Chapter 1
Functions
(函數)

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本章預定授課範圍

1.1 Functions and Their Graphs

1.3 Trigonometric Functions

1.5 Exponential Functions

1.6 Inverse Functions and Logarithms

Useful Notations (常用的數學符號)

1. A set (集合) is a collection of specified objects, and usually denoted by A, B, $C, \cdots$.

$$\mathbb{R} = \{x \,|\, x \text{ is a real number (實數)}\}$$

$$\mathbb{N} = \{x \,|\, x \text{ is a positive integer (正整數)}\}$$

$$\mathbb{Z} = \{x \,|\, x \text{ is an integer (整數)}\}$$

$$\mathbb{Q} = \{x \,|\, x \text{ is a rational number (有理數)}\}$$

2. $x \in A$: x belongs to (屬於) A, i.e., x is an element of the set A.

$$\pi \in \mathbb{R}, \quad 5 \in \mathbb{N}, \quad -7 \in \mathbb{Z} \quad \text{and} \quad \frac{2}{3} \in \mathbb{Q}.$$

$A \subseteq B$: A is a subset (子集合) of B, i.e., if $x \in A$, then $x \in B$.

3. The union (聯集) and intersection (交集) of two sets A and B are defined by

$$A \cup B = \{x \,|\, x \in A \quad \text{or} \quad x \in B\},$$
$$A \cap B = \{x \,|\, x \in A \quad \text{and} \quad x \in B\}.$$

4. $\forall$: for all (對於所有).

5. $\exists$: exist (存在).

6. $\nexists$: does not exist (不存在).

7. s.t. or $\ni$: such that (使得).

8. $\Longrightarrow$: imply that (意指).

9. $\Longleftrightarrow$: if and only if (若且唯若).

10. Greek letters: (常用希臘字符)

$$\begin{array}{llll} \alpha \text{ (alpha)}, & \beta \text{ (beta)}, & \gamma \text{ (gamma)}, & \delta \text{ (delta)}, \\ \varepsilon \text{ (epsilon)}, & \theta \text{ (theta)}, & \lambda \text{ (lambda)}, & \mu \text{ (mu)}, \\ \rho \text{ (rho)}, & \tau \text{ (tau)}, & \phi \text{ (phi)}, & \omega \text{ (omega)}, \cdots \end{array}$$

11. The intervals (區間) in $\mathbb{R}$ are often denoted by I, e.g.,

$$(a,b) = \{x \in \mathbb{R} \,|\, a < x < b\},$$

$$[a,b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\},$$

$$(a,b] = \{x \in \mathbb{R} \,|\, a < x \leq b\},$$

$$[a,b) = \{x \in \mathbb{R} \,|\, a \leq x < b\}.$$

12. i.e.: that is (也就是說).

13. e.g.: for example (舉例來說).

14. w.r.t.: with respect to (關於).

15. Def: Definition (定義), Thm: Theorem (定理), Cor: Corollary (推論).

Section 1.1
Functions and Their Graphs
(函數與其圖形)

Def (實值函數的定義)

Let $X \subseteq \mathbb{R}$ and $Y \subseteq \mathbb{R}$. A real-valued function f from X to Y, denoted by $f: X \to Y$, is a correspondence (對應) that assigns to (指派) each $x \in X$ one unique (唯一的) value $y \in Y$.

(1) The set $X = \mathrm{dom}(f)$ is called the domain (定義域) of f.

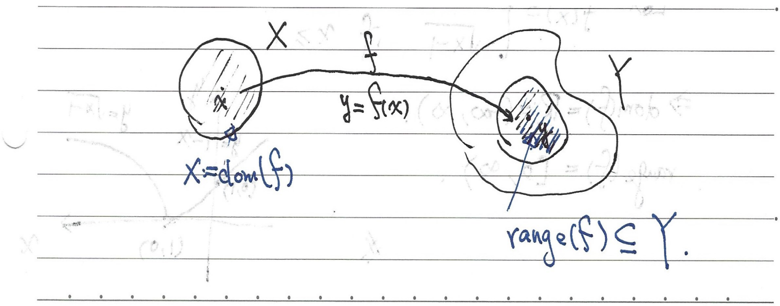
(2) The subset of Y defined by

$$\mathrm{range}(f) = \{y \in Y \,|\, \exists x \in X \text{ s.t. } y = f(x)\}$$

is called the range (值域) of f.

(3) x is the independent variable (自變數) and y is the dependent variable (應變數) of f.

函數映射的示意圖

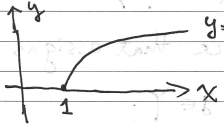


Example 2 : Find dom(f) and range(f).

(a) $f(x) = \sqrt{x-1}$

$\Rightarrow \operatorname{dom}(f) = \{x \in \mathbb{R} \mid x - 1 \geq 0\} = [1, \infty)$

$\operatorname{range}(f) = \{ f(x) \mid x \geq 1 \} = [0, \infty)$

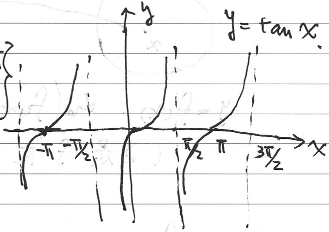


(b) $f(x) = \tan x$

$\Rightarrow \operatorname{dom}(f) = \{ x \in \mathbb{R} \mid x \neq (n+\frac{1}{2})\pi \text{ with } n \in \mathbb{Z} \}$

$\operatorname{range}(f) = \{ f(x) \mid x \neq (n+\frac{1}{2})\pi \text{ with } n \in \mathbb{Z} \}$

$= \mathbb{R} = (-\infty, \infty)$. ※

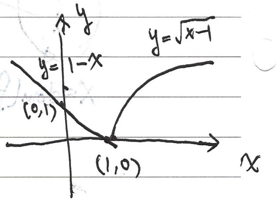


Example 3: (分段函数の定义域与值域)

Let $f(x)=\begin{cases} 1-x & \text{if } x<1 \\ \sqrt{x-1} & \text{if } x\geq 1. \end{cases}$

$\Rightarrow \text{dom}(f)=\mathbb{R}=(-\infty,\infty).$

$\text{range}(f)=[0,\infty).$



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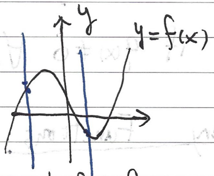
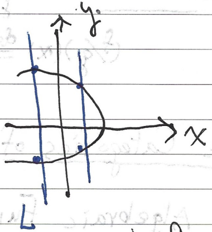
* Graph of a Function :

Thm : (Vertical Line Test)

Any vertical line intersects the graph of $y=f(x)$ at most ~~exactly~~ once.



Graph of $y=f(x)$



Not a function graph!

Def (函數的基本運算; 1/2)

Let f and g be real-valued functions defined on $X \subseteq \mathbb{R}$.

(1) The sum $f+g$ of f and g is defined by

$$(f+g)(x) = f(x) + g(x) \quad \forall\, x \in X.$$

(2) The difference $f-g$ of f and g is defined by

$$(f-g)(x) = f(x) - g(x) \quad \forall\, x \in X.$$

(3) For any $k \in \mathbb{R}$, the constant multiple kf of f is defined by

$$(kf)(x) = k \cdot f(x) \quad \forall\, x \in X.$$

Def (函數的基本運算; 2/2)

(4) The product fg of f and g is defined by

$$(fg)(x) = f(x) \cdot g(x) \quad \forall\, x \in X.$$

(5) The quotient $\frac{f}{g}$ of f and g is defined by

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} \quad \forall\, x \in X,$$

provided that $g(x) \neq 0 \quad \forall\, x \in X$.

1. Algebraic Functions (代數函數)

(a) polynomial (function) of nth degree ($n \in \mathbb{N}$)

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \quad \text{with } a_n \neq 0.$$

(b) rational function (有理函數)

$$f(x) = p(x)/q(x),$$

where $p(x)$ and $q(x) \neq 0$ are polynomials.

(c) radical function (根式函數)

$$f(x) = x^{1/n} = \sqrt[n]{x} \quad \text{with } n \in \mathbb{N}.$$

Note (根式函數的定義域與值域)

Let $f(x) = \sqrt[n]{x}$ with $n \in \mathbb{N}$.

- When n is odd (奇數), we know that

$$\text{dom}(f) = \text{range}(f) = (-\infty, \infty) = \mathbb{R}.$$

- When n is even (偶數), we see that

$$\text{dom}(f) = \text{range}(f) = [0, \infty).$$

2. Trigonometric Functions (三角函數)

$$f(x) = \sin x, \cos x, \tan x, \cot x, \sec x, \csc x.$$

3. Exponential and Logarithmic Functions (指數與對數函數)

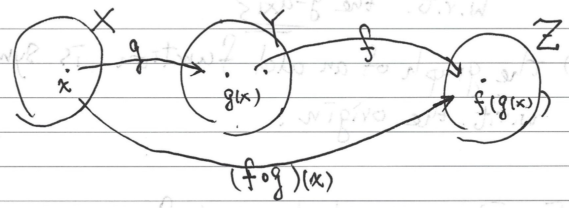
$$f(x) = a^x \quad \text{or} \quad f(x) = \log_a x,$$

where $0 < a \neq 1$. (See Section 1.4 later!)

Def (合成函數的定義)

Let X, Y and Z be subsets of $\mathbb{R}$. The composite function (合成函數) of $f : Y \to Z$ and $g : X \to Y$ is defined by

$$(f \circ g)(x) = f(g(x)) \quad \forall x \in X.$$



Example 4: Let $f(x) = 2x-3$ and $g(x) = \cos x$.

(a) $(f \circ g)(x) = f(g(x)) = 2\cos x - 3 \quad \forall x \in \mathbb{R}$.

(b) $(g \circ f)(x) = g(f(x)) = \cos(2x-3) \quad \forall x \in \mathbb{R}$.

Note: $(f \circ g)(x) \neq (g \circ f)(x)$ in general !

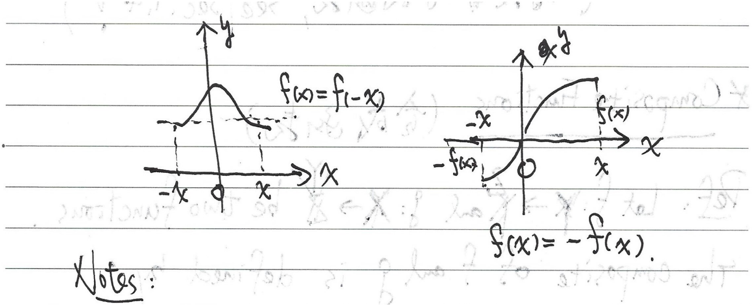
#

Def (Even and Odd Functions)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$.

(1) f is an even function (偶函數) if $f(-x) = f(x) \quad \forall x \in X$.

(2) f is an odd function (奇函數) if $f(-x) = -f(x) \quad \forall x \in X$.



Notes

- The graph of an even function is symmetric w.r.t. the y-axis. (偶函數的圖形對稱於 y-軸)
- The graph of an odd function is symmetric w.r.t. the origin. (奇函數的圖形對稱於原點)

Example 5: Determine whether each function is even, odd or neither.

(a) $f(x) = x^3 - x$ is odd, since $f(-x) = (-x)^3 - (-x)$

$= -x^3 + x = -(x^3 - x) = -f(x)$. $\forall x \in \mathbb{R}$.

(b) $g(x) = 1 + \cos x$ is even because $g(-x) = 1 + \cos(-x)$

$= 1 + \cos x = g(x)$ $\forall x \in \mathbb{R}$.

※

Section 1.3
Trigonometric Functions
(三角函數)

- The functions $f(x) = \sin x$ and $f(x) = \cos x$ are periodic with the period 2π, and their domain and range are given by

$$\mathrm{dom}(f) = \mathbb{R} = (-\infty, \infty), \quad \mathrm{range}(f) = [-1, 1].$$

- The other trigonometric functions are defined by

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

for all $x \in \mathbb{R}$ with $\sin x \neq 0$ or $\cos x \neq 0$.

Useful Identities

- $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$.
- 和角公式:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

- 倍角公式:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta, \quad \sin 2\theta = 2\sin\theta\cos\theta.$$

- 半角公式:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

Useful Inequalities

For any $\theta \neq 0$, we have the following inequalities

$$-|\theta| \leq \sin\theta \leq |\theta|,$$

$$-|\theta| \leq 1 - \cos\theta \leq |\theta|,$$

which will be used in Chapter 2.

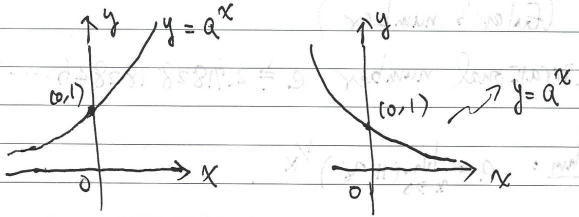
Section 1.5
Exponential Functions
(指數函數)

Def (以 a 為底的指數函數)

The exponential function with base number a is defined by

$$f(x) = a^x \quad \forall\, x \in \mathbb{R},$$

where $0 < a \neq 1$.



$a > 1$ $\qquad\qquad$ $0 < a < 1$

Example 2: Sketch the graphs of $y = 2^x$, $y = (\frac{1}{2})^x$ and $y = 3^x$.

Sol:

x	-3	-2	-1	0	1	2	3	4
2^x	$1/8$	$1/4$	$1/2$	1	2	4	8	16
2^{-x}	8	4	2	1	$1/2$	$1/4$	$1/8$	$1/16$
3^x	$1/27$	$1/9$	$1/3$	1	3	9	27	81

Thm (Basic Properties of a^x)

Let $f(x) = a^x$ with $0 < a \neq 1$. Then

1. $\mathrm{dom}(f) = \mathbb{R} = (-\infty, \infty)$.
2. $\mathrm{range}(f) = (0, \infty)$, i.e., $f(x) = a^x > 0 \quad \forall\, x \in \mathbb{R}$.
3. $f(0) = a^0 = 1$, i.e., the y-intercept of f is $(0, 1)$, and $f(1) = a$.
4. f is one-to-one on $\mathbb{R}$, i.e., $f^{-1} : (0, \infty) \to \mathbb{R} \quad \exists$.

Thm (Laws of Exponents; 指數律)

If $a, b > 0$ and $x, y \in \mathbb{R}$, then

1. $a^x a^y = a^{x+y}$.
2. $(a^x)^y = a^{xy} = (a^y)^x$.
3. $(ab)^x = a^x b^x$.
4. $\dfrac{a^x}{a^y} = a^{x-y}$.
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$.

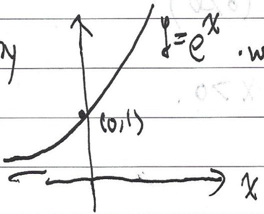
Def (Euler's number; 歐拉數或尤拉數)

The irrational number $e = \lim_{x \to 0} (1 + x)^{1/x} \approx 2.71828182846 \cdots$.

Definition of e^x

For the base number $a = e > 1$, $f(x) = a^x = e^x$ is called the natural exponential function (自然指數函數).

Example 4: The graph of $f(x)=e^x \ \forall x\in\mathbb{R}$ is given by



$y=e^x$, with $e>1$.

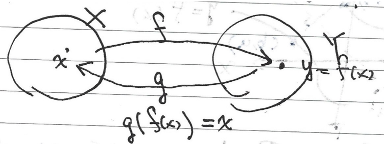
Section 1.6
Inverse Functions and Logarithms
(反函數與對數函數)

Def (反函數的定義)

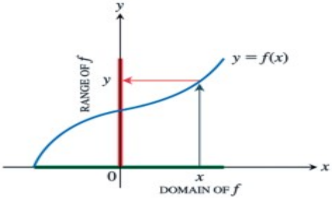
Let $f : X \to Y$ be a function, where $X, Y \subseteq \mathbb{R}$. A function $g : Y \to X$ is called the inverse function (反函數) of f if

$$y = f(x) \quad \forall\, x \in X \Longleftrightarrow g(y) = x \quad \forall\, y \in Y.$$

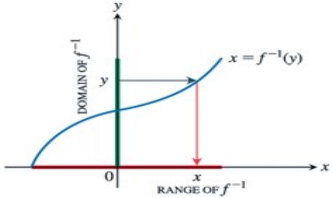
In this case, we often write $g = f^{-1}$. (讀作 f-inverse)



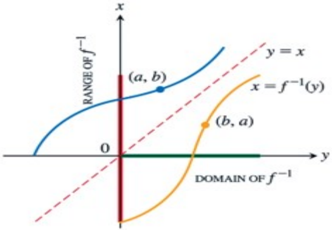
Remark: If the inverse function g $\exists$, then $g = f^{-1}$ is unique.



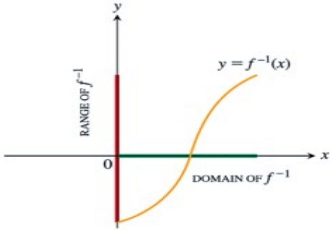
(a) To find the value of f at x, we start at x, go up to the curve, and then over to the y-axis.



(b) The graph of f^{-1} is the graph of f, but with x and y interchanged. To find the x that gave y, we start at y and go over to the curve and down to the x-axis. The domain of f^{-1} is the range of f. The range of f^{-1} is the domain of f.



(c) To draw the graph of f^{-1} in the more usual way, we reflect the system across the line $y = x$.

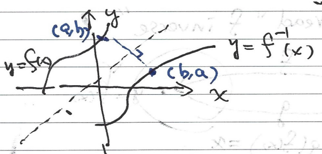


(d) Then we interchange the letters x and y. We now have a normal-looking graph of f^{-1} as a function of x.

FIGURE 1.57 The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

Remarks

1. If f^{-1} $\exists$, then $(f^{-1})^{-1} = f$.
2. In general, it is true that $f^{-1}(x) \neq \frac{1}{f(x)}$.
3. The graph of f^{-1} is a reflection (反射) of the graph of f in the line $y = x$, i.e., $b = f(a) \Longleftrightarrow f^{-1}(b) = a$.



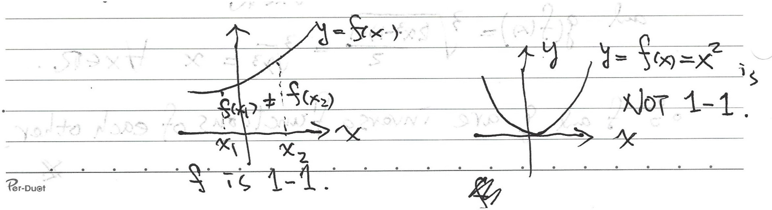
(f 與 f^{-1} 的圖形 反對稱於直線 $y=x$)

Main Questions

- Does f always have an inverse function f^{-1}?
- When does f^{-1} exist for any real-valued function f?

Def (一對一函數的定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$. If $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2 \in X$, then f is an one-to-one function. (一對一函數; 簡寫成 1-1)



Thm (反函數 f^{-1} 的存在性)

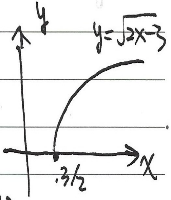
Let f be a real-valued function defined on $X \subseteq \mathbb{R}$. Then

$$f^{-1} \quad \exists \iff f \text{ is one-to-one on } X.$$

Example 3: Find f^{-1} for $f(x)=\sqrt{2x-3}$ $\forall x\in[\frac{3}{2},\infty)$.

Sol: $\because f:[\frac{3}{2},\infty)\to[0,\infty)$ is 1-1

$\therefore f^{-1}:[0,\infty)\to[\frac{3}{2},\infty)$ $\exists$!



$$y=\sqrt{2x-3}\Leftrightarrow y^2=2x-3\Leftrightarrow x=\frac{y^2+3}{2}=f^{-1}(y).$$

Per-Duct

So, the inverse function of f is

$$f^{-1}(x)=\frac{x^2+3}{2}\quad\forall x\in[0,\infty).$$

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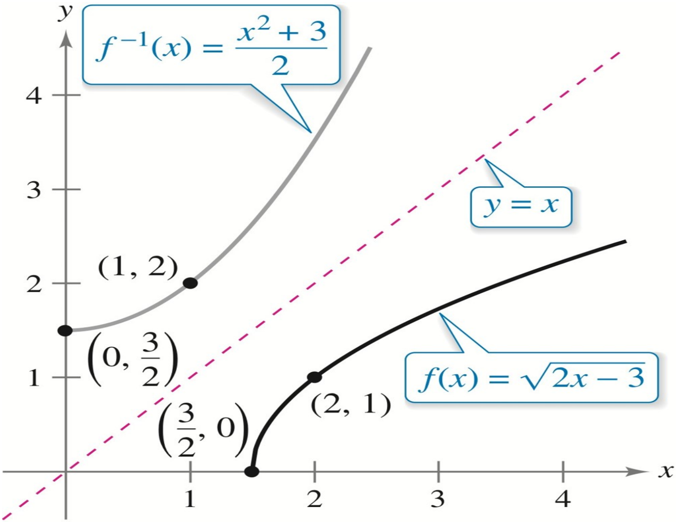
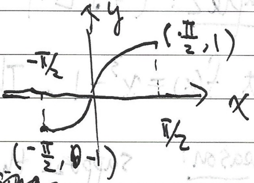
示意圖 (承上頁)



y
$f^{-1}(x) = \frac{x^2+3}{2}$
4
3
2
1
$(1, 2)$
$\left(0, \frac{3}{2}\right)$
$y = x$
$\left(\frac{3}{2}, 0\right)$
$(2, 1)$
$f(x) = \sqrt{2x-3}$
1
2
3
4
x

Example 4: $f(x)=\sin x$ is NOT 1-1 on $\mathbb{R}$ because $f(0)=f(\pi)=0$, but it is 1-1 on $[-\frac{\pi}{2},\frac{\pi}{2}]$.



$\Rightarrow f^{-1}=\sin^{-1}$ ~~$\neq$ arcsin~~

$=\arcsin:[-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$ ∎

- In order to obtain the inverse trigonometric functions, we need to restrict the domains of six trigonometric functions.
- Conventionally, the following functions

$$\begin{aligned}
&\sin : [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1, 1], \quad \cos : [0, \pi] \to [-1, 1] \\
&\tan : (\frac{-\pi}{2}, \frac{\pi}{2}) \to (-\infty, \infty), \quad \cot : (0, \pi) \to (-\infty, \infty) \\
&\sec : [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \to (-\infty, -1] \cup [1, \infty), \\
&\csc : [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}] \to (-\infty, -1] \cup [1, \infty)
\end{aligned}$$

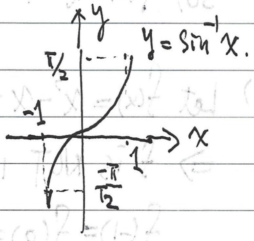
are both 1-1 on the restricted domains.

① $y = \arcsin x = \sin^{-1} x$

$\Leftrightarrow \sin y = x$

Domain: $-1 \le x \le 1$

Range: $-\pi/2 \le y \le \pi/2$

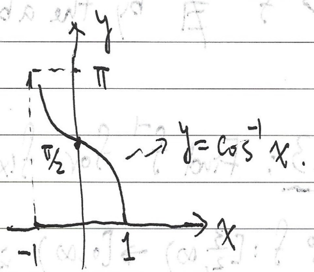


② $y = \arccos x = \cos^{-1} x$

$\Leftrightarrow \cos y = x$

Domain: $-1 \le x \le 1$

Range: $0 \le y \le \pi$.

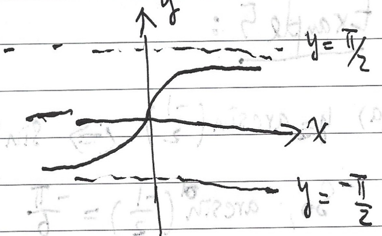


③ $y = \arctan x = \tan^{-1} x$

$\Leftrightarrow \tan y = x$

Domain: $-\infty < x < \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



$y = \tan^{-1} x.$

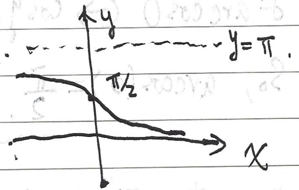
④

$y = \operatorname{arccot} x = \cot^{-1} x.$

$\Leftrightarrow \cot y = x.$

Domain: $-\infty < x < \infty$

Range: $0 < y < \pi.$



$y = \cot^{-1} x.$

⑤ $y = \operatorname{arcsec} x = \sec^{-1} \ldots$

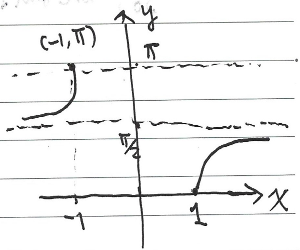
$y = \cot^{-1} x$.

⑤ $y = \operatorname{arcsec} x = \sec^{-1} x$

$\Leftrightarrow \sec y = x$

Domain: $|x| \geq 1$

Range: $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$



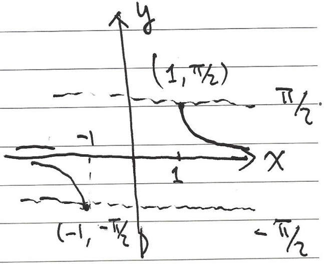
$y = \sec^{-1} x$.

⑥ $y = \operatorname{arccsc} x = \csc^{-1} x$

$\Leftrightarrow \csc y = x$

Domain: $|x| \geq 1$

Range: $-\pi/2 \leq y \leq \pi/2$, $y \neq 0$



Example 5:

(a) $y=\arcsin\left(\frac{-1}{2}\right) \Longleftrightarrow \sin y=\frac{-1}{2} \Longleftrightarrow y=\frac{-\pi}{6}$

So, $\arcsin\left(\frac{-1}{2}\right)=\frac{-\pi}{6}$ #

(b) $y=\arccos 0 \Longleftrightarrow \cos y=0 \Longleftrightarrow y=\pi/2$.

So, $\arccos(0)=\frac{\pi}{2}$. #

(c) $y=\arctan\sqrt{3} \Longleftrightarrow \tan y=\sqrt{3} \Longleftrightarrow y=\frac{\pi}{3}$.

So, $\arctan\sqrt{3}=\pi/3$. #

Since $f(x) = a^x$ is one-to-one on $\mathbb{R}$ for $0 < a \neq 1$, it must have an inverse function $f^{-1} : (0, \infty) \longrightarrow \mathbb{R} = (-\infty, \infty)$.

Definition of $\log_a x$

The inverse function of $f(x) = a^x$, denoted by $f^{-1}(x) = \log_a x$, is called the logarithm function with base $0 < a \neq 1$ (以 a 為底的對數函數). Moreover, we have

$$y = \log_a x \quad \forall\, x > 0 \Longleftrightarrow a^y = x \quad \forall\, y \in \mathbb{R}.$$

Other Logarithmic Functions

For the exponential function $f(x) = e^x$ or $f(x) = 10^x$ defined on $\mathbb{R}$, it also has an inverse function $f^{-1} : (0, \infty) \longrightarrow \mathbb{R} = (-\infty, \infty)$!

Definition of $\ln x$ and $\log x$

(1) The inverse function of $f(x) = e^x$, denoted by $f^{-1}(x) = \ln x$, is called the natural logarithm function (自然對數函數). Moreover, we have

$$y = \ln x \quad \forall\, x > 0 \Longleftrightarrow e^y = x \quad \forall\, y \in \mathbb{R}.$$

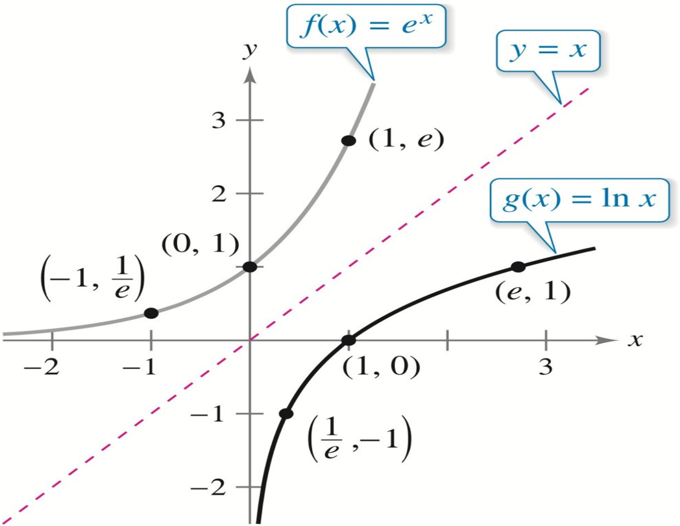
(2) We often call $\log x = \log_{10} x$ the common logarithm function (常用對數函數).

Thm (Basic Properties of $\ln x$)

Let $f(x) = \ln x$. Then

1. $\text{dom}(f) = (0, \infty)$.
2. $\text{range}(f) = \mathbb{R} = (-\infty, \infty)$.
3. $f(1) = ln 1 = 0$, i.e., the x-intercept of f is $(1, 0)$, and $f(e) = \ln e = 1$.
4. f is one-to-one on $(0, \infty)$, i.e., $f^{-1} : \mathbb{R} \to (0, \infty) \quad \exists$.
5. $\ln(e^x) = x$ for $x \in \mathbb{R}$, and $e^{\ln x} = x$ for $x > 0$.
6. Change of Base: $\log_a x = \dfrac{\ln x}{\ln a}$ for $x > 0$ and $0 < a \neq 1$.

示意圖 (承上頁)



$f(x) = e^x$
$y = x$
$g(x) = \ln x$
y
x
3
2
−1
−2
−2
−1
3
(0, 1)
(1, e)
$\left(-1, \frac{1}{e}\right)$
(1, 0)
(e, 1)
$\left(\frac{1}{e}, -1\right)$

Thm (Laws of Logarithms; 對數律)

If $x > 0$, $y > 0$ and $z \in \mathbb{R}$, then

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln(x^z) = z \cdot \ln x$.

Example 5 (續):

(a), (b), (c) 自行閱讀.

(d) $\ln \dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln\left[(x^2+3)^2\right] - \ln\left[x(x^2+1)^{1/3}\right]$.

$= 2\ln(x^2+3) - \ln x - \dfrac{1}{3}\ln(x^2+1)$. #

Example 6: Solve the equations.

(a). $7 = e^{x+1} \Rightarrow \ln 7 = (x+1)\ln e = x+1 \Rightarrow x = \ln 7 - 1$ #

(b) $\ln(2x-3) = 5 \Rightarrow 2x-3 = e^5 \Rightarrow x = \frac{1}{2}(e^5+3)$. #

Thank you for your attention!