Chapter 2
Limits and Continuity
(極限與連續性)

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本章預定授課範圍

2.2 Limit of a Function and Limit Laws

2.4 One-Sided Limits

2.5 Continuity

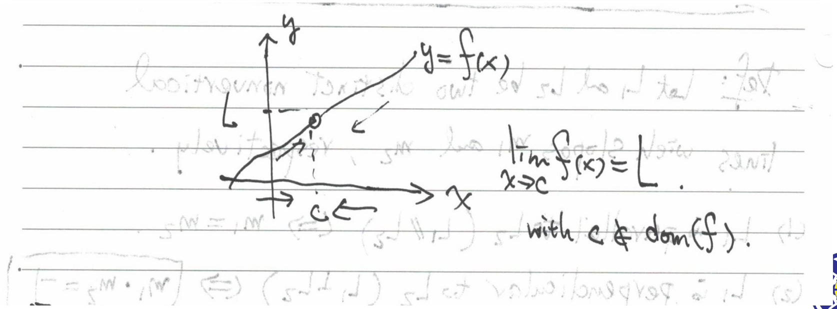
2.6 Limits Involving Infinity; Asymptotes of Graphs

Section 2.2
Limit of a Function and Limit Laws
(函數的極限值與極限法則)

Informal Definition of a Limit

Def (函數極限值的非正式定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \notin X$. If $f(x)$ becomes arbitrarily close to a unique number $L \in \mathbb{R}$ as x approaches c from either side, then the limit of f is L as x approaches c, denoted by $\lim_{x \to c} f(x) = L$.



The following functions defined by

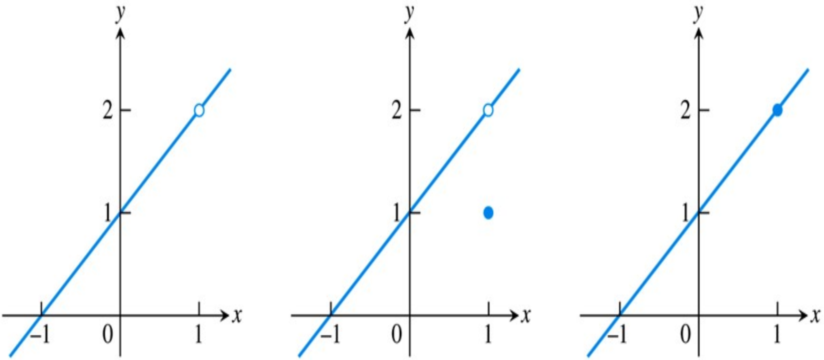
(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ for $x \neq 1$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

have the limit $L = 2$ as x approaches $c = 1$.

Example 2 的示意圖



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

Nonexistence of a Limit (1/3)

Type I (在 c 點兩邊的極限值不相等)

If $f(x) \to L_1$ and $f(x) \to L_2$, where $L_1 \neq L_2$, as x approaches c from either side, then $\lim_{x \to c} f(x) \quad \nexists$.

In this case, we say that f has a jump discontinuity at $x = c$.

Type II (函數值在 c 點附近無上下界)

If $f(x)$ increases (遞增) or decreases (遞減) without bound as x approaches c from either side, then $\lim_{x \to c} f(x) \quad \nexists$.

In this case, we say that f has an infinite discontinuity at $x = c$.

Type III (函數值在 c 點附近震盪)

If $f(x)$ oscillates (震盪) between two fixed values as x approaches c from either side, then $\lim_{x\to c} f(x) \nexists$.

In this case, we say that f has an oscillating discontinuity at $x = c$.

Example 4 (在 $x = 0$ 處的極限值不存在)

For the following functions defined by

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0, \end{cases}$

the limit of these functions does not exist as x approaches $c = 0$.

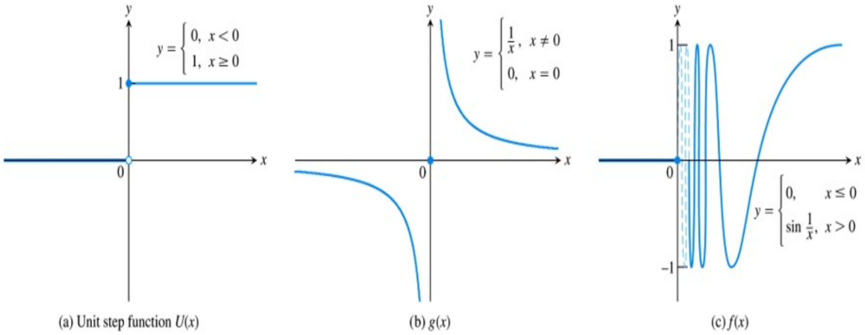


FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

Def (函數極限值的正式定義)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \notin X$. Then

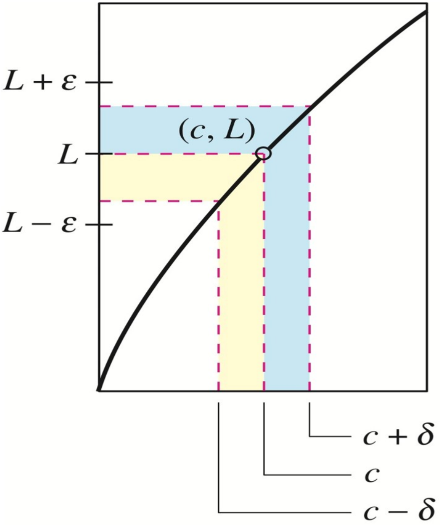
$$\lim_{x \to c} f(x) = L.$$

$$\Longleftrightarrow \forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. if } 0 < |x - c| < \delta \text{ (and } x \in X\text{)},$$

$$\text{then } |f(x) - L| < \varepsilon.$$

Note: this is also called the ε-δ definition of a limit.

極限的正式定義



Example 6: Find a $\delta > 0$ s.t. if $0 < |x-3| < \delta$,

then $|(2x-5)-1| < \varepsilon = 0.01$.

Sol: Want $|(2x-5)-1| = |2x-6| = 2|x-3| < 0.01$

$\Rightarrow |x-3| < \frac{1}{2}(0.01) = 0.005 = \delta$

So, we may choose $\delta = 0.005$.

Remark: Any $\delta \in (0, 0.005]$ also works in this example!

Example 7: Show that $\lim_{x \to 2} (3x-2) = 4$.

Pf: Let $f(x) = 3x-2$ and $L=4$.

Given $\varepsilon > 0$ arbitrarily.

Choose $\delta = \varepsilon/3 > 0$.

If $0 < |x-2| < \delta$, then $|f(x) - L| = |(3x-2)-4| = 3|x-2|$

$$< 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

Thus, $\lim_{x \to 2} (3x-2) = 4$ by the ε-δ Def.

#

Thm (Basic Limit Laws; 1/2)

Let $b, c \in \mathbb{R}$ and let f and g be real-valued functions with

$$\lim_{x \to c} f(x) = L, \quad \lim_{x \to c} g(x) = K.$$

(1) $\lim_{x \to c} b = b$ and $\lim_{x \to c} |x| = |c|$.

(2) $\lim_{x \to c} x^n = c^n$ and $\lim_{x \to c} \left[f(x)\right]^n = L^n \quad \forall\, n \in \mathbb{N}$.

(3) $\lim_{x \to c} [b \cdot f(x)] = b \cdot \left[\lim_{x \to c} f(x)\right] = b \cdot L$.

(4) $\lim_{x \to c} \left[f(x) \pm g(x)\right] = \left[\lim_{x \to c} f(x)\right] \pm \left[\lim_{x \to c} g(x)\right] = L \pm K$.

Thm (Basic Limit Laws; 2/2)

(5) $\lim_{x\to c}\left[f(x)\cdot g(x)\right] = \left[\lim_{x\to c} f(x)\right]\cdot\left[\lim_{x\to c} g(x)\right] = L\cdot K.$

(6) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{K}$ if $K \neq 0$.

(7) $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L}$ $\forall\, n \in \mathbb{N}$, where $L \geq 0$ if n is even.

(8) The Limit of $f \circ g$: (合成函數的極限值)

If $\lim_{x\to K} f(x) = f(K)$, then $\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(K)$.

Example 5 (極限法則的例子)

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + 4\left(\lim_{x \to c} x^2\right) - \lim_{x \to c} 3 = c^3 + 4c^2 - 3.$

(b) $\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)} = \frac{c^4 + c^2 - 1}{c^2 + 5}.$

(c) $\lim_{x \to (-2)} \sqrt{4x^2 + 3} = \sqrt{4(-2)^2 + 3} = \sqrt{19}.$

Example 6 (計算有理函數的極限)

Applying the limit laws, we immediately obtain

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{-1 + 4 - 3}{1 + 5} = \frac{0}{6} = 0.$$

Thm (Limits of Elementary Functions; 1/2)

Let c be a real number in the domain of the given function.

(1) If $p(x)$ is a polynomial, then $\lim_{x \to c} p(x) = p(c)$.

(2) If $r(x) = p(x)/q(x)$ is a rational function with $q(c) \neq 0$, then $\lim_{x \to c} r(x) = r(c) = p(c)/q(c)$.

(3) $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c} \quad \forall\, n \in \mathbb{N}$, where $c \geq 0$ when n is even and $c \in \mathbb{R}$ when n is odd.

Thm (Limits of Elementary Functions; 2/2)

(4) Limits of 6 trigonometric functions are given by

$$\lim_{x\to c}\sin x=\sin c,\quad \lim_{x\to c}\cos x=\cos c,\quad \lim_{x\to c}\tan x=\tan c,$$
$$\lim_{x\to c}\cot x=\cot c,\quad \lim_{x\to c}\sec x=\sec c,\quad \lim_{x\to c}\csc x=\csc c.$$

(5) $\lim_{x\to c} a^x = a^c$ for $a>0$ and $c\in\mathbb{R}$.

(6) $\lim_{x\to c}\ln x=\ln c$ for $c>0$.

Thm (化簡函數後求極限值)

If $\exists\,\delta > 0$ s.t. $f(x) = g(x) \quad \forall\, x \in (c-\delta, c) \cup (c.c+\delta)$, then

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Note: 將 $f(x)$ 簡化為 $g(x)$ 後，兩者在 c 點附近的極限值相等!

Example 7 (分子和分母消去公因式)

Evaluate the following limit

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

The direct substitution gives

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \frac{1 + 1 - 2}{1 - 1} = \frac{0}{0}. \qquad (\mathsf{X})$$

Solution of Example 7

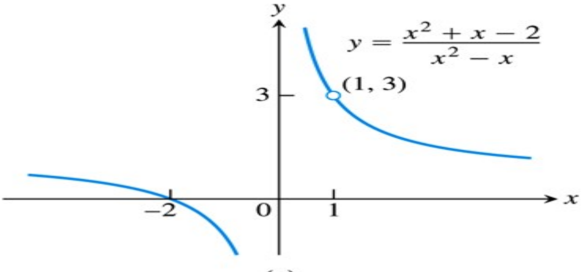
Note that the given function can be simplified as

$$f(x) = \frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x} = g(x) \quad \text{for } x \neq 1.$$

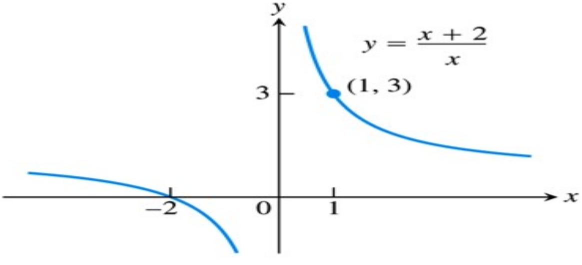
Thus, from above Thm, we see that

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \frac{1+2}{1} = 3.$$

Example 7 的示意圖



(a)



(b)

Use the rationalizing technique (有理化技巧) to evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}.$$

A simple calculation leads to

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \frac{10-10}{0} = \frac{0}{0}. \qquad (\text{X})$$

Solution of Example 9

Applying the rationalizing technique, we obtain

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0} \left(\frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \right) \\
&= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+100}+10)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} \\
&= \frac{1}{\sqrt{0^2+100}+10} = \frac{1}{20} = 0.05.
\end{aligned}$$

Thm 4 (The Sandwich Theorem; 三明治定理)

If $\exists\,\delta > 0$ s.t. $h(x) \le f(x) \le g(x) \quad \forall\, x \in (c-\delta, c) \cup (c, c+\delta)$, and

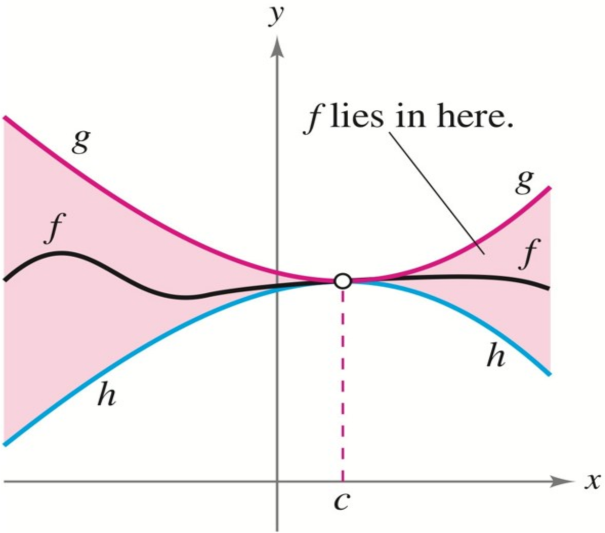
$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x),$$

then $\lim_{x\to c} f(x) = L$.

Note: this theorem is also called the Squeeze Theorem (夾擠定理或夾擊定理).

Thm 4 的示意圖

$$h(x) \le f(x) \le g(x)$$



Find $\lim_{x \to 0} u(x)$ if the function $u(x)$ satisfies the following inequality

$$h(x) \equiv 1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \equiv g(x)$$

for all $x \neq 0$.

Solution of Example 10

We first notice that

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 = \lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = \lim_{x\to 0} g(x).$$

Thus, it follows from the Sandwich Thm that

$$\lim_{x\to 0} u(x) = 1.$$

Example 11 (三明治定理的例子)

Prove the following statements using the Sandwich Theorem.

(a) $\lim_{\theta\to 0} \sin\theta = 0$.

(b) $\lim_{\theta\to 0} \cos\theta = 1$.

(c) For any real-valued function f, $\lim_{x\to c} |f(x)| = 0 \Longrightarrow \lim_{x\to c} f(x) = 0$.
In fact, it is also true that $\lim_{x\to c} |f(x)| = 0 \Longleftrightarrow \lim_{x\to c} f(x) = 0$.

Proof of Example 11

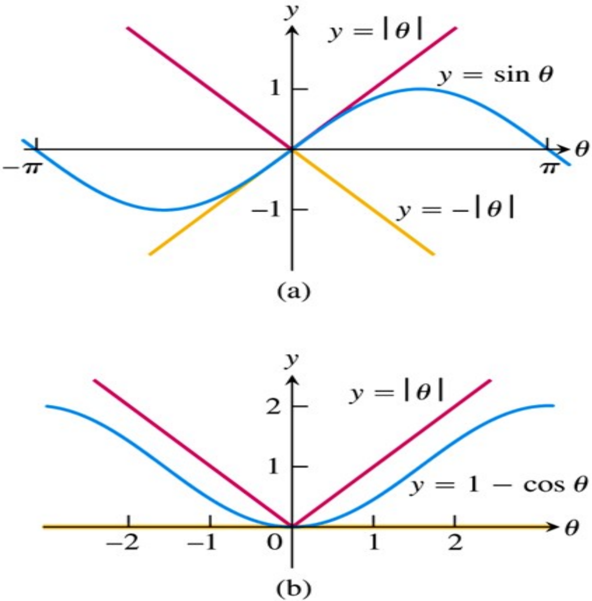
(a) Since $-|\theta| \leq \sin\theta \leq |\theta|$ for $\theta \in \mathbb{R}$ (ssee Section 1.3), and $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, it follows from the Sandwich Thm that $\lim_{\theta\to 0}\sin\theta = 0$.

(b) From the Sandwich Thm and $-|\theta| \leq 1-\cos\theta \leq |\theta|$ for $\theta \in \mathbb{R}$ (see Section 1.3), we obtain $\lim_{\theta\to 0}(1-\cos\theta) = 0$ and hence

$$\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0}\Big[1-(1-\cos\theta)\Big] = 1-0 = 1.$$

(c) The result follows from $-|f(x)| \leq f(x) \leq |f(x)|$ for $x \neq c$ and the Sandwich Thm.

Example 11 的示意圖



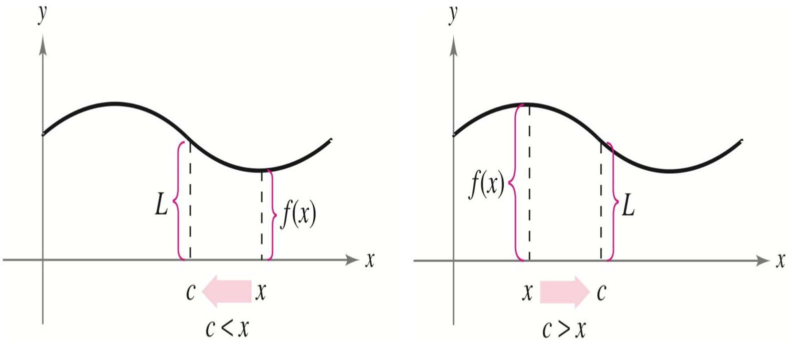
Section 2.4
One-Sided Limits
(單邊極限)

Def (單邊極限值的定義)

(1) f has the limit L from the right (or the right-hand limit L; 右極限值) at c, denoted by $\lim_{x \to c^+} f(x) = L$, if $f(x) \to L$ as $x \to c$ from the right.

(2) f has the limit L from the left (or the left-hand limit L; 左極限值) at c, denoted by $\lim_{x \to c^-} f(x) = L$, if $f(x) \to L$ as $x \to c$ from the left.

單邊極限值的示意圖



Existence of a Limit

Thm 6 (函數極限值存在的等價條件)

Suppose that f is defined on an open interval containing c, except possibly at c itself. Then

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x).$$

(f 在 c 點的極限值為 $L \iff$ 左右極限值均為 L)

Notice that $y = f(x) = \sin\frac{1}{x}$ is defined on the open intervals $(-\infty, 0)$ and $(0, \infty)$, respectively, and $f(0)$ is NOT well-defined.

Since the graph of f oscillates between -1 and 1 on both sides of $x = 0$, we know that

$$\lim_{x\to 0^-} f(x) \text{ and } \lim_{x\to 0^+} f(x) \nexists.$$

Therefore, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin\frac{1}{x} \nexists$.

Example 4 的示意圖

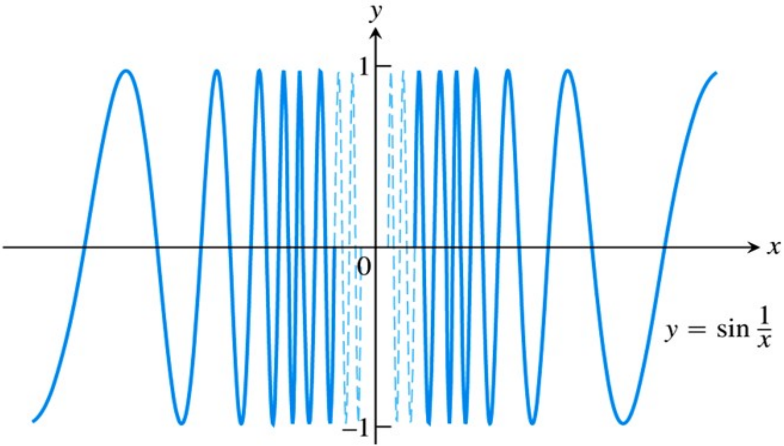


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Def (區間端點的極限)

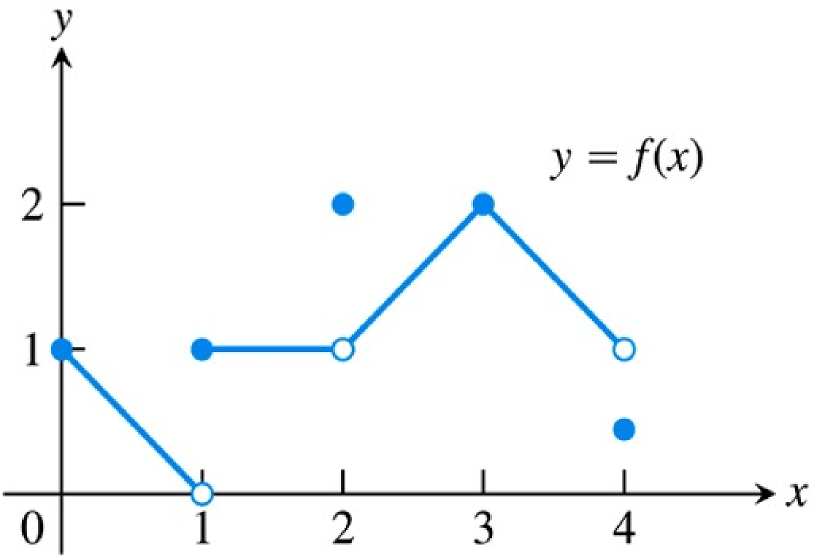
(1) If f is defined on an open interval (c, b) and NOT defined on on an open interval (a, c), then the limit of f at the endpoint c is denoted by

$$\lim_{x \to c} f(x) := \lim_{x \to c^+} f(x).$$

(2) If f is defined on an open interval (a, c) and NOT defined on on an open interval (c, b), then the limit of f at the endpoint c is denoted by

$$\lim_{x \to c} f(x) := \lim_{x \to c^-} f(x).$$

Example 1 (端點極限的例子)



Solution of Example 1

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1 = f(0)$. So, $\lim_{x \to 0} f(x) = 1$ by Def.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0 \neq 1 = \lim_{x \to 1^+} f(x)$. So, $\lim_{x \to 1} f(x) \nexists$ by Thm 6.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1 = \lim_{x \to 2^+} f(x)$. So, $f(2) = 2 \neq 1 = \lim_{x \to 2} f(x) \exists$ by Thm 6. In this case, we say that f has a removable discontinuity (可移除不連續點) at $x = 2$.

At $x = 3$: From Thm 6, we see that $\lim_{x \to 3} f(x) = 2 = f(3)$ and hence we say that f has a continuity (連續點) at $x = 3$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1 \neq f(4)$. So, $\lim_{x \to 4} f(x) = \lim_{x \to 4^-} f(x) = 1$ by Def.

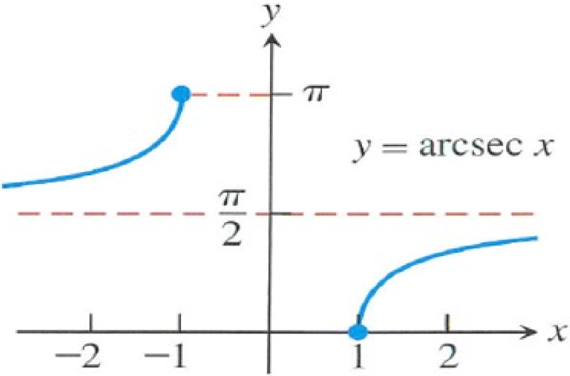


FIGURE 2.27 The arcsec function has limits at $x = \pm 1$.

Solution of Example 2

For the function $f(x) = \operatorname{arcsec} x$, we see that

(i) $\lim_{x \to c} f(c) = f(c)$ for any $c \in (-\infty, -1) \cup (1, \infty)$.

(ii) Since f is NOT defined on $(-1, 1)$, it follows that the endpoint limits of f at $x = \pm 1$ are

$$\lim_{x \to (-1)} f(x) = \lim_{x \to (-1)^-} f(x) = \operatorname{arcsec}(-1) = \pi,$$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1^+} f(x) = \operatorname{arcsec}(1) = 0.$$

Thm (Some Special Limits)

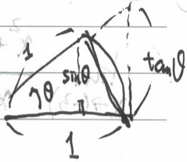
(1) $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. (see Theorem 7)

(2) $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$. (see Example 5a)

(3) $\lim_{x \to 0} (1+x)^{1/x} = e$.

pf: ① Note that for $0<\theta<\frac{\pi}{2}$, we have

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$



$$\Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \quad \left(同乘以 \frac{2}{\sin\theta}\right)$$

$$\Rightarrow \quad \cos\theta < \frac{\sin\theta}{\theta} < 1 \quad \forall\, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), \text{ since}$$

$\cos\theta=\cos(-\theta)$ and $\frac{\sin\theta}{\theta}=\frac{\sin(-\theta)}{-\theta}$ are even.

From the Squeeze Thm and $\lim\limits_{\theta\to 0}\cos\theta=\lim\limits_{\theta\to 0}1=1 \Rightarrow$

$$\boxed{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1}$$

#

Example 5a (特殊三角函數的極限)

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta}\times\frac{1+\cos\theta}{1+\cos\theta}\right)$$

$$= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}$$

$$= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right) = (1)(0) = 0.$$

Use Theorem 7 to show that

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}.$$

Solution of Example 5b

We fist observe that

$$\frac{\sin 2x}{5x} = \frac{\sin 2x}{2x} \cdot \frac{2x}{5x} \quad \text{for } x \neq 0.$$

If we let $\theta = 2x$, then $\theta \to 0$ as $x \to 0$ and thus

$$\lim_{x\to 0} \frac{\sin 2x}{5x} = \left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right)\left(\lim_{x\to 0} \frac{2x}{5x}\right) = (1)\left(\frac{2}{5}\right) = \frac{2}{5}.$$

Section 2.5
Continuity
(連續性)

Def (實值函數的連續性)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$.

(1) f is continuous (連續的; 簡寫為 conti.) at c if $\lim_{x \to c} f(x) = f(c)$.

(2) f is conti. on I if it is conti. at each $c \in I$.

(3) f is everywhere conti. (處處連續) if it is conti. on $\mathbb{R} = (-\infty, \infty)$.

Def (函數的不連續性)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$.

(1) f has a discontinuity (不連續點; 簡寫為 disconti.) at c if it is NOT conti. at c.

(2) A disconti. of f at c is called removable (可移除的) if f can be made conti. at c by redefining $f(c)$. Otherwise, the disconti. at c is called nonremovable (不可移除的).

Note: the jump disconti., infinite disconti. and oscillating disconti. are nonremovable.

One-Sided Continuity

Def (單邊連續的定義)

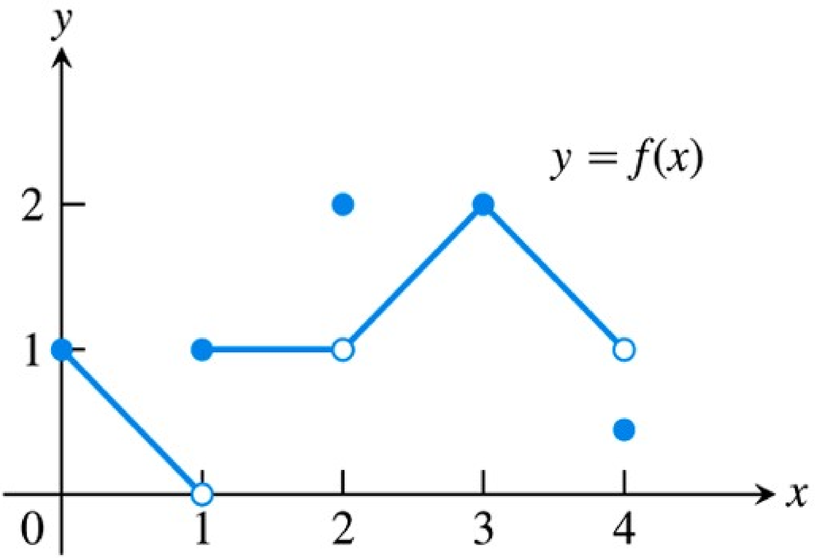
(1) f is right-conti. or conti. from the right (佑蓮續) at c if $\lim_{x \to c^+} f(x) = f(c)$.

(2) f is left-conti. or conti. from the left (左蓮續) at c if $\lim_{x \to c^-} f(x) = f(c)$.

Remark

f is conti. at $c \iff f$ is right-conti. and left-conti. at c.
(f 在 c 點連續 $\iff f$ 在 c 點右蓮續且左連續)

Example 1 (判斷單點的連續性)



At $x = 0$: f is right-conti. at $x = 0$ because $\lim_{x \to 0^+} f(x) = 1 = f(0)$.

At $x = 1$: f has a jump disconti. at $x = 1$ because $\lim_{x \to 1^-} f(x) = 0 \neq 1 = f(1) = \lim_{x \to 1^+} f(x)$. In fact, f is right-conti. at $x = 1$.

At $x = 2$: f has a removable disconti. at $x = 2$ because it can be made conti. at $x = 2$ by redefining $f(2) := \lim_{x \to 2} f(x) = 1$.

At $x = 3$: f is conti. at $x = 3$ because $\lim_{x \to 3} f(x) = 2 = f(3)$.

At $x = 4$: f is NOT left-conti. at $x = 4$ because $\lim_{x \to 4^-} f(x) = 1 \neq f(4)$, but it can be made left-conti. there by redefining $f(4) := \lim_{x \to 4^-} f(x) = 1$.

The greatest integer fuction is defined by

$$f(x) = \lfloor x \rfloor := n \quad \forall\, x \in \mathbb{R},$$

where n is the largest integer s.t. $n \le x$.

Example 4 的示意圖

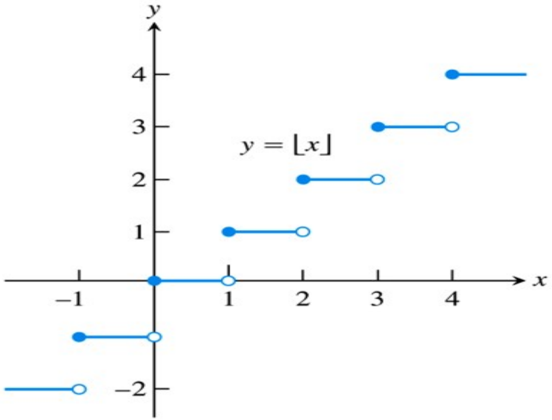


FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

Let $f(x) = \lfloor x \rfloor$ be the greatest integer function (最大整數函數) defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$, we have

$$\lim_{x \to n^-} f(x) = n - 1 \quad \text{and} \quad \lim_{x \to n^+} f(x) = n = f(n).$$

So, f is right-conti. and has a jump disconti. at each $n \in \mathbb{Z}$, but NOT left-conti. there! In addition, f is conti. at every point $x \in \mathbb{R} \setminus \mathbb{Z}$.

Def (在閉區間上的連續性)

We say that f is conti. on $I = [a, b]$ if the following conditions hold:

1. f is conti. on the open interval (a, b).
2. f is right-conti. at a, i.e., $\lim_{x \to a^+} f(x) = f(a)$.
3. f is left-conti. at b, i.e., $\lim_{x \to b^-} f(x) = f(b)$.

Example 2 (在閉區間上的連續性)

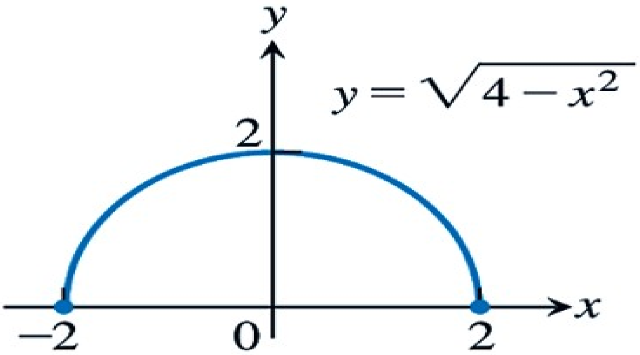


FIGURE 2.37 A function that is continuous over its domain (Example 2).

Solution of Example 2

Note that the domain of $f(x) = \sqrt{4 - x^2}$ is $D = [-2, 2]$.

Since f is conti. on the open interval $(-2, 2)$, right-conti. at $x = -2$ and left-conti. at $x = 2$, we conclude that f is conti. on the closed interval D.

Thm (連續函數的性質)

1. If f and g are conti. at c and $b \in \mathbb{R}$, then $f \pm g$, bf, fg and f/g with $g(c) \neq 0$ are conti. at c, respectively.
2. If g is conti. at c and f is conti. at $g(c)$, then $(f \circ g)(x) = f(g(x))$ is conti. at c. (see Theorem 10)
3. Elementary functions and inverse trigonometric functions are conti. on their domains.

Note: these properties are also true for the one-sided continuity!

(a) The polynomial $P(x)$ is everywhere conti., since $\lim_{x \to c} P(x) = P(c)$ for any $c \in \mathbb{R}$.

(b) The rational function $P(x)/Q(x)$ is conti. at every point $c \in \mathbb{R}$ with $Q(c) \neq 0$, since its limit at c satisfies

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} \quad \exists.$$

(c) The function $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ is everywhere conti., since f is a polynomial for $x \neq 0$ and $\lim_{x \to 0} f(x) = 0 = f(0)$.

Example 8 (判斷合成函數的連續性)

(a) The funtion $f(x) = \sqrt{x^2 - 2x - 5}$ is conti. on its domain $\text{dom}(f) = (-\infty, 1 - \sqrt{6}] \cup [1 + \sqrt{6}, \infty)$.

(b) $y = \dfrac{x^{2/3}}{1 + x^4}$ is everywhere continuous, since we know that $1 + x^4 > 0 \quad \forall\, x \in \mathbb{R}$.

(c) $y = \left| \dfrac{x - 2}{x^2 - 2} \right|$ is conti. for all $x \neq \pm\sqrt{2}$.

(d) $y = \left| \dfrac{x \sin x}{x^2 + 2} \right|$ is everywhere conti., since $x^2 + 2 > 0 \quad \forall\, x \in \mathbb{R}$.

Example 9 (計算合成還數的極限值)

(a) Applying the continuity of the composite function, we have

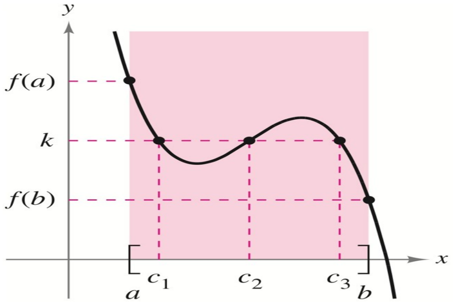
$$\lim_{x\to\pi/2} \cos\left(2x+\sin\left(\frac{3\pi}{2}+x\right)\right) = \cos(\pi+\sin(2\pi)) = \cos\pi = -1.$$

(b) $\displaystyle \lim_{x\to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+x}\right) = \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}.$

(c) $\displaystyle \lim_{x\to 0} \sqrt{x+1}e^{\tan x} = \sqrt{0+1}\cdot e^0 = (1)(1) = 1.$

Thm 11 (I.V.T.; 中間值定理)

If f is conti. on $[a, b]$, $f(a) \neq f(b)$ and k is any number between $f(a)$ and $f(b)$, then $\exists\, c \in [a, b]$ s.t. $f(c) = k$.



Example 11 (中間值定理的應用)

Use the I.V.T. to show that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution in the closed interval $[0, 2]$.

Since $f(x) = \sqrt{2x+5} + x^2 - 4$ is conti. on $\text{dom}(f) = [-5/2, \infty)$, it follows that f is also conti. on $[0, 2]$. Moreover, it is easily seen that $f(0) = \sqrt{5} - 4 \approx -1.76 < 0 < 3 = \sqrt{9} = f(2)$. Thus, for the value $k = 0$ in the I.V.T., $\exists\, c \in (0, 2)$ s.t. $f(c) = k = 0$, i.e., the nonlinear equation (非線性方程式) $f(x) = 0$ has at least one solution $c \in (0, 2)$.

Example 11 的示意圖

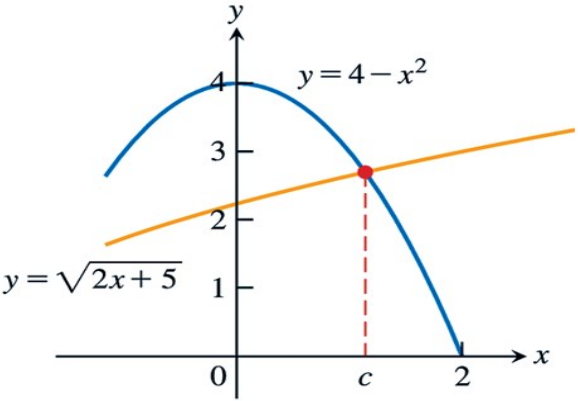


FIGURE 2.46 The curves $y = \sqrt{2x+5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\sqrt{2x+5} + x^2 - 4 = 0$ (Example 11).

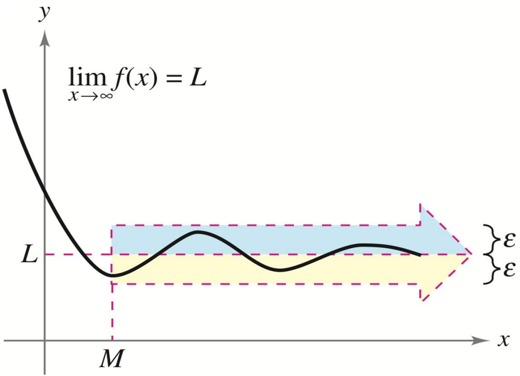
Section 2.6
Limits Involving Infinity; Asymptotes of Graphs
(無窮極限: 函數圖形的漸近線)

Def (無窮遠處的極限)

(1) $\lim_{x\to\infty} f(x) = L \Longleftrightarrow \forall\, \varepsilon > 0$, $\exists\, M > 0$ s.t. if $x > M$, then $|f(x) - L| < \varepsilon$.

(2) $\lim_{x\to-\infty} f(x) = L \Longleftrightarrow \forall\, \varepsilon > 0$, $\exists\, N < 0$ s.t. if $x < N$, then $|f(x) - L| < \varepsilon$.

示意圖 (承上頁)



Thm (重要的極限法則)

(1) If $r > 0$ is a rational number and $c \in \mathbb{R}$, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0 = \lim_{x \to -\infty} \frac{c}{x^r}.$$

(2) $\lim_{x \to \infty} e^{-x} = 0 = \lim_{x \to -\infty} e^{x}$.

HW: read Example 1, Section 2.6 by yourself!

Let $\varepsilon > 0$ be given arbitrarily. Choose $M, N \in \mathbb{R}$ with

$$M > \left(\frac{|c|}{\varepsilon}\right)^{1/r} > 0 \quad \text{and} \quad N < -\left(\frac{|c|}{\varepsilon}\right)^{1/r} < 0.$$

Thus we have the following inequalities

$$\frac{|c|}{M^r} < \varepsilon \quad \text{and} \quad \frac{|c|}{(-N)^r} < \varepsilon. \qquad \text{(Check!)}$$

If $x > M(> 0)$ or $x < N(< 0)$, then

$$\left|\frac{c}{x^r} - 0\right| = \frac{|c|}{x^r} < \frac{|c|}{M^r} < \varepsilon \quad \text{or} \quad \left|\frac{c}{x^r} - 0\right| = \frac{|c|}{|x|^r} = \frac{|c|}{(-x)^r} < \frac{|c|}{(-N)^r} < \varepsilon.$$

So, it follows from the Def. that

$$\lim_{x \to \infty} \frac{c}{x^r} = 0 = \lim_{x \to -\infty} \frac{c}{x^r}.$$

(a) Applying above Thm, we see that

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}=\frac{5+0-0}{3+0}=\frac{5}{3}.$$

(b) Applying above Thm, we see that

$$\lim_{x\to-\infty}\frac{11x+2}{2x^3-1}=\lim_{x\to-\infty}\frac{(11/x^2)+(2/x^3)}{2-(1/x^3)}=\frac{0+0}{2-0}=0.$$

Example 3 的示意圖

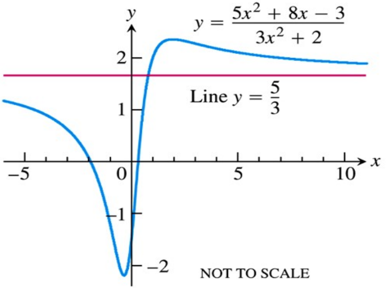
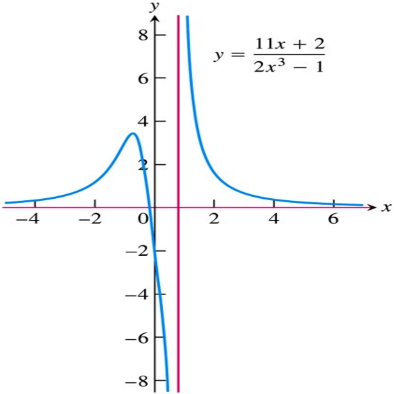


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

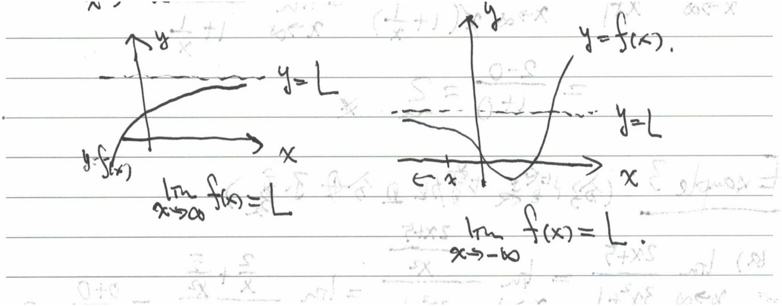


Horizontal Asymptotes

Def (水平漸近線的定義)

A line $y = L$ is callled a horizontal asymptote (水平漸近線) of the graph of f if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$



Example 4 (水平漸近線的例子)

Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Since the limits of f as $x \to \pm\infty$ are given by

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3-2}{x^3+1} = \lim_{x\to\infty} \frac{1-(2/x^3)}{1+(1/x^3)} = 1 \quad \text{and}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^3-2}{(-x)^3+1} = \lim_{x\to-\infty} \frac{x^3-2}{-x^3+1} = -1,$$

the horizontal asymptotes of the graph of f are $y = 1$ and $y = -1$, respectively.

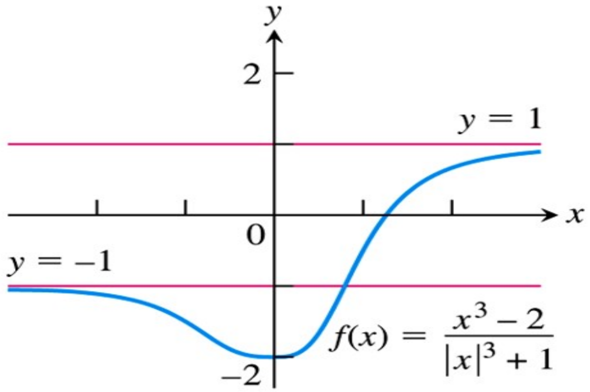


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

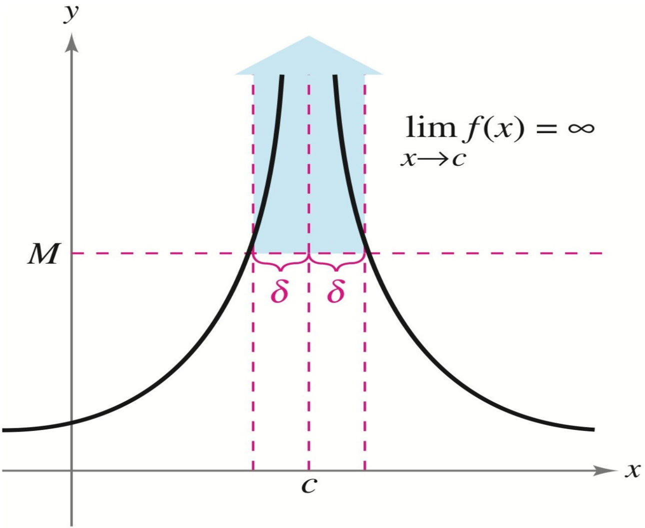
Infinite Limits (1/2)

Def (無窮極限值的定義; 1/2)

(1) $\lim_{x \to c} f(x) = \infty \iff \forall\, M > 0,\ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $f(x) > M$.

(2) $\lim_{x \to c} f(x) = -\infty \iff \forall\, N < 0,\ \exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $f(x) < N$.

示意圖 (承上頁)

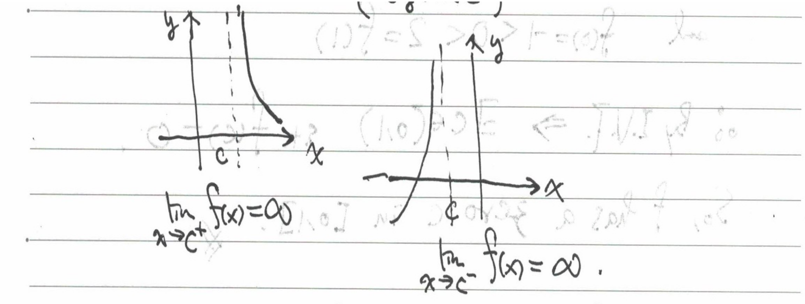


Def (無窮極限值的定義; 2/2)

(3) $\lim_{x\to c^+} f(x) = \infty$ $\left(\text{or } \lim_{x\to c^-} f(x) = \infty\right) \iff \forall\, M > 0,\ \exists\, \delta > 0$
s.t. if $c < x < c + \delta$ (or $c - \delta < x < c$), then $f(x) > M$.

(4) $\lim_{x\to c^+} f(x) = -\infty$ $\left(\text{or } \lim_{x\to c^-} f(x) = -\infty\right) \iff \forall\, N < 0,\ \exists\, \delta > 0$
s.t. if $c < x < c + \delta$ (or $c - \delta < x < c$), then $f(x) < N$.

示意圖 (承上頁)



重要口訣 (切記!)

若 $+0$ 與 -0 分別代表接近零的正數與負數, 則

- $\dfrac{1}{+0} = \infty, \qquad \dfrac{1}{-0} = -\infty$
- $\dfrac{1}{\infty} = \dfrac{1}{-\infty} = 0,$

其中，$+\infty = \infty$ 和 $-\infty$ 分別為正負無窮遠處的符號。

Example 11 (利用口訣求無窮極限)

The following one-sided limits are given by

$$\lim_{x \to 1^+} \frac{1}{x-1} = \frac{1}{+0} = \infty,$$
$$\lim_{x \to 1^-} \frac{1}{x-1} = \frac{1}{-0} = -\infty.$$

Example 11 (利用口訣求無窮極限)

The following one-sided limits are given by

$$\lim_{x\to 1^+} \frac{1}{x-1} \left(= \frac{1}{+0}\right) = \infty,$$

$$\lim_{x\to 1^-} \frac{1}{x-1} \left(= \frac{1}{-0}\right) = -\infty.$$

(a) The one-sided limits of $\sec x$ at $x = -\pi/2$ are

$$\lim_{x\to(-\pi/2)^-} \sec x = \lim_{x\to(-\pi/2)^-} \frac{1}{\cos x} = \frac{1}{-0} = -\infty,$$
$$\lim_{x\to(-\pi/2)^+} \sec x = \lim_{x\to(-\pi/2)^+} \frac{1}{\cos x} = \frac{1}{+0} = \infty.$$

(b) The one-sided limits of $\tan x$ at $x = \pi/2$ are

$$\lim_{x\to(\pi/2)^-} \tan x = \lim_{x\to(\pi/2)^-} \frac{\sin x}{\cos x} = \frac{1}{+0} = \infty,$$
$$\lim_{x\to(\pi/2)^+} \tan x = \lim_{x\to(\pi/2)^+} \frac{\sin x}{\cos x} = \frac{1}{-0} = -\infty.$$

(a) The one-sided limits of $\sec x$ at $x = -pi/2$ are

$$\lim_{x \to (-\pi/2)^-} \sec x = \lim_{x \to (-\pi/2)^-} \frac{1}{\cos x} \left(= \frac{1}{-0} \right) = -\infty,$$

$$\lim_{x \to (-\pi/2)^+} \sec x = \lim_{x \to (-\pi/2)^+} \frac{1}{\cos x} \left(= \frac{1}{+0} \right) = \infty.$$

(b) The one-sided limits of $\tan x$ at $x = pi/2$ are

$$\lim_{x \to (\pi/2)^-} \tan x = \lim_{x \to (\pi/2)^-} \frac{\sin x}{\cos x} \left(= \frac{1}{+0} \right) = \infty,$$

$$\lim_{x \to (\pi/2)^+} \tan x = \lim_{x \to (\pi/2)^+} \frac{\sin x}{\cos x} \left(= \frac{1}{-0} \right) = -\infty.$$

Example 8 (利用三明治定理求水平漸近線)

Using the Sandwich Theorem, find a horizontal asymptote of the graph of the curve

$$y = f(x) = 2 + \frac{\sin x}{x}$$

for all $x \neq 0$.

Since it is easily seen that

$$0 \le \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{|x|} \le \frac{1}{|x|}$$

for all $x \neq 0$, we immediately obtain

$$\lim_{x\to\pm\infty} \left|\frac{\sin x}{x}\right| = 0 \Longleftrightarrow \lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

by the Sandwich Thm and $\lim_{x\to\pm\infty} \frac{1}{|x|} = 0$. Thus, $y = 2$ is the horizontal asymptote of the curve $y = f(x)$ because we further have

$$\lim_{x\to\pm\infty} f(x) = 2 + \lim_{x\to\pm\infty} \frac{\sin x}{x} = 2 + 0 = 2.$$

Def (鉛直漸近線或垂直漸近線)

If one of the following infinite limits

$$\lim_{x\to c^+} f(x) = \pm\infty, \quad \lim_{x\to c^-} f(x) = \pm\infty,$$

is satisfied, then the line $x = c$ is a vertical asymptote (垂直漸近線) of the graph of f.

Thm (判斷垂直漸近線的位置)

If f and g are conti. on an open interval I containing c, where $g(x) \neq 0 \quad \forall x \in I \setminus \{c\}$. If the values $f(c)$ and $g(c)$ both satisfy

$$f(c) \neq 0 \quad \text{and} \quad g(c) = 0,$$

then $\dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

Example 17 (求函數的水平與垂直漸進線)

Find the horizontal and vertical asymptotes of the graph of the function

$$f(x) = \frac{8}{x^2 - 4} = \frac{8}{(x+2)(x-2)}$$

for all $x \neq \pm 2$.

(1) The x-axis ($y = 0$) is a horizontal asymptote of f because

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{8}{x^2-4} = \lim_{x\to\infty} \frac{8/x^2}{1-(4/x^2)} = 0.$$

(2) The graph of f has two vertical asymptotes at $x = -2$ and $x = 2$, respectively, sice

$$\lim_{x\to(-2)^-} f(x) = \frac{8}{+0} = \infty, \quad \lim_{x\to(-2)^+} f(x) = \frac{8}{-0} = -\infty,$$
$$\lim_{x\to 2^-} f(x) = \frac{8}{-0} = -\infty, \quad \lim_{x\to 2^+} f(x) = \frac{8}{+0} = \infty.$$

(1) The x-axis ($y = 0$) is a horizontal asymptote of f because

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{8}{x^2-4} = \lim_{x\to\infty} \frac{8/x^2}{1-(4/x^2)} = 0.$$

(2) The graph of f has two vertical asymptotes at $x = -2$ and $x = 2$, respectively, sice

$$\lim_{x\to(-2)^-} f(x) \left(= \frac{8}{+0}\right) = \infty, \qquad \lim_{x\to(-2)^+} f(x) \left(= \frac{8}{-0}\right) = -\infty,$$

$$\lim_{x\to 2^-} f(x) \left(= \frac{8}{-0}\right) = -\infty, \qquad \lim_{x\to 2^+} f(x) \left(= \frac{8}{+0}\right) = \infty.$$

The functions $\sec x = \dfrac{1}{\cos x}$ and $\tan x = \dfrac{\sin x}{\cos x}$ both have infinitely many vertical asymptotes at $x = (2n+1)\dfrac{\pi}{2} = (n + \dfrac{1}{2})\pi$ with $n \in \mathbb{Z}$, since $\cos x = 0$ but 1 and $\sin x$ are nonzero at these points.

For example, it is easily seen that

$$\lim_{x \to (-\pi/2)^-} \sec x = -\infty, \quad \lim_{x \to (-\pi/2)^+} \sec x = \infty,$$
$$\lim_{x \to (\pi/2)^-} \tan x = \infty, \quad \lim_{x \to (\pi/2)^+} \tan x = -\infty.$$

So, $\sec x$ and $\tan x$ have vertical asymptotes at $x = \dfrac{-\pi}{2}$ and $x = \dfrac{\pi}{2}$, respectively.

Example 19 的示意圖

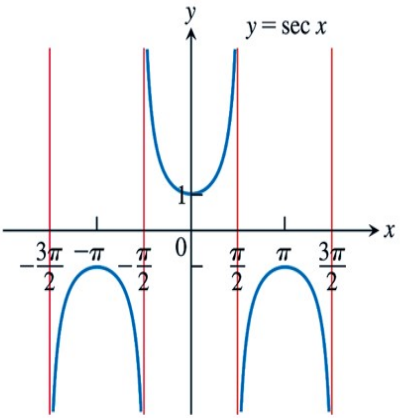


FIGURE 2.68 The graphs of sec x and tan x have infinitely many vertical asymptotes (Example 19).

Thm (Properties of Infinite Limits)

Suppose that $\lim_{x \to c} f(x) = \pm\infty$ and $\lim_{x \to c} g(x) = L \neq 0$.

1. $\lim_{x \to c} [f(x) \pm g(x)] = \pm\infty$.
2. $\lim_{x \to c} [f(x)g(x)] = \pm\infty$ if $L > 0$.
3. $\lim_{x \to c} [f(x)g(x)] = \mp\infty$ if $L < 0$.
4. $\lim_{x \to c} \dfrac{g(x)}{f(x)} = 0$.

Def (斜漸近線的定義)

A line $y = mx + b$ with slope $m \neq 0$ is called an oblique or slant asymptote (斜漸近線) of the graph of f if either

$$\lim_{x\to\infty}[f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty}[f(x) - (mx + b)] = 0.$$

Note: If $p(x)/q(x)$ is a rational function with $\deg(p) = \deg(q) + 1$, applying the method of long division (長除法), we obtain

$$\frac{p(x)}{q(x)} = (mx + b) + \frac{r(x)}{q(x)},$$

where $r(x)$ is a polynomial with $\deg(r) < \deg(q)$.

Find an oblique asymptote of the graph of the rational function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

for all $x \neq 2$. Notice that $x = 2$ is the only vertical asymptote of the graph of f, but there is no any horizontal asymptote for f!

Applying the long division, we obtain

$$f(x) = \frac{x^2-3}{2x-4} = \left(\frac{1}{2}x+1\right) + \frac{1}{2x-4}.$$

Then $y = \frac{x}{2} + 1$ is an oblique asymptote of the graph of f because

$$\begin{aligned}\lim_{x\to\infty}\left[f(x) - \left(\frac{x}{2}+1\right)\right] &= \lim_{x\to\infty}\frac{1}{2x-4} \\ &= 0 = \lim_{x\to-\infty}\left[f(x) - \left(\frac{x}{2}+1\right)\right].\end{aligned}$$

Example 10 的示意圖

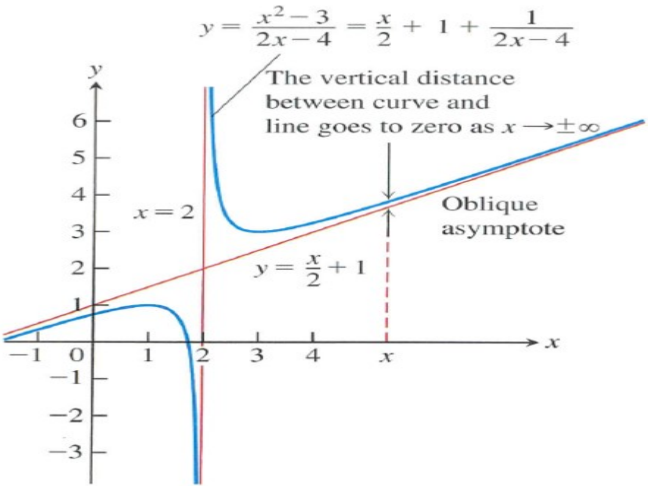


FIGURE 2.58 The graph of the function in Example 10 has an oblique asymptote.

Thank you for your attention!