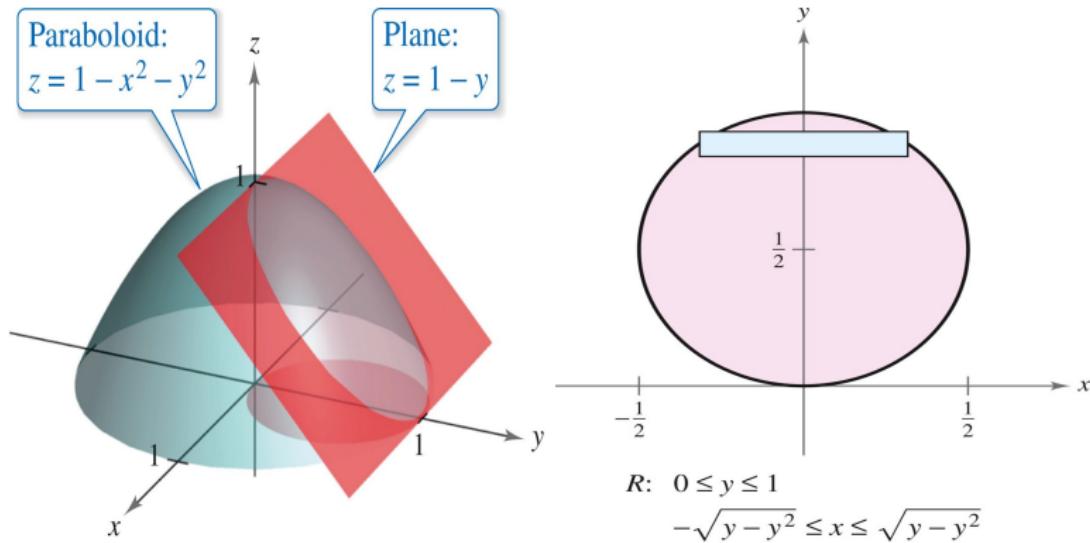
$$\Leftrightarrow x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4} = \left(\frac{1}{2}\right)^2$$

Write $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \text{ and } x = -\sqrt{y-y^2} \text{ to } x = \sqrt{y-y^2}\}$

$$-\sqrt{y-y^2} \leq x \leq \sqrt{y-y^2} \}$$



Example 5 的示意圖



$$\text{f}(x,y) = 1-x^2-y^2 \geq 1-y = g(x,y) \quad \forall (x,y) \in R.$$

$$\begin{aligned}
 V &= \iint_R f dA - \iint_R g dA = \iint_R (f-g) dA = \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y-y^2-x^2) dx dy \\
 &= \int_0^1 \left[(y-y^2)x - \frac{1}{3}x^3 \right]_{x=-\sqrt{y-y^2}}^{x=\sqrt{y-y^2}} dy = \left(\frac{4}{3} \right) \int_0^1 (y-y^2)^{3/2} dy.
 \end{aligned}$$

Per-Dust



$$= \left(\frac{4}{3} \right) \left(\frac{1}{8} \right) \int_0^1 (4y - 4y^2)^{3/2} dy = \frac{1}{6} \int_0^1 [1 - (2y - 1)^2]^{3/2} dy.$$

Let $2y-1 = \sin\theta$. Then $y = \frac{\sin\theta + 1}{2}$ and $dy = \frac{1}{2} \cos\theta d\theta$.

$$\Rightarrow V = \frac{1}{6} \int_{-\pi/2}^{\pi/2} \cos^3\theta \left(\frac{1}{2} \cos\theta\right) d\theta = \frac{1}{6} \int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{2} d\theta.$$

$$= \frac{1}{6} \int_0^{\pi/2} \cos^4\theta d\theta \quad (\because \frac{\cos^4\theta}{2} \text{ is even})$$

$$= \underline{\underline{\left(\frac{1}{6} \right)}} \underline{\underline{\left(\frac{3\pi}{16} \right)}} = \underline{\underline{\frac{\pi}{32}}} *$$



The Average Value of $f(x, y)$

Def. (雙自變量函數的平均值)

If $f(x, y)$ is integrable over a closed and bounded region \mathcal{R} , then the average value of f over \mathcal{R} is

$$f_{av} \equiv \frac{1}{\text{area}(\mathcal{R})} \iint_{\mathcal{R}} f(x, y) dA.$$

Recall (單自變量函數的平均值)

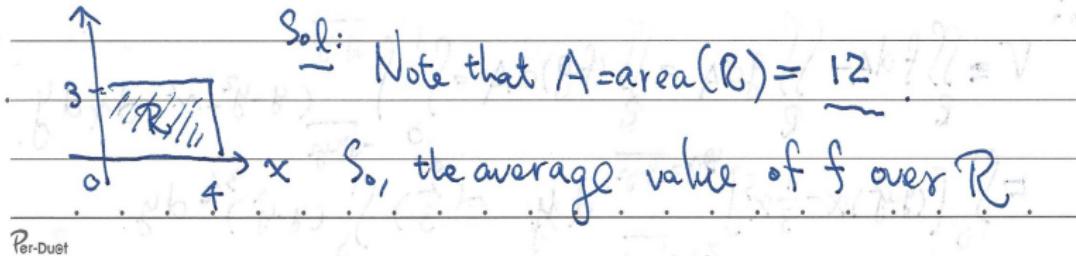
The average value of an integrable function $y = f(x)$ on $[a, b]$ is

$$f_{av} \equiv \frac{1}{b-a} \int_a^b f(x) dx.$$



Example 6: Find the average value of $f(x,y) = \frac{1}{2}xy$

over $R = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4, 0 \leq y \leq 3\}$.



Per-Duet



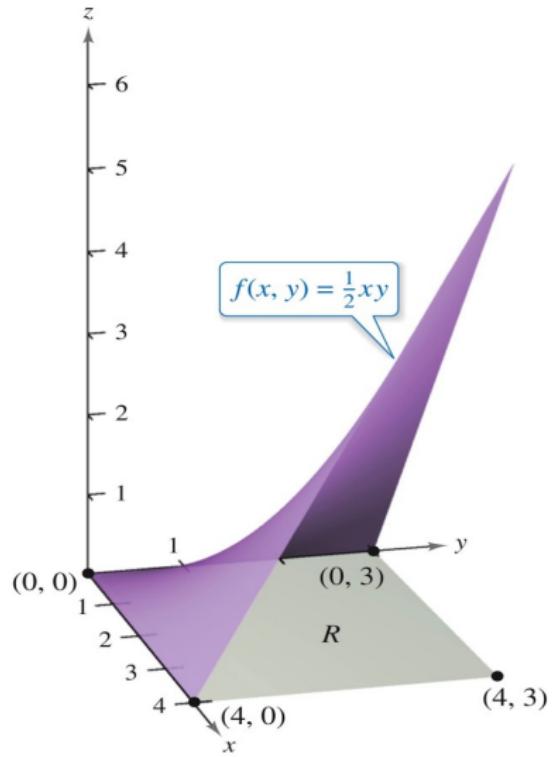
$$\Rightarrow \text{is } \iint_D f_{xy} = \frac{1}{12} \iint_D (\frac{1}{2}xy) dA = \frac{1}{12} \int_0^4 \int_0^3 (\frac{1}{2}xy) dy dx.$$

$$= \frac{1}{12} \int_0^4 (\frac{1}{4}xy^2) \Big|_0^3 dx = \left(\frac{1}{12} \right) \left(\frac{9}{4} \right) \int_0^4 x dx$$

$$= \left(\frac{3}{16} \right) \left[\left(\frac{1}{2}x^2 \right) \Big|_0^4 \right] = \underline{\underline{\frac{3}{2}}} \quad *$$



Example 6 的示意圖



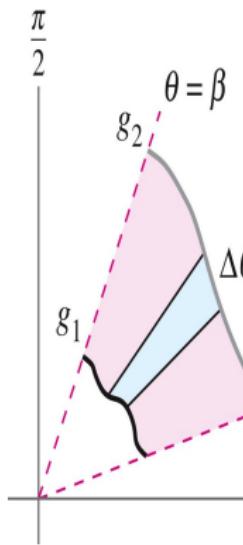
Section 12.3

Change of Variables: Polar Coordinates

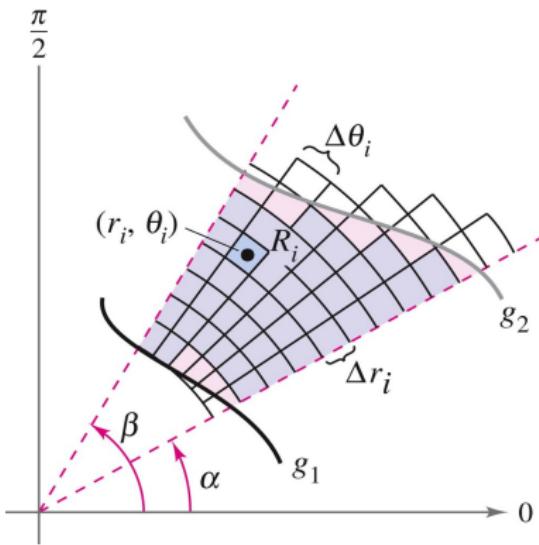
(變數變換：極坐標)



平面極坐標區域的分割



Fixed bounds for θ :
 $\alpha \leq \theta \leq \beta$
Variable bounds for r :
 $0 \leq g_1(\theta) \leq r \leq g_2(\theta)$

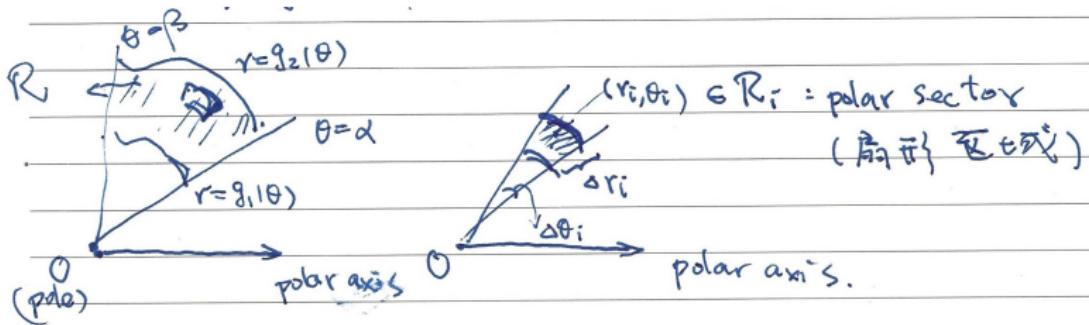


Let $f(x, y)$ be conti. on a closed and bounded region $\mathcal{R} \subseteq \mathbb{R}^2$. If we can rewrite \mathcal{R} as a polar region defined by

$$\mathcal{R} = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, 0 \leq g_1(\theta) \leq r \leq g_2(\theta)\},$$

then consider an inner partition of \mathcal{R} as

$$\Delta = \{R_i \mid R_i \text{ is a polar sector lying insise } \mathcal{R}, 1 \leq i \leq n\}.$$



- For each $i = 1, 2, \dots, n$ and any $(r_i, \theta_i) \in R_i$, the area of R_i is

$$\begin{aligned}\Delta A_i &= \frac{1}{2} \left(r_i + \frac{\Delta r_i}{2} \right)^2 \Delta \theta_i - \frac{1}{2} \left(r_i - \frac{\Delta r_i}{2} \right)^2 \Delta \theta_i \\ &= \frac{1}{2} (2r_i)(\Delta r_i)\Delta \theta_i = r_i \cdot \Delta r_i \cdot \Delta \theta_i.\end{aligned}$$

- The Riemann sum of f associated with Δ is given by

$$\sum_{i=1}^n f(x_i, y_i) \Delta A_i = \sum_{i=1}^n f(r_i \cos \theta_i, r_i \sin \theta_i) r_i \cdot \Delta r_i \cdot \Delta \theta_i.$$



Thm 12.3 (直角坐標積分轉換為極坐標積分)

If $g_1(\theta), g_2(\theta)$ are conti. on $[\alpha, \beta]$ with $0 \leq \beta - \alpha < 2\pi$, and $f(x, y)$ is conti. on the polar region

$$\mathcal{R} = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, 0 \leq g_1(\theta) \leq r \leq g_2(\theta)\},$$

then the double integral of f over \mathcal{R} is given by

$$\begin{aligned}\iint_{\mathcal{R}} f(x, y) dA &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i, y_i) \Delta A_i \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(r_i \cos \theta_i, r_i \sin \theta_i) r_i \Delta r_i \Delta \theta_i; \\ &= \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.\end{aligned}$$



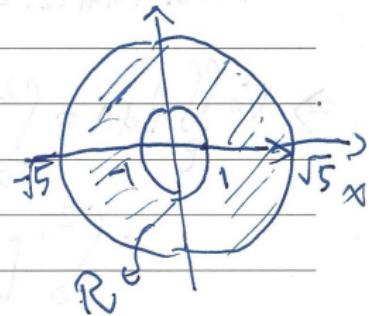
Example 2: Evaluate $\iint_R (x^2+y) dA$,

where R is the annular region between two circles

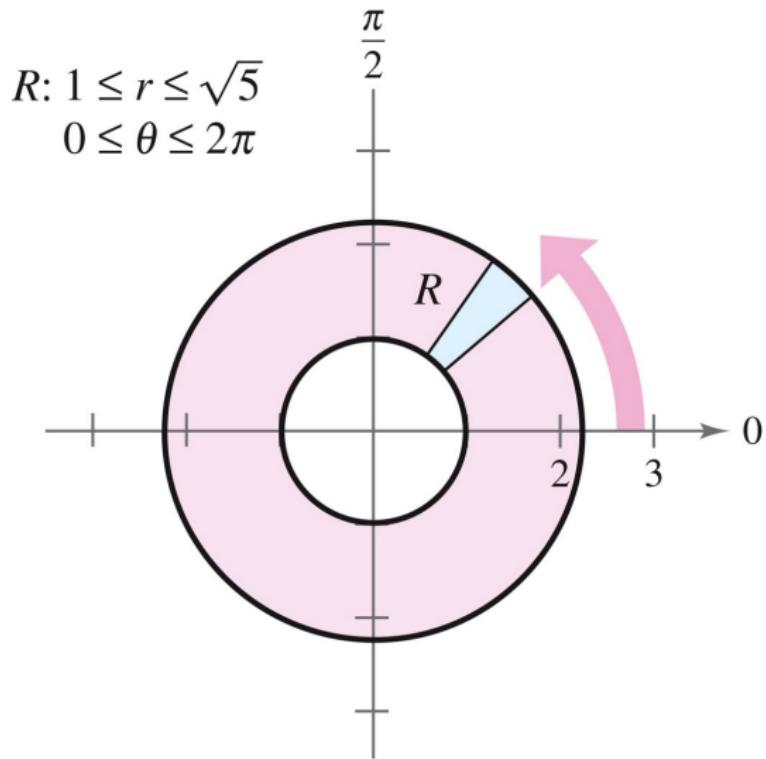
$$x^2+y^2=1 \text{ and } x^2+y^2=5.$$

Sol: Let $x=r\cos\theta$ and $y=r\sin\theta$ for $0 \leq \theta \leq 2\pi$.

$$\Rightarrow R \text{ rewrite } R = \{(r,\theta) | 0 \leq \theta \leq 2\pi, 1 \leq r \leq \sqrt{5}\}.$$



Example 2 的示意圖



$$\begin{aligned}
 & \int_0^{\frac{\pi}{2}} \iint_R f(x)(x^2+y^2) dA = \int_0^{2\pi} \int_0^{\sqrt{5}} (r^2 \cos^2 \theta + r^2 \sin^2 \theta) r dr d\theta. \\
 &= \int_0^{2\pi} \int_1^{\sqrt{5}} (r^3 \cos^2 \theta + r^2 \sin^2 \theta) dr d\theta = \int_0^{2\pi} \left(\frac{r^4}{4} \cos^2 \theta + \frac{r^3}{3} \sin^2 \theta \right) \Big|_1^{\sqrt{5}} d\theta. \\
 &= \int_0^{2\pi} \left(6 \cos^2 \theta + \frac{5\sqrt{5}-1}{3} \sin \theta \right) d\theta = \int_0^{2\pi} \left(3 + 3 \cos 2\theta + \frac{5\sqrt{5}-1}{3} \sin \theta \right) d\theta \\
 &= \left(3\theta + \frac{3}{2} \sin 2\theta - \frac{5\sqrt{5}-1}{3} \cos \theta \right) \Big|_0^{2\pi} = 6\pi. \quad \text{***}
 \end{aligned}$$



Example 3: Find the volume of the solid region

bounded above by $z = \sqrt{16 - x^2 - y^2}$ and

below by the circular region $R_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$.

Sol: Note that $f(x, y) = \sqrt{16 - x^2 - y^2} \geq 0 \quad \forall (x, y) \in R_1$.

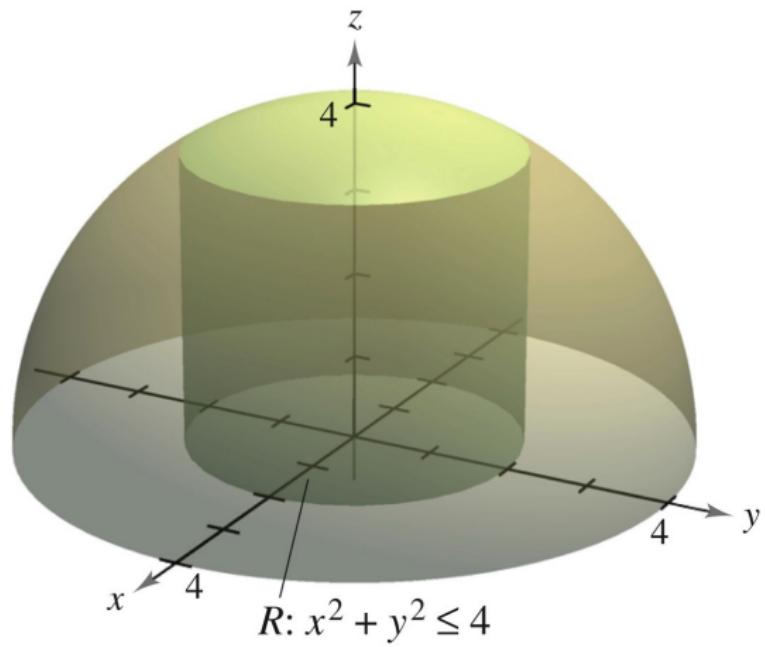
$\Rightarrow V = \iint_R f(x, y) dA \geq 0$. because f is anti. on R .

Rewrite $R_1 = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 2\}$.



Example 3 的示意圖

Surface: $z = \sqrt{16 - x^2 - y^2}$



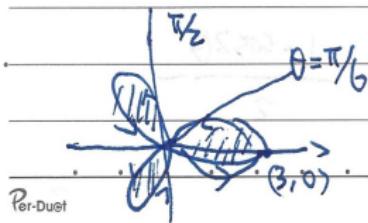
From Thm 12.2, let $x = r \cos \theta$ and $y = r \sin \theta$ for $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} \Rightarrow V &= \iint_R f dA = \int_0^{2\pi} \int_0^2 \sqrt{16-r^2} (r) dr d\theta \\ &= \int_0^{2\pi} \left[\left(-\frac{1}{2} \right) \left(\frac{2}{3} \right) (16-r^2)^{\frac{3}{2}} \right]_0^2 d\theta \\ &= \int_0^{2\pi} \left(\frac{-1}{3} \right) (12^{\frac{3}{2}} - 64) d\theta = \int_0^{2\pi} \left(\frac{8}{3} \right) (8 - 3\sqrt{3}) d\theta. \\ &= \underline{\frac{16\pi}{3}} (8 - 3\sqrt{3}) \stackrel{?}{=} \underline{46.979} \end{aligned}$$



Example 4: Find the area of the polar region enclosed

by $r = f(\theta) = 3 \cos 3\theta$ for $0 \leq \theta \leq 2\pi$.

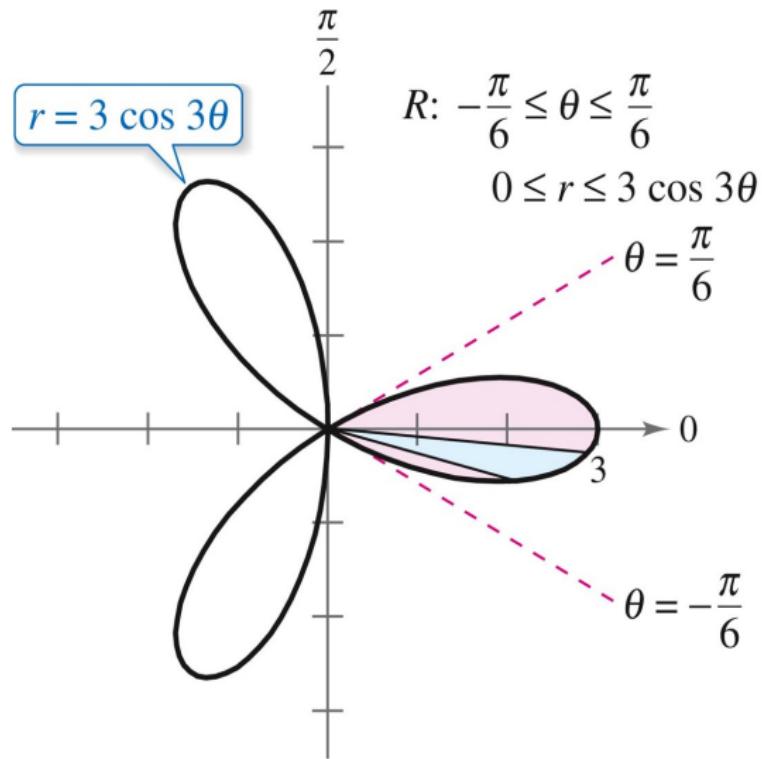


Sol: $A = \text{area}(R) = \int_{0}^{\pi/6} \frac{1}{2} [3 \cos 3\theta]^2 d\theta$.

Let $R_1 = \{(r, \theta) \mid 0 \leq \theta \leq \pi/6, 0 \leq r \leq 3 \cos 3\theta\}$.



Example 4 的示意圖



$$\text{So, } A = \text{area}(R) = 6 \text{ area}(R_1) = 6 \iint_{R_1} 1 dA.$$

$$= 6 \int_0^{\pi/6} \int_0^{\sqrt[3]{\cos 3\theta}} r dr d\theta = 6 \int_0^{\pi/6} \left(\frac{r^2}{2} \Big|_0^{\sqrt[3]{\cos 3\theta}} \right) d\theta.$$

$$= 27 \int_0^{\pi/6} \cos^2(3\theta) d\theta = \frac{27}{2} \int_0^{\pi/6} (1 + \cos 6\theta) d\theta.$$

$$= \frac{27}{2} (\theta + \frac{1}{6} \sin 6\theta) \Big|_0^{\pi/6} = \frac{27}{2} \left(\frac{\pi}{6} \right) = \frac{9\pi}{4}.$$



Section 12.5

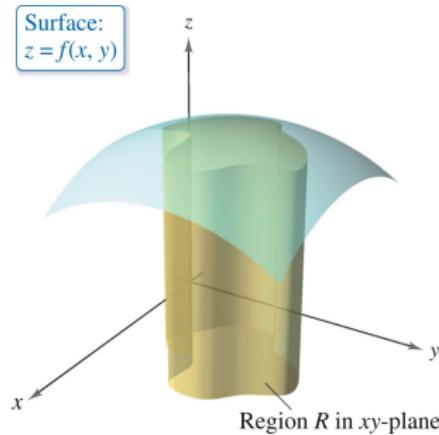
Surface Area

(空間曲面的面積)

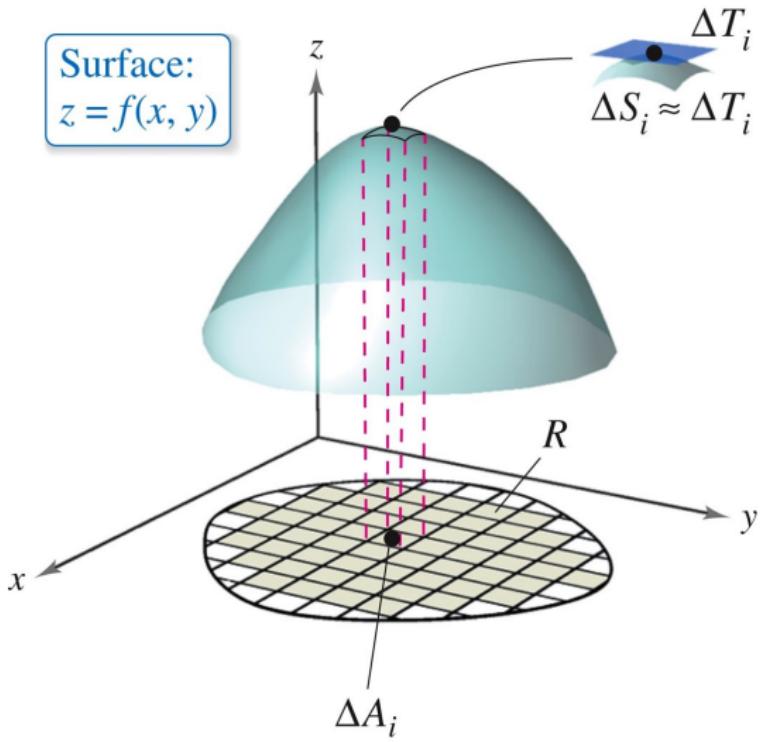


A Question

How to evaluate the area of the surface $z = f(x, y)$ over a closed and bounded plane region \mathcal{R} ?



如何求得空間曲面的表面積?



Derivation of the Surface Area (1/3)

Given a real-valued function $f(x, y)$ s.t. f_x and f_y are conti. on a closed and bounded region \mathcal{R} . Choose an inner partition of \mathcal{R} as

$$\Delta = \{R_i \mid R_i \text{ is a small rectangle lying inside } \mathcal{R}, 1 \leq i \leq n\}.$$

Recall (兩空間向量所生成平行四邊形的面積)

The area of the parallelogram (平行四邊形) generated by two vectors u and v in \mathbb{R}^3 is

$$A = \|u \times v\| = \|v \times u\|.$$



Derivation of the Surface Area (2/3)

Assume that the area of R_i is $\Delta A_i = \Delta x_i \cdot \Delta y_i$ for $i = 1, 2, \dots, n$. If we choose $(x_i, y_i) \in R_i$, the i th surface area of f over R_i is

$$\begin{aligned}\Delta S_i &\approx \Delta T_i = \|u \times v\| \\&= \|\langle \Delta x_i, 0, f_x(x_i, y_i) \Delta x_i \rangle \times \langle 0, \Delta y_i, f_y(x_i, y_i) \Delta y_i \rangle\| \\&= \left\| \begin{vmatrix} i & j & k \\ \Delta x_i & 0 & f_x \Delta x_i \\ 0 & \Delta y_i & f_y \Delta y_i \end{vmatrix} \right\| \\&= \|-f_x(\Delta x_i)(\Delta y_i)i - f_y(\Delta x_i)(\Delta y_i)j + (\Delta x_i)(\Delta y_i)k\| \\&= \|-f_x(\Delta A_i)i - f_y(\Delta A_i)j + (\Delta A_i)k\| \\&= \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2} (\Delta A_i),\end{aligned}$$

for each $i = 1, 2, \dots, n$.



Derivation of the Surface Area (3/3)

Therefore, the surface area of $z = f(x, y)$ over \mathcal{R} is given by

$$\begin{aligned} S &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \Delta S_i \\ &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \sqrt{1 + [f_x(x_i, y_i)]^2 + [f_y(x_i, y_i)]^2} (\Delta A_i) \\ &= \iint_{\mathcal{R}} \sqrt{1 + [f_x(x, y)]^2 + [f_y(x, y)]^2} dA. \end{aligned}$$



Example 2: Find the area of the surface ~~XX~~

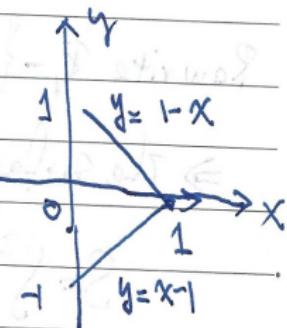
$$z = f(x, y) = 1 - x^2 + y \text{ that lies above the}$$

triangular region R with vertices $(1, 0, 0), (0, 1, 0)$,

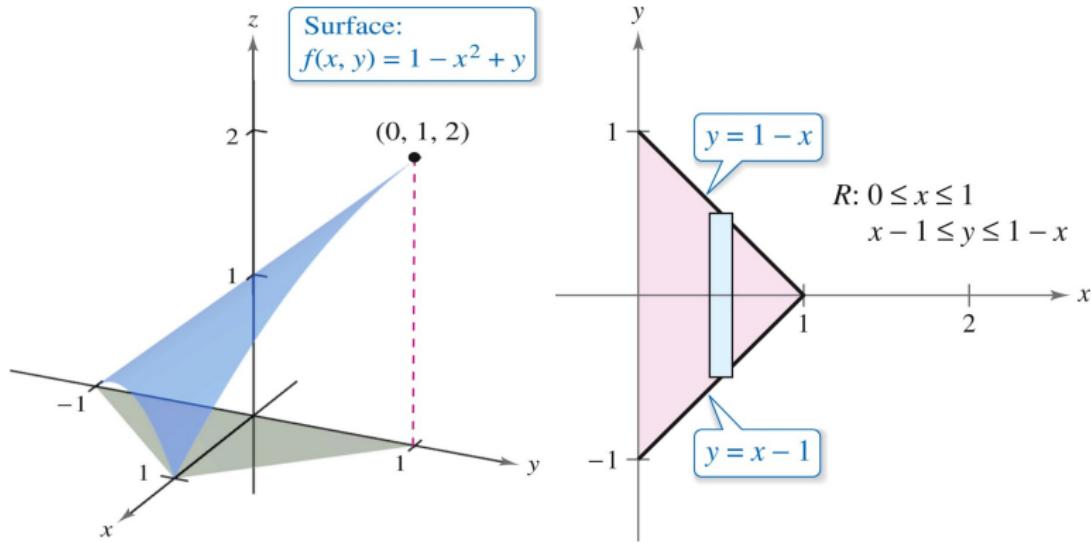
and $(0, 1, 0)$.

Sol: Write $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, x-1 \leq y \leq 1-x\}$.

$\therefore f_x = -2x$ and $f_y = 1$ are conti. on R



Example 2 的示意圖



\therefore the surface area of $z = f(x, y)$ over R is

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA = \iint_R \sqrt{1 + (-2x)^2 + 1^2} dA = \iint_R \sqrt{2 + 4x^2} dA.$$

$$= \int_0^1 \int_{x-1}^{1-x} \sqrt{2+4x^2} dy dx = \int_0^1 \left[y \sqrt{2+4x^2} \right]_{y=x-1}^{y=1-x} dx$$

$$= \int_0^1 2(1-x) \sqrt{2+4x^2} dx = \int_0^1 (2\sqrt{2+4x^2} - 2x \sqrt{2+4x^2}) dx.$$

$$= \int_0^1 \left[(2\sqrt{2})\sqrt{1+2x^2} - \frac{1}{4}(8x)\sqrt{2+4x^2} \right] dx$$

$$= x\sqrt{2+4x^2} + \ln|\sqrt{2}x + \sqrt{1+2x^2}| - \frac{1}{6}(2+4x^2)^{\frac{3}{2}} \Big|_0^1$$

$$= \sqrt{6} + \ln(\sqrt{2} + \sqrt{3}) - \sqrt{6} + \frac{1}{6}(2)^{\frac{3}{2}} = \ln(\sqrt{2} + \sqrt{3}) + \frac{\sqrt{2}}{3} \approx 1.618$$



Homework

Applying the formula (上學期講過的公式)

$$\int \sec^3 \theta \, d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C,$$

and the transformation $x = \frac{1}{\sqrt{2}} \tan \theta$, we obtain

$$\begin{aligned}\int \sqrt{1+2x^2} \, dx &= \frac{1}{\sqrt{2}} \int \sec^2 \theta \sqrt{1+\tan^2 \theta} \, d\theta \\ &= \frac{1}{2\sqrt{2}} \left(x \sqrt{2+4x^2} + \ln \left| \sqrt{2}x + \sqrt{1+2x^2} \right| \right) + C,\end{aligned}$$

where C is a constant of integration.

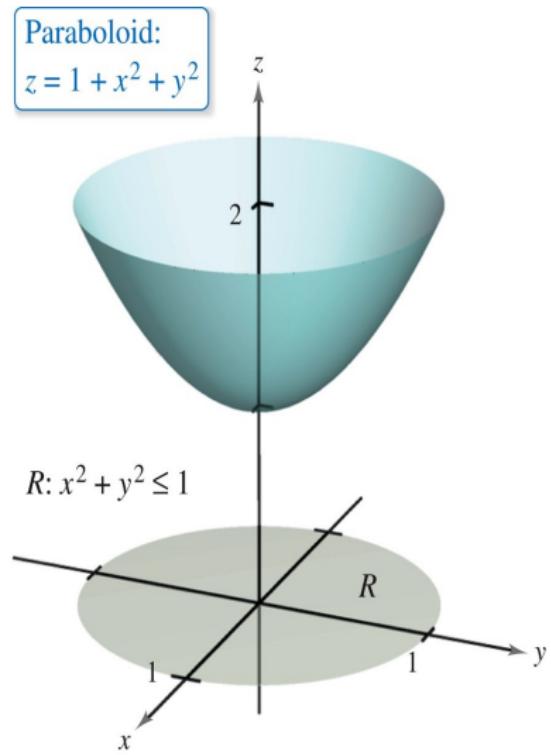


例題 3. (利用極坐標變換求表面積)

Find the surface area of $z = f(x,y) = 1 + x^2 + y^2$ over
the unit circle $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.



Example 3 的示意圖



Sol: $\because f_x = 2x$ and $f_y = 2y$ are conti. on R

$$\therefore S = \iint_R \sqrt{1+(2x)^2 + (2y)^2} \, dA = \iint_R \sqrt{1+4(x^2+y^2)} \, dA.$$

Rewrite $R - \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$.

\Rightarrow The surface area of $z=f(x,y)$ over R is

$$S = \int_0^{2\pi} \int_0^1 \sqrt{1+4r^2} \cdot r \, dr \, d\theta = \int_0^{2\pi} \left(\frac{1}{8} \right) \left(\frac{2}{3} \right) (1+4r^2)^{\frac{3}{2}} \Big|_0^1 \, d\theta.$$

$$= \int_0^{2\pi} \frac{5\sqrt{5}-1}{12} \, d\theta = \frac{\pi(5\sqrt{5}-1)}{6} \approx 5.33$$



Q8

Example 4: Find the surface area of the hemisphere. 半球面

$$z = f(x, y) = \sqrt{25 - x^2 - y^2}$$

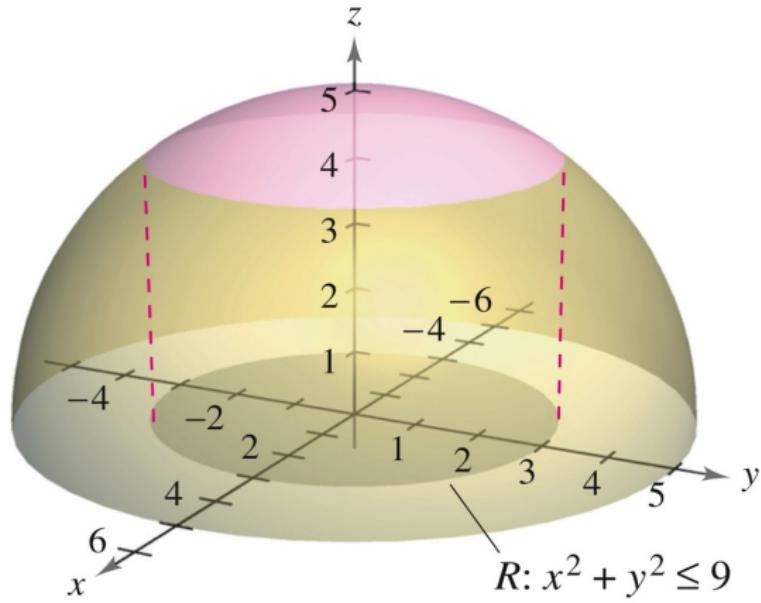
$$\text{over } R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}.$$



Example 4 的示意圖

Hemisphere:

$$f(x, y) = \sqrt{25 - x^2 - y^2}$$



Sol: Note that $f(x,y) \geq 0$ if $25-x^2-y^2 \geq 0$ or $x^2+y^2 \leq 25$

~~ad~~ $f_x = \frac{-x}{\sqrt{25-x^2-y^2}}$ ad $f_y = \frac{-y}{\sqrt{25-x^2-y^2}}$ are conti on \mathbb{R} .

$$\Rightarrow \sqrt{(f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{25-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{25-x^2-y^2}}\right)^2}.$$

$$= \sqrt{\frac{25}{25-x^2-y^2}} = \frac{5}{\sqrt{25-x^2-y^2}}.$$



If we rewrite $R = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 3\}$, then the surface area of $z = f(x, y)$ over R is given by

$$S = \iint_R \frac{5}{\sqrt{25-x^2-y^2}} dA = \int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25-r^2}} \cdot r dr d\theta.$$

$$= \int_0^{2\pi} \left(\frac{-5}{2} \right) (2) (25-r^2)^{\frac{1}{2}} \Big|_0^3 d\theta = \int_0^{2\pi} 5 d\theta = \underline{\underline{10\pi}}.$$



Section 12.6

Triple Integrals and Applications

(三重積分及其應用)

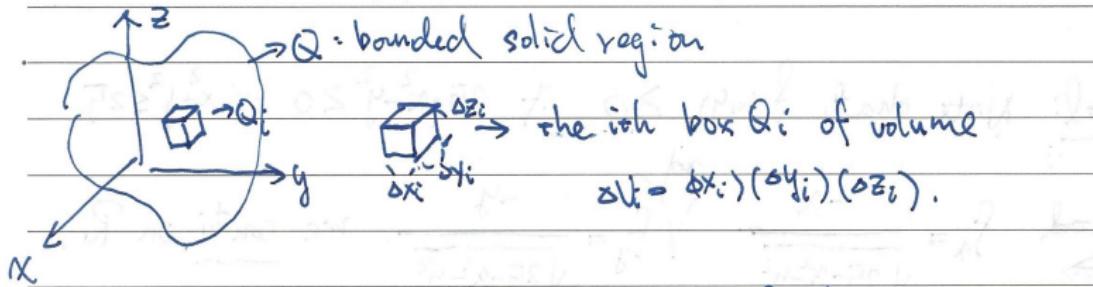


Triple Integrals (1/3)

Let $f(x, y, z)$ be defined on a bounded solid region $Q \subseteq \mathbb{R}^3$. Choose an inner partition Δ of Q defined as

$$\Delta = \{Q_i \mid Q_i \text{ is a small box lying entirely within } Q, 1 \leq i \leq n\},$$

where each box Q_i is of volume $\Delta V_i = \Delta x_i \cdot \Delta y_i \cdot \Delta z_i \quad \forall i$.



Triple Integrals (2/3)

- If d_i is the length of the diagonal of Q_i for $i = 1, 2, \dots, n$, the norm of Δ is defined by $\|\Delta\| \equiv \max_{1 \leq i \leq n} d_i > 0$.
- The Riemann sum of $f(x, y, z)$ associated with Δ is

$$\sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i = \sum_{i=1}^n f(x_i, y_i, z_i) \Delta x_i \cdot \Delta y_i \cdot \Delta z_i,$$

where $(x_i, y_i, z_i) \in Q_i$ is selected arbitrarily for $i = 1, 2, \dots, n$.



Def. (三重積分的定義)

Let $f(x, y, z)$ be defined on a bounded solid region $Q \subseteq \mathbb{R}^3$.

- (1) f is integrable over Q , if the triple integral of f over Q

$$\iiint_Q f(x, y, z) dV \equiv \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_i, y_i, z_i) \Delta V_i \quad \exists$$

for any inner partition Δ of Q .

- (2) The volume of Q is defined by $V = \iiint_Q 1 dV$.



Thm (三重積分存在性的充分條件)

If $f(x, y, z)$ is conti. on a bounded solid region $Q \subseteq \mathbb{R}^3$, then f is integrable over Q , i.e., the triple integral

$$\iiint_Q f(x, y, z) dV \quad \exists.$$



Thm (Properties of Triple Integrals)

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are integrable over a bounded solid region $Q \subseteq \mathbb{R}^3$.

$$(1) \iiint_Q [c \cdot f(x, y, z)] dV = c \cdot \iiint_Q f(x, y, z) dV \quad \forall c \in \mathbb{R}.$$

$$(2) \iiint_Q [f(x, y, z) \pm g(x, y, z)] dV = \left(\iiint_Q f(x, y, z) dV \right) \pm \left(\iiint_Q g(x, y, z) dV \right).$$

$$(3) \iiint_Q f(x, y, z) dV \geq 0 \text{ if } f(x, y, z) \geq 0 \quad \forall (x, y, z) \in Q.$$

$$(4) \iiint_Q f(x, y, z) dV \geq \iiint_Q g(x, y, z) dV \text{ if } f(x, y, z) \geq g(x, y, z) \quad \forall (x, y, z) \in Q.$$

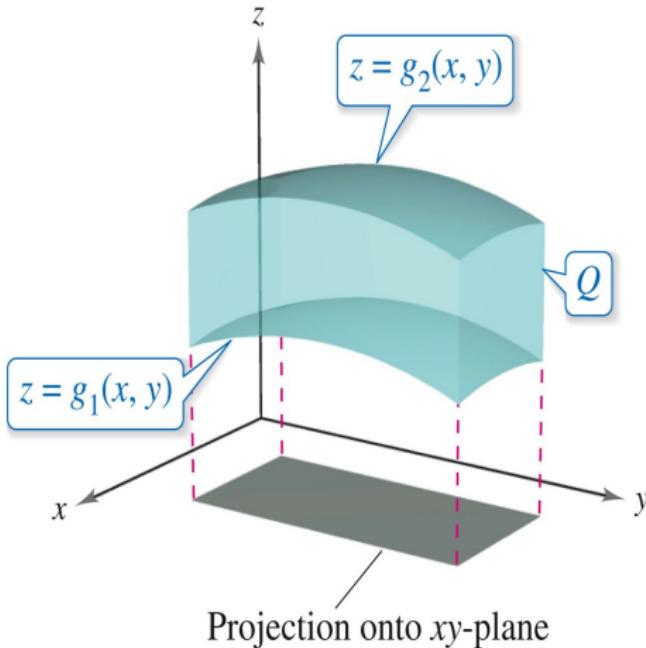
$$(5) \iiint_Q f(x, y, z) dV = \iiint_{Q_1} f(x, y, z) dV + \iiint_{Q_2} f(x, y, z) dV, \text{ where } Q_1 \text{ and } Q_2 \text{ are}$$

nonoverlapping solid subregions of Q with $Q = Q_1 \cup Q_2$.



Main Question

How to evaluate $\iiint_Q f(x, y, z) dV$ via some iterated integrals?



Thm 12.4 (Fubini's Theorem)

Suppose that $f(x, y, z)$ is conti. on a bounded solid region Q , and that h_1, h_2, g_1, g_2 are conti. functions.

- (1) If $Q = \{(x, y, z) \in \mathbb{R}^3 \mid a \leq x \leq b, h_1(x) \leq y \leq h_2(x), g_1(x, y) \leq z \leq g_2(x, y)\}$, then

$$\iiint_Q f(x, y, z) dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz dy dx.$$

- (2) If $Q = \{(x, y, z) \in \mathbb{R}^3 \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), g_1(x, y) \leq z \leq g_2(x, y)\}$, then

$$\iiint_Q f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{g_1(x, y)}^{g_2(x, y)} f(x, y, z) dz dx dy.$$



Example 1: Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z) dz dy dx$.

$$\text{S.o.l: } \int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z) dz dy dx = \int_0^2 \int_0^x e^x (yz + z^2) \Big|_{\substack{z=x+y \\ z=0}} dy dx$$

$$= \int_0^2 \int_0^x (x^2 + 3xy + 2y^2) dy dx = \int_0^2 e^x \left(x^2 y + \frac{3xy^2}{2} + \frac{2y^3}{3} \right) \Big|_{y=0} dy dx$$

$$= \int_0^2 e^x \left(x^3 + \frac{3}{2}x^3 + \frac{2}{3}x^3 \right) dx = \frac{19}{6} \int_0^2 x^3 e^x dx$$

$$= \frac{19}{6} e^x (x^3 - 3x^2 + 6x - 6) \Big|_0^2 = \frac{19}{6} e^2 (2e^2 + 6)$$

$$= 19 \left(\frac{e^2}{3} + 1 \right) \doteq 65.797$$



Homework

Applying the I.B.P. formula to evaluate

$$\int x^3 e^x dx = e^x(x^3 - 3x^2 + 6x - 6) + C,$$

where C is a constant of integration.



Example 2: Find the volume of the ellipsoid (半椭圆面) 体

given by $4x^2 + 4y^2 + z^2 = 16$.

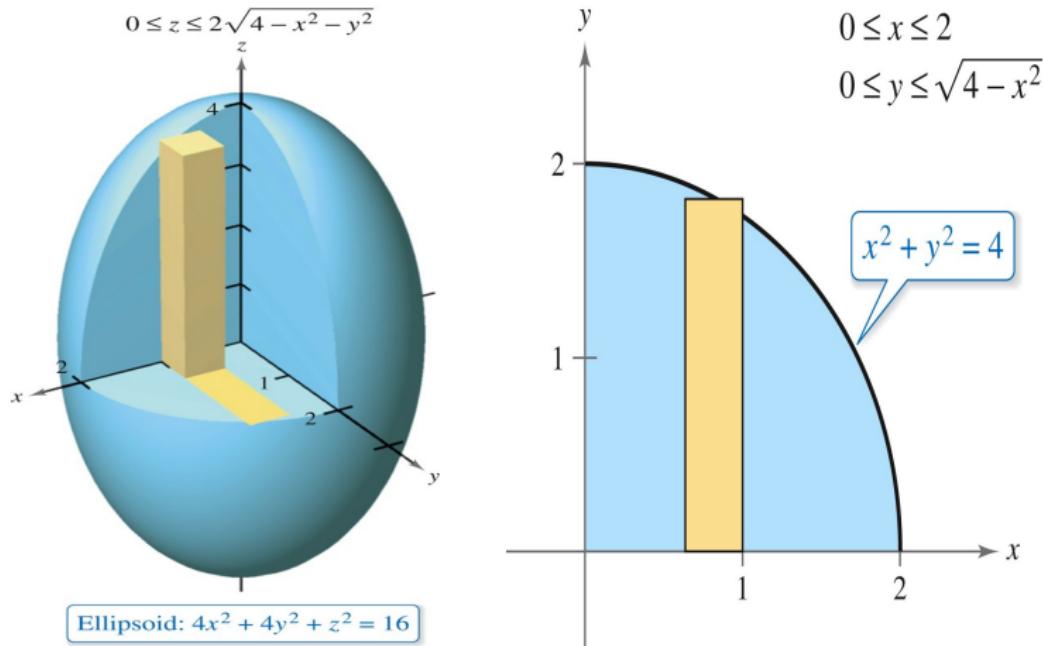
Sol: Let $Q_1 = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq 2, 0 \leq y \leq \sqrt{4-x^2}, 0 \leq z \leq 2\sqrt{4-x^2-y^2}\}$.

be the portion of the ellipsoid in the first quadrant.

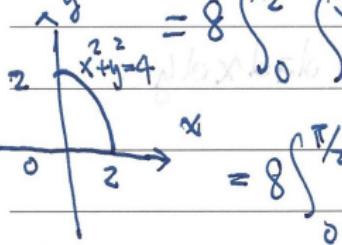
$$\Rightarrow V = 8 \iiint_{Q_1} dV = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{2\sqrt{4-x^2-y^2}} dz dy dx$$



Example 2 的示意圖



$$= 8 \int_0^2 \int_0^{\sqrt{4-x^2}} 2\sqrt{4-x^2-y^2} dy dx.$$



$$= 8 \int_0^{\frac{\pi}{2}} \int_0^2 2\sqrt{4-r^2} \cdot r dr d\theta.$$

$$= (-8) \int_0^{\frac{\pi}{2}} \frac{2}{3} (4-r^2)^{\frac{3}{2}} \Big|_0^2 dr = \left(\frac{16}{3} \right) (8) \left(\frac{\pi}{2} \right) = \underline{\underline{\frac{64\pi}{3}}}.$$

Q. 1

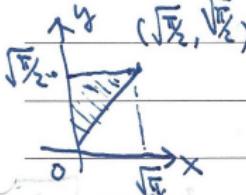


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Example 3: (改變積分順序才能算出體積分值)

Evaluate $\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^{\sqrt{\frac{\pi}{2}}} \int_x^3 z^2 y^2 dx \sin(y^2) dz dy dx$.

Sol: Note that $Q = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x \leq \sqrt{\frac{\pi}{2}}, x \leq y \leq \sqrt{\frac{\pi}{2}}, 0 \leq z \leq 3\}$.

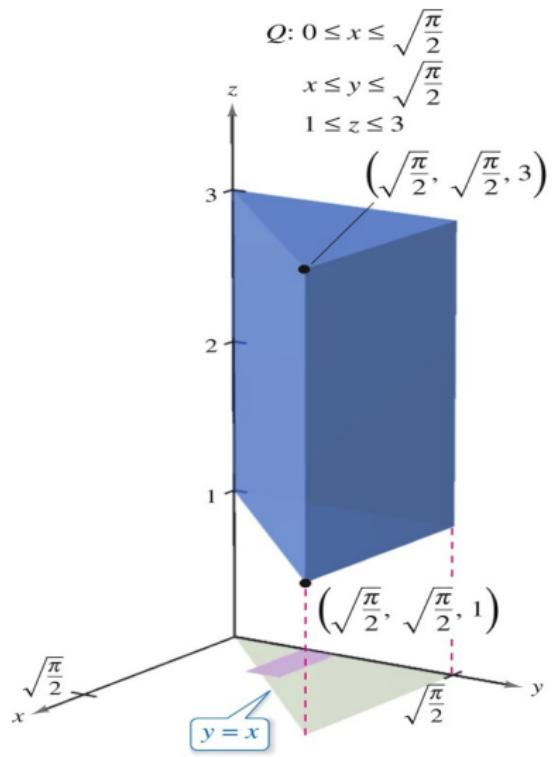

$$= \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq y \leq \sqrt{\frac{\pi}{2}}, 0 \leq x \leq y, 0 \leq z \leq 3\}$$

$$\text{So, } I = \iiint_Q \sin(y^2) dV = \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^y \int_0^3 \sin(y^2) dz dx dy$$

$$= \int_0^{\sqrt{\frac{\pi}{2}}} \int_0^y 2\sin(y^2) dx dy = \int_0^{\sqrt{\frac{\pi}{2}}} 2y \sin(y^2) dy = -\cos(y^2) \Big|_0^{\sqrt{\frac{\pi}{2}}} = 1$$



Example 3 的示意圖

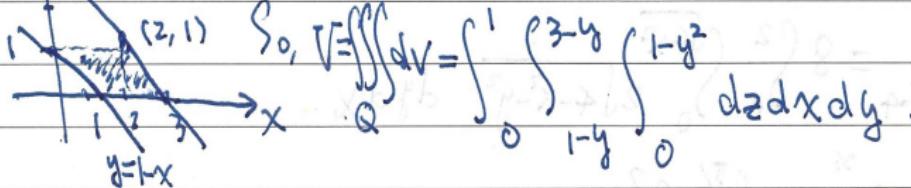


Example 4: Set up a triple integral for the volume of each solid region.

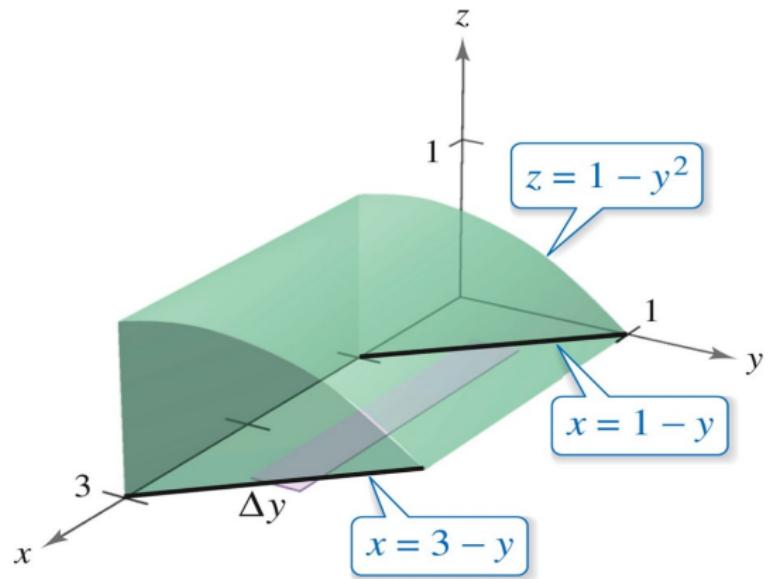
(a) The region Q in the first octant bounded above by $z=1-y^2$ and lying between the planes $x+y=1$ and $x+y=3$.

Sol:

Consider $Q = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq y \leq 1, 1-y \leq x \leq 3-y, 0 \leq z \leq 1-y^2\}$.



Example 4(a) 的示意圖



$$Q: \begin{aligned} 0 &\leq z \leq 1 - y^2 \\ 1 - y &\leq x \leq 3 - y \\ 0 &\leq y \leq 1 \end{aligned}$$



(c) The region Q bounded below by $z = x^2 + y^2$ and above by $x^2 + y^2 + z^2 = 6$.

Sol: Note that $x^2 + y^2 + z^2 = 6$ and $z = x^2 + y^2$

$$\Leftrightarrow z + z^2 = 6 \Rightarrow (z+3)(z-2) = 0$$

$$\Rightarrow z = -3 \text{ or } z = 2 \Rightarrow z = 2.$$

($z = -3$)

So, the two surfaces intersect $z = 2$. and hence the solid region Q lies above the plane region

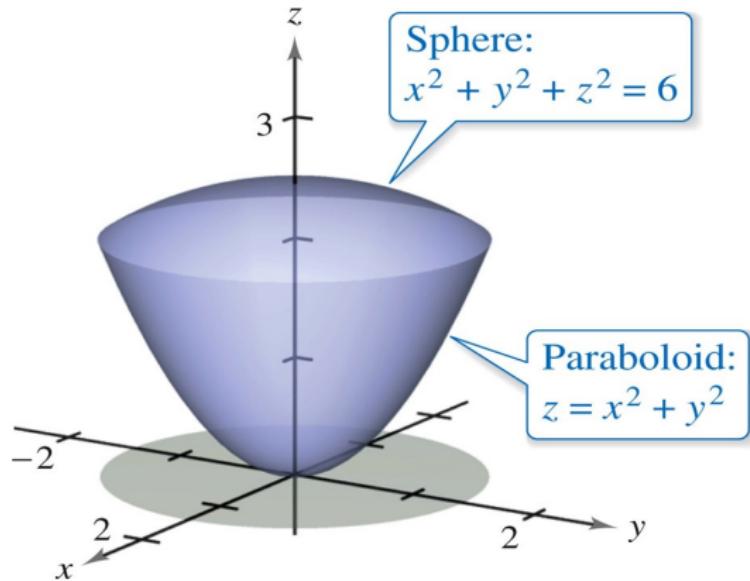
$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2\} \quad \text{if } x, y \in \mathbb{R}$$

If we write $Q = \{(x, y, z) \in \mathbb{R}^3 \mid -\sqrt{2} \leq x \leq \sqrt{2}, -\sqrt{2-x^2} \leq y \leq \sqrt{2-x^2}, x^2 + y^2 \leq z \leq \sqrt{6-x^2-y^2}\}$

$$\text{then } V = \iiint_Q dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} dz dy dx.$$



Example 4(c) 的示意圖



$$\begin{aligned}Q: \quad & x^2 + y^2 \leq z \leq \sqrt{6 - x^2 - y^2} \\& -\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2} \\& -\sqrt{2} \leq x \leq \sqrt{2}\end{aligned}$$



Section 12.7

Triple Integrals in Other Coordinates

(在其他坐標上的三重積分)



Main Goal

In this section, we will transform the triple integral

$$\iiint_Q f(x, y, z) dV$$
 in the rectangular coordinates to some iterated integral in other coordinate systems.

Type I: Change of Variables in Cylindrical Coordinates

Type II: Change of Variables in Spherical Coordinates



Type I: Cylindrical Coordinates

Thm (柱面坐標的變數變換公式)

If $f(x, y, z)$ is conti. on a solid region defined, in the cylindrical coordinates, as

$$Q = \{(r, \theta, z) \mid \theta_1 \leq \theta \leq \theta_2, 0 \leq g_1(\theta) \leq r \leq g_2(\theta), h_1(r, \theta) \leq z \leq h_2(r, \theta)\},$$

where g_1, g_2, h_1, h_2 are conti. functions, then

$$\iiint_Q f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r, \theta)}^{h_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta.$$



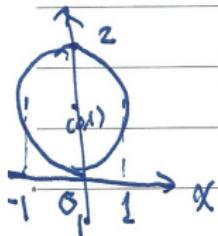
Example 1:

Find the volume of the solid region Q

cut from $x^2 + y^2 + z^2 = 4$ by the cylinder $x^2 + (y-1)^2 = 1$.

Sol:

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r^2 = x^2 + y^2$, and hence



$$x^2 + (y-1)^2 = 1 \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow r^2 - 2r\sin\theta = 0$$

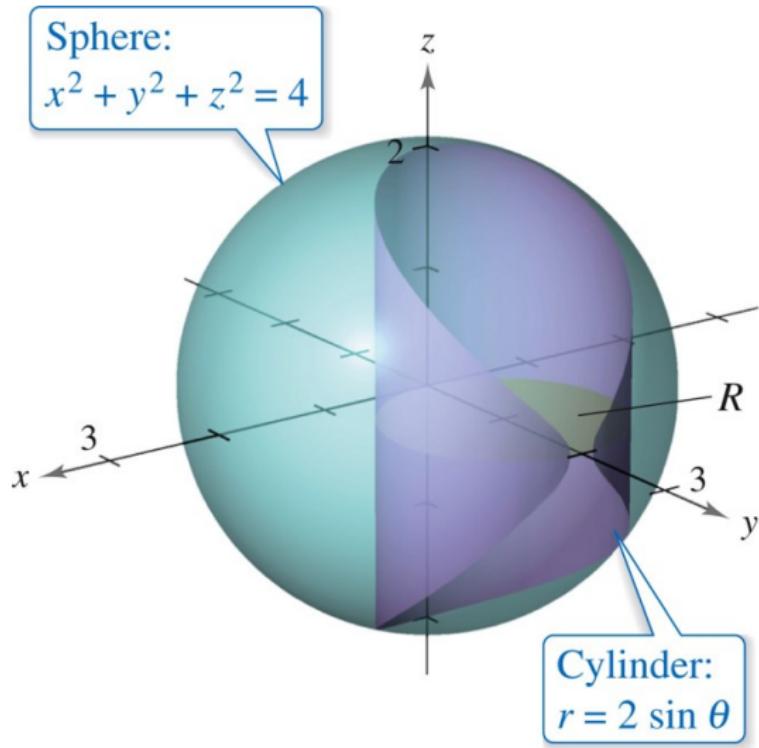
$$\Rightarrow r(r - 2\sin\theta) = 0 \Rightarrow r = 2\sin\theta \text{ for } 0 \leq \theta \leq \pi$$

$$\therefore x^2 + y^2 + z^2 = 4 \text{ and } r^2 = x^2 + y^2.$$

$$\therefore z^2 = 4 - r^2 \Rightarrow -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}$$



Example 1 的示意圖



If we consider the solid subregion of Q defined by

$$Q_1 = \{(r, \theta, z) \mid 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 2 \sin \theta, -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}\},$$

then the volume of the solid region Q is

$$\begin{aligned} V &= 2 \cdot \text{volume}(Q_1) = 2 \iiint_{Q_1} 1 \, dV \\ &= 2 \int_0^{\frac{\pi}{2}} \int_0^{2 \sin \theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \int_0^{2 \sin \theta} 2r\sqrt{4-r^2} \, dr \, d\theta \\ &= 2 \int_0^{\frac{\pi}{2}} \left(\frac{-2}{3} \right) (4-r^2)^{3/2} \Big|_{r=0}^{r=2 \sin \theta} \, d\theta. \end{aligned}$$



Since $1 - \sin^2 \theta = \cos^2 \theta$, it follows that

$$\begin{aligned} V &= 2 \int_0^{\frac{\pi}{2}} \left(\frac{-2}{3} \right) (8\cos^3 \theta - 8) d\theta \\ &= \frac{32}{3} \int_0^{\frac{\pi}{2}} (1 - \cos^3 \theta) d\theta \\ &= \frac{32}{3} \left(\theta - \sin \theta + \frac{\sin^3 \theta}{3} \right) \Big|_0^{\frac{\pi}{2}} \quad (\text{Check!}) \\ &= \frac{32}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right) = \frac{16}{9}(3\pi - 4). \end{aligned}$$



Type II: Spherical Coordinates

Thm (球面坐標的變數變換公式)

If $f(x, y, z)$ is conti. on a solid region defined, in the spherical coordinates, as

$$Q = \{(\rho, \theta, \phi) \mid \rho_1 \leq \rho \leq \rho_2, \theta_1 \leq \theta \leq \theta_2, \phi_1 \leq \phi \leq \phi_2\},$$

where $\rho_1 \geq 0$, $0 \leq \theta_2 - \theta_1 \leq 2\pi$ and $0 \leq \phi_1 \leq \phi_2 \leq \pi$, then

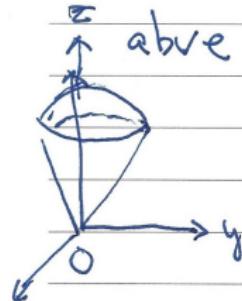
$$\iiint_Q f(x, y, z) dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta.$$



例 4 Find the volume of the solid region

Q bounded below by the cone $z^2 = x^2 + y^2$ and

above by the sphere $x^2 + y^2 + z^2 = 9$.



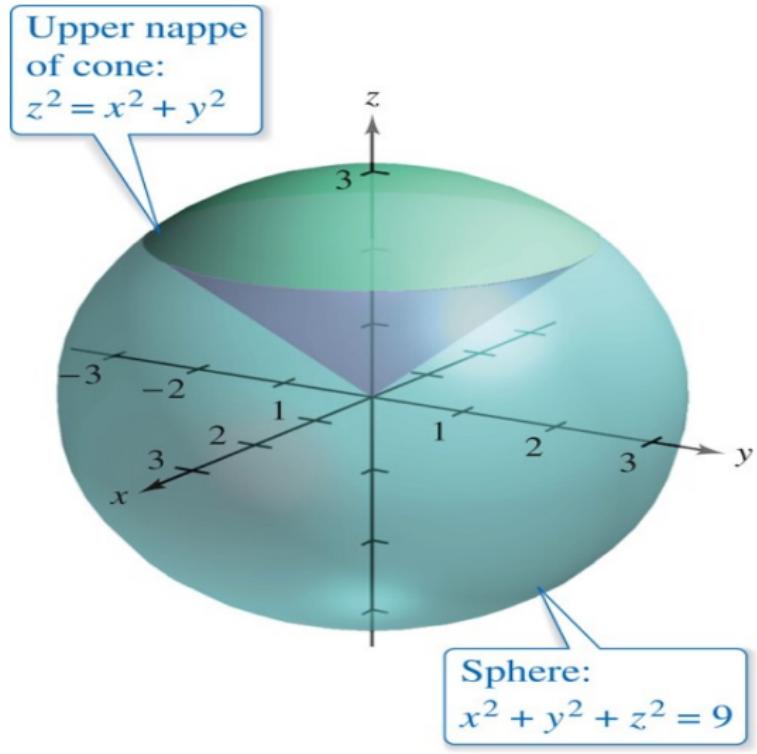
Sol: Note that
 $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{9} = 3$.

These two surfaces intersect when

$$x^2 + y^2 + z^2 = 9 \text{ and } z^2 = x^2 + y^2$$



Example 4 的示意圖



$$\text{Given } 2z^2 = 9 \Rightarrow z = \frac{3}{\sqrt{2}} = \rho \cos \phi = 3 \cos \phi$$

$$\Rightarrow \cos \phi = \frac{1}{\sqrt{2}} \Rightarrow \underline{\phi = \frac{\pi}{4}}$$

If we write $Q = \{(p, \theta, \phi) \mid 0 \leq p \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{4}\}$,
 then $V = \iiint_Q dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 p^2 \sin \phi \, dp \, d\phi \, d\theta$



$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left(\frac{p^3}{3} \sin\phi \right) \Big|_{p=0}^{p=3} d\phi d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} q \sin\phi d\phi d\theta \\
 &= \int_0^{2\pi} \left(-q \cos\phi \right) \Big|_0^{\frac{\pi}{4}} d\theta = \int_0^{2\pi} \left(\frac{-q}{\sqrt{2}} + q \right) d\theta \\
 &= (2\pi) \left(q - \frac{q}{\sqrt{2}} \right) = q\pi(2 - \sqrt{2})
 \end{aligned}$$



Section 12.8

Change of Variables: Jacobians

(變數變換: 雅克比)



Def. (Jacobian 行列式的定義; 1/2)

- (1) If $x = g(u, v)$ and $y = h(u, v)$ have conti. first partial derivatives, the Jacobian of x and y w.r.t. u and v is defined by

$$\begin{aligned}\frac{\partial(x, y)}{\partial(u, v)} &\equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad (\text{二階行列式值}) \\ &= \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}.\end{aligned}$$



Def. (Jacobian 行列式的定義; 2/2)

(2) If $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = k(u, v, w)$ have conti. first partial derivatives, the Jacobian (determinant) of x , y and z w.r.t. u , v and w is defined by

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \quad (\text{三階行列式值})$$



Example 1 (極坐標的 Jacobian 行列式)

The Jacobian of the polar coordinate transformation

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

w.r.t. r and θ is given by

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

for any $r \neq 0$.



Example (球面坐標的 Jacobian 行列式)

The Jacobian of the spherical coordinate transformation

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

w.r.t. ρ , θ and ϕ is given by

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \rho^2 \sin \phi \geq 0$$

for $0 \leq \phi \leq \pi$. (Check!)



Def. (坐標變換的定義; 1/2)

- (1) Let \mathcal{R} be a closed and bounded region in the xy -plane, and \mathcal{S} be a closed and bounded region in the uv -plane. A function $T: \mathcal{S} \rightarrow \mathcal{R}$ is called a coordinate transformation if \exists two conti. functions $x = g(u, v)$ and $y = h(u, v)$ defined on \mathcal{S} s.t.

$$T(u, v) = (x, y) = (g(u, v), h(u, v))$$

is one-to-one on \mathcal{S} .



Def. (坐標變換的定義; 2/2)

- (2) Let Q be a solid region in the (x, y, z) -space, and \mathcal{S} be a solid region in the (u, v, w) -space. A function $T: \mathcal{S} \rightarrow Q$ is called a coordinate transformation if \exists conti. functions $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = k(u, v, w)$ defined on \mathcal{S} s.t.

$$T(u, v, w) = (x, y, z) = (g(u, v, w), h(u, v, w), k(u, v, w))$$

is one-to-one on \mathcal{S} .



$\text{Ex. } ((u+v)^{\frac{1}{2}}, (u+v)^{\frac{1}{2}}) = (u, v) = (v, u)^T \text{ pt horizto}$

Example 2 : (求坐標變換函數)

Let R be a bounded plane region bounded by the lines

$$x-2y=-4, x-2y=0, x+y=1 \text{ and } x+y=4.$$

Find a bounded region S in the uv -plane and a coordinate transformation T s.t. $T(u, v) = (x, y) \quad \forall (u, v) \in S$



Sol:

Let $u = x+y$ and $v = x-2y$. Then we need

to solve $\begin{cases} u = x+y \\ v = x-2y \end{cases}$ for x and y .

e.g. $(x,y) \in R \Leftrightarrow (u,v) \in (0,1) \text{ and } u > v \Rightarrow T \text{ holds}$

$$\Rightarrow 2u + v = 3x \text{ and } u - v = 3y$$

$$\Rightarrow x = \frac{1}{3}(2u+v) \text{ and } y = \frac{1}{3}(u-v)$$

If x, y satisfying Note that the boundaries of R
become

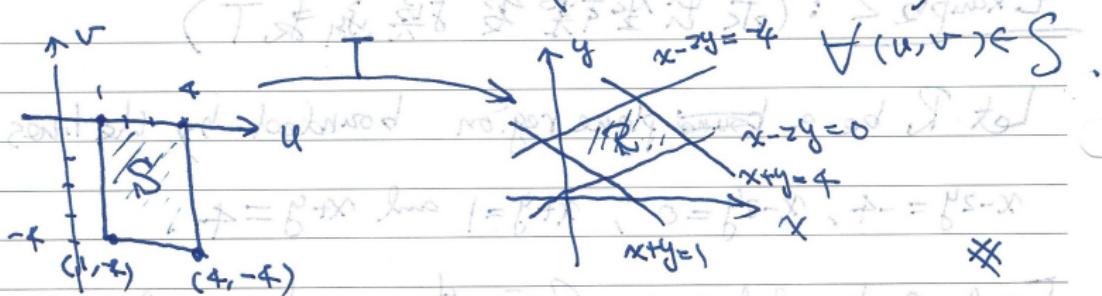


xy-plane	uv-plane
$x+y=1$	$u=1$
$x+y=4$	$u=4$
$x-2y=0$	$v=0$
$x-2y=-4$	$v=-4$

If we let $S = \{(u, v) \mid 1 \leq u \leq 4 \text{ and } -4 \leq v \leq 0\}$, then

the coordinate transformation $T: S \rightarrow R$ is

$$\text{defined by } T(u, v) = (x, y) = \left(\frac{1}{3}(2u+v), \frac{1}{3}(u-v) \right),$$



Thm 12.5 (Change of Variables for Double Integrals)

Suppose that $T(u, v) = (x, y) = (g(u, v), h(u, v))$ is one-to-one on \mathcal{S} , where g and h have conti. first partial derivatives. If $f(x, y)$ is conti. on \mathcal{R} and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \forall (u, v) \in \mathcal{S}$, then

$$\iint_{\mathcal{R}} f(x, y) dx dy = \iint_{\mathcal{S}} f(g(u, v), h(u, v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv.$$



Example 3: Given a plane region in the xy -plane

$$R = \{(x,y) \in \mathbb{R}^2 \mid x+y=1, x+y=4, x-2y=0, x-2y=-4\}$$

$$\text{Evaluate } \iint_R 3xy \, dA = \iint_R 3xy \, dx \, dy. \quad (\text{Why?})$$

Sol: Let $u = x+y$ and $v = x-2y$.

$$\Rightarrow x = g(u,v) = \frac{1}{3}(2u+v) \quad \text{and} \quad y = h(u,v) = \frac{1}{3}(u-v)$$

xy-plane	uv-plane
$x+y=1$	$u=1$
$x+y=4$	$u=4$
$x-2y=0$	$v=0$
$x-2y=-4$	$v=-4$

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{vmatrix}$

 $= \frac{-2}{9} - \frac{1}{9} = \frac{-1}{3}$

If we write $S = \{(u,v) \mid 1 \leq u \leq 4, -4 \leq v \leq 0\}$, then



from Thm 12.5 $\Rightarrow \iint_R 3xy \, dx \, dy = \iint_{R'} 3 \left[\frac{1}{3}(2u+v) \right] \left[\frac{1}{3}(uv) \right] \left(\frac{1}{3} \right) \, du \, dv$

$$= \frac{1}{9} \int_1^4 \int_{-4}^4 (2u^2 - uv - v^2) \, dv \, du$$

$$= \frac{1}{9} \int_1^4 \left(2u^2v - \frac{u}{2}v^2 - \frac{1}{3}v^3 \right) \Big|_{v=-4}^{v=0} \, du = \int_1^4 (8u^2 + 8u - \frac{64}{3}) \, du$$

$$= \frac{1}{9} \left(\frac{8}{3}u^3 + 4u^2 - \frac{64}{3}u \right) \Big|_1^4 = \frac{164}{9}$$



Example 4: Evaluate $\iint_R (x+y)^2 \sin^2(x-y) dA$,

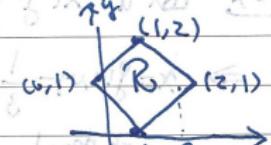
where R is the square with vertices $(0,1)$, $(1,2)$, $(2,1)$ and $(1,0)$, respectively.

Sol: Let $u = x+y$ and $v = x-y$.

$$\Rightarrow x = \frac{1}{2}(u+v) \text{ and } y = \frac{1}{2}(u-v)$$

Note that the region R is bounded by the lines

$$x+y=1, x+y=3, x-y=-1, x-y=1$$



$$\Rightarrow \begin{array}{|c|c|} \hline \text{xy-plane} & \text{uv-plane} \\ \hline x+y=1 & u=1 \\ \hline x+y=3 & u=3 \\ \hline x-y=-1 & v=-1 \\ \hline x-y=1 & v=1 \\ \hline \end{array} \text{ and } \begin{vmatrix} \frac{\partial(x,y)}{\partial(u,v)} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



If we write $S = \{(u,v) \mid 1 \leq u \leq 3, -1 \leq v \leq 1\}$, then

$$\iint_R (x+y)^2 \sin^2(x-y) dA = \int_1^3 \int_{-1}^1 u^2 \sin^2 v \left| \frac{1}{2} \right| du dv$$

$$= \frac{1}{2} \int_{-1}^1 \left(\frac{u^3}{3} \sin^2 v \right) \Big|_{u=1}^{u=3} dv = \int_{-1}^1 \frac{1}{2} \cdot \frac{26}{3} \sin^2 v dv$$

$$= \frac{13}{3} \int_{-1}^1 \frac{1 - \cos 2v}{2} dv = \frac{13}{6} \int_{-1}^1 (1 - \cos 2v) dv$$

$$= \frac{13}{6} \left(v - \frac{1}{2} \sin 2v \right) \Big|_{v=-1}^{v=1} = \frac{13}{6} \left[1 - \frac{1}{2} \sin 2 + 1 + \frac{1}{2} \sin(-2) \right]$$

$$= \frac{13}{6} (2 - \sin 2)$$



Thm (Change of Variables for Triple Integrals)

Suppose $T(u, v, w) = (x, y, z) = (g(u, v, w), h(u, v, w), k(u, v, w))$ is one-to-one on \mathcal{S} , where g, h and k have conti. first partial derivatives. If $f(x, y, z)$ is conti. on Q and $\frac{\partial(x, y, z)}{\partial(u, v, w)} \neq 0$, then

$$\iiint_Q f(x, y, z) dx dy dz = \iiint_{\mathcal{S}} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw.$$

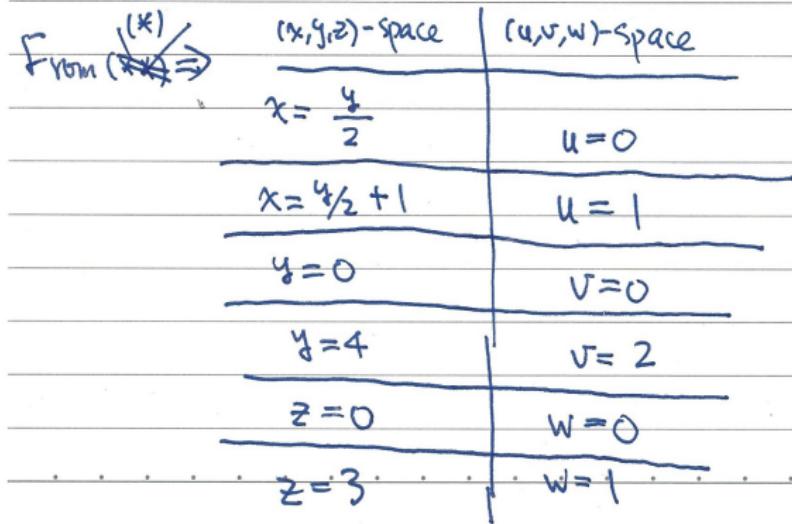


Example: (三重積分の変数変換)

Evaluate $\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz$

Sol: Let $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$ and $w = \frac{z}{3}$. — (*)

$\Rightarrow x = u+v$, $y = 2v$ and $z = 3w$ — (**).



$$\text{From } (**) \Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

So, it follows from Thm 12.5 that

$$\begin{aligned} & \int_0^3 \int_0^{y_1} \int_{y_1}^{y_2+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx dy dz \\ &= \int_0^1 \int_0^2 \int_0^1 (u+w) \cdot 6 du dv dw \quad \left. \begin{array}{l} u=1 \\ v=0 \end{array} \right\} dvol_w \\ &= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dw dw \quad \left. \begin{array}{l} (12) \\ (12) \end{array} \right\} \\ &= 12 \left. \begin{array}{l} (12) \\ \left(\frac{1}{2}w + \frac{1}{3}w^2 \right) \end{array} \right\}_0^1 = 12 \end{aligned}$$



Thank you for your attention!

