Chapter 3
Derivatives
(導數或是導函數)

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本章預定授課範圍

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Section 3.1
Tangent Lines and the Derivative at a Point
(切線與單點的導數)

Tangent Line Problem (切線問題)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$. What is the slope (斜率) m of the tangent line (切線) to the graph of f at the point $P(c, f(c))$?

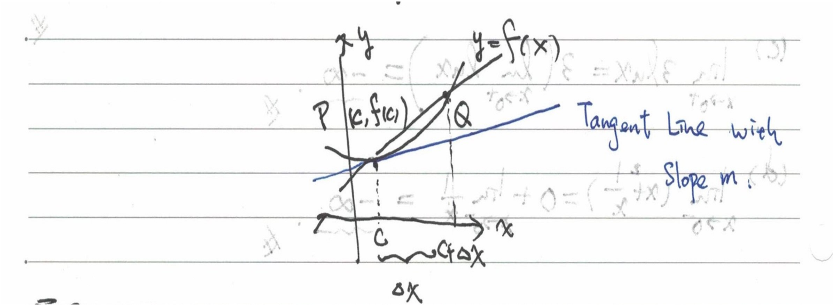
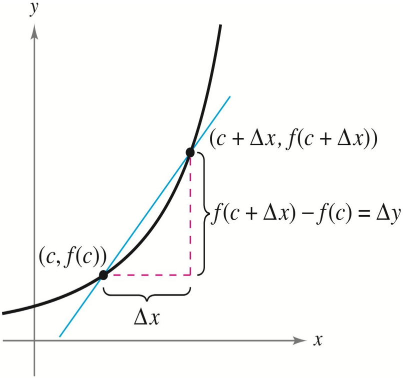
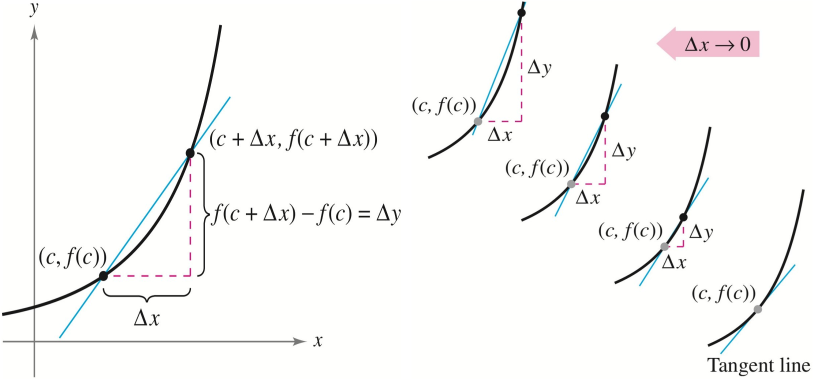


示意圖 (承上頁)




y
x
(c, f(c))
(c + Δx, f(c + Δx))
f(c + Δx) − f(c) = Δy
Δx
Δx → 0
Δy
Δx
Tangent line

Observation

Since the slope of the secant line (割線) passing through $P(c, f(c))$ and $Q(c+\Delta x, f(c+\Delta x))$ is

$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x},$$

the slope m of the tangent line at P is determined by considering

$$m = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}.$$

Def (切線的定義)

If the following limit of the form

$$m = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \exists,$$

then the line passing through $P(c, f(c))$ with slope m is called a tangent line to the graph of f at P.

Equation of a Tangent Line

Remark (切線方程式)

Equation of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ is given by

$$y - f(c) = m(x - c). \quad \text{(Point-Slope Form; 點斜式)}$$

Example 1 (使用定義計算切線斜率)

(a) Find the slope of the curve $y = \frac{1}{x}$ at $x = a \neq 0$. What is the slope at $x = -1$?

(b) Where does the slope $m = \frac{-1}{4}$?

(c) What happens to the tangent lime to the curve at $(a, 1/a)$ as a changes?

(a) The slope of the curve $y = f(x) = 1/x$ at $x = a \neq 0$ is

$$m = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{a+\Delta x} - \frac{1}{a}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(\Delta x)a(a + \Delta x)} = \frac{-1}{a(a + 0)} = \frac{-1}{a^2} < 0.$$

So, the slope at $x = -1$ is $m = -1/(-1)^2 = -1$.

(b) When $\dfrac{-1}{a^2} = m = \dfrac{-1}{4}$, we see that $a^2 = 4$ or $a = \pm 2$. That is, the curve has slope $m = -1/4$ at $(-2, -1/2)$ and $(2, 1/2)$.

(c) Note that the slope of a tangent line at $(a, 1/a)$ is $-1/a^2$. So, the tangent line becomes increasingly steep (陡峭的) as $a \to 0$ because

$$\lim_{a \to 0} \frac{-1}{a^2} = -\infty,$$

and it becomes more and more horizontal as $a \to \pm\infty$ because

$$\lim_{a \to -\infty} \frac{-1}{a^2} = 0 = \lim_{a \to \infty} \frac{-1}{a^2}.$$

Example 1 的示意圖

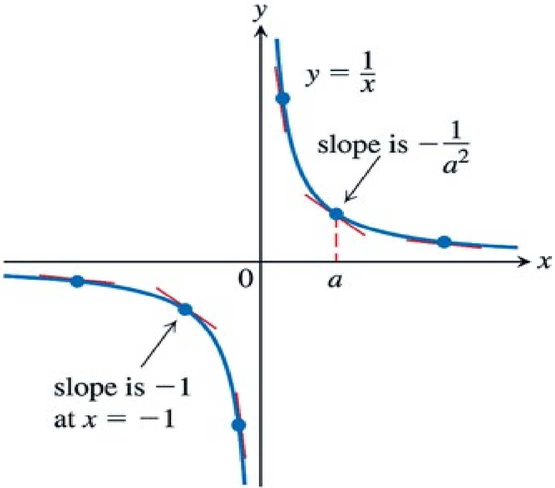


FIGURE 3.2 The tangent line slopes, steep near the origin, become more gradual as the point of tangency moves away

Section 3.2
The Derivative as a Function
(導函數)

Def (導函數的定義)

(1) The derivative (導函數) of f w.r.t. x is a function f' whose value at $x \in \text{dom}(f)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

(2) f is differentiable (可微分; 簡寫為 diff.) at $x \in \text{dom}(f)$ if the derivative $f'(x)$ $\exists$.

(3) f is diff. on $I = (a, b)$ if it is diff. at each $x \in I$.

Notes

- If $S = \{x \in \text{dom}(f) \mid f'(x) \exists\}$, the first derivative f' can be regarded as a function defined on S.
- For any $y = f(x)$, the derivative is often denoted by

$$f'(x) = y'(x) = \frac{df(x)}{dx} = \frac{dy}{dx} = D_x f(x) = D_1 f(x).$$

Use Def. to differentiate the real-valued fuction

$$f(x) = \frac{x}{x-1}$$

for all $x \neq 1$.

Solution of Example 1

Applying the Def. of the derivative, we see that

$$
\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{\frac{z}{z-1}-\frac{x}{x-1}}{z-x} \\
&= \lim_{z\to x} \frac{\frac{-(z-x)}{(z-1)(x-1)}}{z-x} = \lim_{z\to x} \frac{-1}{(z-1)(x-1)} \\
&= \frac{-1}{(x-1)^2} < 0 \quad \text{for all } x \neq 1.
\end{aligned}
$$

Example 2 (利用定義求切線的斜率與方程式)

Let $y = f(x) = \sqrt{x}$ for all $x \geq 0$.

(a) Find the derivative of f for $x > 0$.

(b) Find the tangent line to the curve $y = f(x)$ at $x = 4$.

(a) For $x > 0$, the derivative of f w.r.t. x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) Since the slope of the tangent line at $x = 4$ is $m = \frac{1}{2\sqrt{4}} = \frac{1}{4}$, the equation of the tangent line at $(4, 2)$ is gicen by

$$y - 2 = \frac{1}{4}(x - 4) \quad \text{or} \quad y = \frac{1}{4}x + 1.$$

Example 2 的示意圖

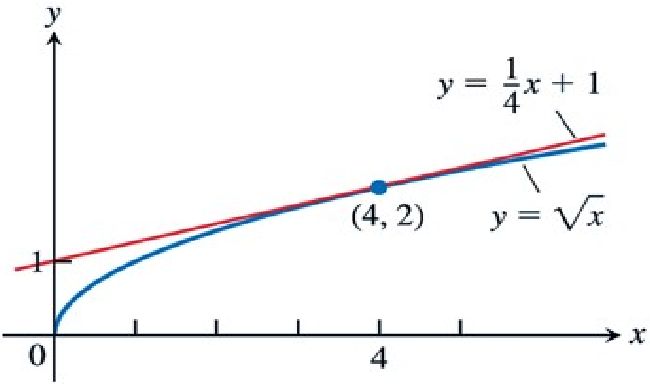


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent line at (4, 2). The tangent line's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Def (閉區間上的可微分性)

f is diff. on $I = [a, b]$ if it is diff. on (a, b), the right-hand derivative (右導數) at $x = a$

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \quad \exists,$$

and the left-hand derivative (左導數) at $x = b$

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} = \lim_{x \to b^-} \frac{f(x) - f(b)}{x - b} \quad \exists.$$

Remark (內點可微分性的等價條件)

Let f be a real-valued function defined on $I = (a, b)$ with $c \in I$. Then it is true that

$$
\begin{aligned}
& f \text{ is diff. at } x = c. \\
\Longleftrightarrow\ & f'_+(c) = f'_-(c) = f'(c) \quad \exists. \\
\Longleftrightarrow\ & \exists \text{ a function } \varepsilon_1 = \varepsilon_1(c, \Delta x) \to 0 \text{ as } \Delta x \to 0 \text{ s.t.} \\
& f(c + \Delta x) = f(c) + f'(c) \cdot \Delta x + \varepsilon_1 \cdot \Delta x.
\end{aligned}
$$

Differentiability v.s. Continuity

Thm 1 (可微分 $\Longrightarrow$ 連續)

Let f be a real-valued function defined on $X \subseteq \mathbb{R}$ with $c \in X = \text{dom}(f)$. If f is diff. at c, then f is conti. at c.

(f 在 $x = c$ 處可微分 $\Longrightarrow$ f 在 $x = c$ 處必定連續!)

Since f is diff. at $c \in \mathrm{dom}(f)$, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \exists.$$

Then the continuity of f at $x = c$ follows, since

$$\begin{aligned} \lim_{x \to c} f(x) &= \lim_{x \to c} \left[f(c) + \frac{f(x) - f(c)}{x - c} \cdot (x - c) \right] \\ &= f(c) + f'(c) \cdot 0 = f(c). \end{aligned}$$

Example 4 (Thm 1 的反例)

Show that $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ is conti. at $x = 0$, but it is NOT diff. at $x = 0$.

(此範例說明函數的連續點不一定是可微分點!)

Since it is easily seen that $\lim_{x\to 0} f(x) = 0 = f(0)$, f is conti. at $x = 0$.

Because the slope of a line $y = mx + n$ is m, we see that
$f'(x) = -1 \quad \forall x \in (-\infty, 0)$ and $f'(x) = 1 \quad \forall x \in (0, \infty)$.

But, f is NOT diff. at $x = 0$ because the one-sided derivatives of f at $x = 0$ are
$f'_-(0) = \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$ and
$f'_+(0) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$,
respectively, and hence we know that $f'(0) \nexists$.

Example 4 的示意圖

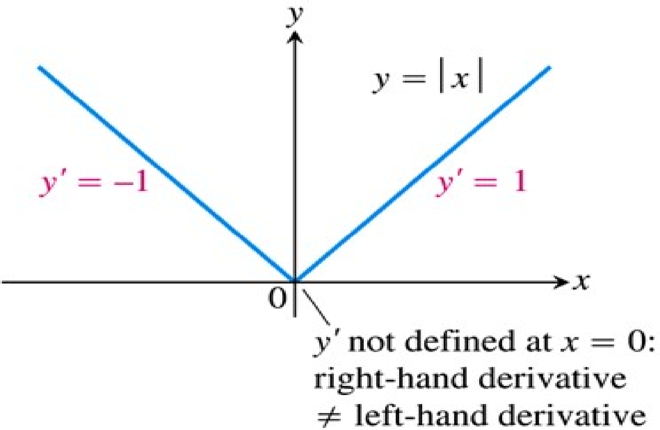


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

Example (Thm 1 的反例)

The real-valued function defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is conti. at $x = 0$, but it is NOT diff. at $x = 0$. Why?

Remarks (重要口訣, 切記!)

1. 函數的可微分點必定是連續點，詳見 Thm 1。
2. 但是，函數的連續點不一定是可微分點! 詳見 Example 4。

Sections 3.3
Differentiation Rules
(微分法則)

Thm (Basic Differentiation Rules)

Let f and g be diff. functions of x and $c \in \mathbb{R}$. Then

(1) $\frac{d}{dx}[c] = 0.$

(2) $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$ for $n \in \mathbb{R}$.

(3) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x).$

(4) $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x).$

(5) $\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) = $ (前微)(後不微) $+$ (前不微)(後微).

(6) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} = \frac{\text{(子微)(母不微) - (子不微)(母微)}}{\text{(母)}^2}.$

Example 1 (冪法則的例子)

Applying the Power Rule (2), we immediately obtain

(a) $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2.$

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3} = \dfrac{2}{3\sqrt[3]{x}}.$

(c) $\dfrac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}.$

(d) $\dfrac{d}{dx}(\dfrac{1}{x^4}) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = \dfrac{-4}{x^5}.$

Applying the differentiation rules (1)–(4), we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) = (x^3)' + \frac{4}{3}(x^2)' - 5(x)' + (1)'$$
$$= 3x^{3-1} + \frac{4}{3}\cdot 2x^{2-1} + 5x^{1-1} + 0 = 3x^2 + \frac{8}{3}x - 5.$$

Thm (自然指數函數的微分公式)

The derivative of $f(x) = e^x$ is the function f itself, i.e.,

$$\frac{dy}{dx} = f'(x) = e^x \quad \forall\, x \in \mathbb{R}.$$

Equivalent Def. of Euler number e

The number e is the base number of an exponential function s.t. the slope of the tangent line at $(0, 1)$ is 1, i.e., it satisfies

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - e^0}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

In Section 1.5, we know that f satisfies the following property:

$$f'(0) = \lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Thus, we immediately obtain

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = e^x\left(\lim_{h\to 0} \frac{e^h - 1}{h}\right) = e^x \cdot 1 = e^x$$

for all $x \in \mathbb{R}$.

Example 6 (乘法法則的例子)

Applying the Product Rule (5), we see that

(a) $y = \frac{1}{x}(x^2 + e^x) \Longrightarrow$

$$y' = \frac{-1}{x^2}(x^2 + e^x) + \frac{1}{x}(2x + e^x) = 1 + (x-1)\frac{e^x}{x^2}.$$

(b) $y = e^{2x} = e^x \cdot e^x \Longrightarrow$

$$y' = (e^x)'e^x + e^x(e^x)' = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}.$$

Applying the Quotient Rule (6), we obtain

(a) $y = \dfrac{t^2-1}{t^3+1} \Longrightarrow$

$$\frac{dy}{dt} = \frac{(2t)(t^3+1)-(t^2-1)(3t^2)}{(t^3+1)^2} = \frac{-t^4+3t^2+2t}{(t^3+1)^2}.$$

(b) $y = e^{-x} \Longrightarrow \dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{1}{e^x}\right) = \dfrac{0-1\cdot e^x}{(e^x)^2} = \dfrac{-1}{e^x} = -e^{-x}.$

Example 8 (化簡後使用冪法則求導數)

Find the derivative of the following function

$$y = \frac{(x-1)(x^2-2x)}{x^4}$$

for all $x \neq 0$.

We first rewrite y as

$$y = \frac{(x-1)(x^2-2x)}{x^4} = \frac{x^3-3x^2+2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Applying the Power Rule (2), the derivative of y is given by

$$y' = -x^{-2} + 6x^{-3} - 6x^{-4} = \frac{-1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}$$

for all $x \neq 0$.

Higher-Order Derivatives (高階導函數)

Let $y = f(x)$ be a diff. function of x. Then

- first derivative: $y' = f'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}[f(x)]$.
- second derivative: $y'' = f''(x) = \dfrac{d^2y}{dx^2} = \dfrac{d^2}{dx^2}[f(x)] := \dfrac{d}{dx}(y')$.
- third derivative: $y''' = f'''(x) = \dfrac{d^3y}{dx^3} = \dfrac{d^3}{dx^3}[f(x)] := \dfrac{d}{dx}(y'')$.
- fourth derivative:
  $y^{(4)} = f^{(4)}(x) = \dfrac{d^4y}{dx^4} = \dfrac{d^4}{dx^4}[f(x)] := \dfrac{d}{dx}(y''')$.

  $\vdots$
- nth derivative: $y^{(n)} = f^{(n)}(x) = \dfrac{d^ny}{dx^n} = \dfrac{d^n}{dx^n}[f(x)] := \dfrac{d}{dx}(y^{(n-1)})$
  for $n \in \mathbb{N}$. Here, we denote $y^{(0)} = y = f(x)$.

Section 3.5
Derivatives of Trigonometric Functions
(三角函數的微分法則)

Thm (Derivatives of Elementary Functions)

(1) $\dfrac{d}{dx}[\sin x] = \cos x, \quad \dfrac{d}{dx}[\cos x] = -\sin x.$

(2) $\dfrac{d}{dx}[\tan x] = \sec^2 x, \quad \dfrac{d}{dx}[\cot x] = -\csc^2 x.$

(3) $\dfrac{d}{dx}[\sec x] = \sec x \tan x, \quad \dfrac{d}{dx}[\csc x] = -\csc x \cot x.$

(4) $\dfrac{d}{dx}[e^x] = e^x, \quad \dfrac{d}{dx}[\,\ln|x|\,] = \dfrac{1}{x}$ for $x \neq 0$.

Note: see also Example 5 for the proof of $\dfrac{d}{dx}(\tan x)$ in (2).

(1)

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x}\\ &= \lim_{\Delta x \to 0} \frac{\sin x \cos(\Delta x) + \cos x \sin(\Delta x) - \sin x}{\Delta x}\\ &= \sin x\left(\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x}\right) + \cos x\left(\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}\right)\\ &= (\sin x)(0) + \cos x \cdot 1 = \cos x.\end{aligned}$$

(2) $\dfrac{d}{dx}[\tan x] = \dfrac{d}{dx}\left[\dfrac{\sin x}{\cos x}\right] = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$

(3) $\dfrac{d}{dx}[\sec x] = \dfrac{d}{dx}\left[\dfrac{1}{\cos x}\right] = \dfrac{(0)\cos x - 1 \cdot (-\sin x)}{\cos^2 x} = \left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{\cos x}\right) = \sec x \tan x.$

Note (另類的證明)

Applying the rule (1) of above Thm, we immediately obtain

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta - \sin(0)}{\theta - 0} = \frac{d}{d\theta}(\sin\theta)\Big|_{\theta=0} = \cos(0) = 1,$$

which is the same as Theorem 7 of Section 2.4.

Example 1 (與 $\sin x$ 微分有關的例子)

(a) $y = x^2 - \sin x \Longrightarrow y' = 2x - \cos x.$

(b) $y = e^x \sin x \Longrightarrow y' = (e^x)' \sin x + e^x (\sin x)' = e^x \sin x + e^x \cos x.$

(c) $y = \dfrac{\sin x}{x} \Longrightarrow y' = \dfrac{(\sin x)'(x) - (\sin x)(x)'}{x^2} = \dfrac{x \cos x - \sin x}{x^2}.$

Example 2 (與 $\cos x$ 微分有關的例子)

(a) $y = 5e^x + \cos x \Longrightarrow y' = 5e^x - \sin x$.

(b) $y = \sin x \cos x \Longrightarrow$
$y' = (\sin x)'(\cos x) + (\sin x)(\cos x)' = \cos^2 x - \sin^2 x$.

(c) $y = \dfrac{\cos x}{1 - \sin x} \Longrightarrow$
$$y' = \frac{(\cos x)'(1 - \sin x) - (\cos x)(1 - \sin x)'}{(1 - \sin x)^2} =$$
$$\frac{(-\sin x)(1 - \sin x) - (\cos x)(-\cos x)}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2} =$$
$$\frac{1}{1 - \sin x}.$$

Since all of six trigonometric functions are conti. at every point in their domains, we immediately obtain

$$\lim_{x\to 0}\frac{\sqrt{2+\sec x}}{\cos(\pi-\tan x)}=\frac{\sqrt{2+1}}{\cos(\pi-0)}=\frac{\sqrt{3}}{-1}=-\sqrt{3}.$$

Section 3.6
The Chain Rule
(連鎖律)

Thm 2 (The Chain Rule; 連鎖律)

If $y = f(u)$ is a diff. function of u and $u = g(x)$ is a diff. function of x, then $y = f(g(x))$ is a diff. function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}\Big[f(g(x))\Big] = f'(g(x)) \cdot g'(x).$$

Example 1 (連鎖律的例子)

Find the first derivative $\dfrac{dy}{dx}$ if the function is given by

$$y = (3x^2 + 1)^2.$$

Solution of Example 1

Let $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$. Applying the Chain Rule, we see that

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot 6x = 12x \cdot u \\ &= 12x(3x^2 + 1) = 36x^3 + 12x.\end{aligned}$$

Differentiate the following real-valued function

$$y = \sin(x^2 + e^x)$$

with respect to the independent variable x.

Solution of Example 3

Let $y = f(u) = \sin u$ and $u = g(x) = x^2 + e^x$. Then

$$y' = f'(g(x)) \cdot g'(x) = \cos(x^2 + e^x) \cdot (2x + e^x) = (2x + e^x)\cos(x^2 + e^x)$$

by the Chain Rule.

Thm (Derivatives of e^x and e^u)

If $u = u(x)$ is a diff. function of x, then

(1) $\frac{d}{dx}e^x = e^x \quad \forall\, x \in \mathbb{R}$.

(2) $\frac{d}{dx}e^u = e^u \cdot u\,'$. (切記!)

Example 4 (利用連鎖律求 e^u 的微分)

To differentiate the real-valued function

$$y = e^{\cos x},$$

we consider $y = f(u) = e^u$ and $u = g(x) = \cos x$. Thus, it follows from the Chain Rule that

$$y' = e^u \cdot u' = e^{\cos x} \cdot (\cos x)' = -e^{\cos x} \sin x.$$

Thm (Power Chain Rule; 冪連鎖律, 切記!)

If $u(x)$ is a diff. function of x, then $y = \Big[u(x)\Big]^n$ is also a diff. function of x for any $n \in \mathbb{R}$. Furthermore, we have

$$y' = n \cdot \Big[u(x)\Big]^{n-1} \cdot u'(x).$$

(a) $\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \cdot (5x^3 - x^4)' = 7(15x^2 - 4x^3)(5x^3 - x^4)^6.$

(b) $y = \frac{1}{3x - 2} = (3x - 2)^{-1} \Longrightarrow$
$y' = (-1)(3x - 2)^{-2} \cdot 3 = \frac{-3}{(3x - 2)^2}.$

(c) $\frac{d}{dx}\sin^5 x = \frac{d}{dx}(\sin x)^5 = 5\sin^4 x \cos x.$

(d) $\frac{d}{dx}e^{\sqrt{3x+1}} = e^{\sqrt{3x+1}} \cdot \left[(3x + 1)^{1/2}\right]' = e^{\sqrt{3x+1}} \cdot \frac{1}{2}(3x + 1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x + 1}}e^{\sqrt{3x+1}}.$

Section 3.7
Implicit Differentiation
(隱微分)

Representations of a function

- Explicit Form (顯式):

$$y = f(x),$$

where x is the independent variable (自變量) and y is the dependent variable (應變量).

- Implicit Form (隱式):

$$F(x, y) = 0,$$

where we assume that $y = y(x)$ is a diff. function of x.

[Q]: How to fined $y' = \dfrac{dy}{dx}$ under the implicit form?
Ans: apply the technique of Implicit Differentiation!

Find the slope of the tangent line to the graph of a circle

$$x^2 + y^2 = 25$$

at the point $(3, -4)$.

Example 2 的示意圖

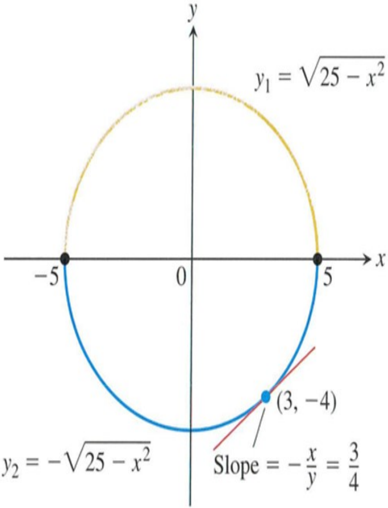


FIGURE 3.31 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

Assume that $y = y(x)$ is a diff. function of x. Differetiating w.r.t. x on both sides of the equation, we have

$$\frac{d}{dx}\left(x^2 + [y(x)]^2\right) = \frac{d}{dx}(25) \Longrightarrow 2x + 2y\frac{dy}{dx} = 0.$$

Then $\dfrac{dy}{dx} = \dfrac{-2x}{2y} = \dfrac{-x}{y}$ and hence the slope of the circle at $(3, -4)$ is $m = \dfrac{dy}{dx}\Big|_{(3,-4)} = \dfrac{-x}{y}\Big|_{(3,-4)} = \dfrac{-3}{-4} = \dfrac{3}{4}$.

Example 3 (隱微分的例子)

Find the first derivative $\frac{dy}{dx}$ if x and y satisfy the equation

$$y^2 = x^2 + \sin(xy).$$

Solution of Example 3

Assume that $y = y(x)$ is a diff. function of x. Differentiating the given equation implicitly, we obtain

$$
\begin{aligned}
\frac{d}{dx}[y(x)]^2 &= \frac{d}{dx}(x^2) + \frac{d}{dx}[\sin(x \cdot y(x))] \\
\Longrightarrow 2y\frac{dy}{dx} &= 2x + \cos(xy) \cdot \left(y + x\frac{dy}{dx}\right). \\
\Longrightarrow [2y - x\cos(xy)]\frac{dy}{dx} &= 2x + y\cos(xy). \\
\Longrightarrow \frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}.
\end{aligned}
$$

Example 3 的示意圖

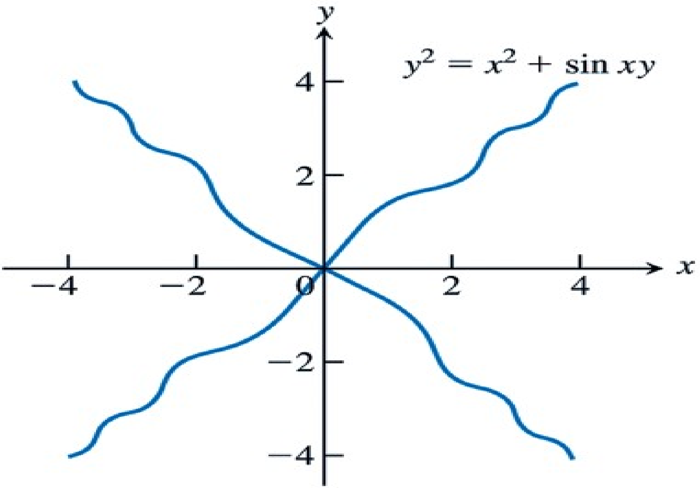


FIGURE 3.32 The graph of the equation in Example 3.

Find the second derivative $\frac{d^2y}{dx^2}$ if x and y satisfy the equation

$$2x^3 - 3y^2 = 8.$$

Solution of Example 4

Assume that $y = y(x)$ is a diff. function of x. Then

$$6x^2 - 6y \cdot y' = 0 \Longrightarrow y' = \frac{6x^2}{6y} = \frac{x^2}{y}.$$

Furthermore, the second derivative is given by

$$\begin{aligned} y'' &= \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2 \cdot y'}{y^2} \\ &= \frac{2x}{y} - \frac{x^2}{y^2} \cdot \frac{x^2}{y} = \frac{2x}{y} - \frac{x^4}{y^3} \quad \text{for } y \neq 0. \end{aligned}$$

Example 5 (利用隱微分求切線與法線)

(a) Show that $P(2,4)$ lies on the curve $x^3 + y^3 - 9xy = 0$.

(b) Find the tangent line and normal line (法線) to the curve at the point P.

(a) It is true because $(2)^3 + (4)^3 - 9(2)(4) = 72 - 72 = 0$.

(b) Assume that $y = y(x)$ is a diff. function of x. Then

$$3x^2 + 3y^2 y' - 9(y + xy') = 0 \Longrightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.$$

Since the slopes of the tangent line and normal line at $P(2,4)$ are $m_1 = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{4}{5}$ and $m_2 = \frac{-5}{4}$, we know that

$$y = 4 + \frac{4}{5}(x - 2) = \frac{4}{5}x + \frac{12}{5} \text{ and } y = \frac{-5}{4}x + \frac{13}{2}$$

are the equations of the tangent line and normal line at P.

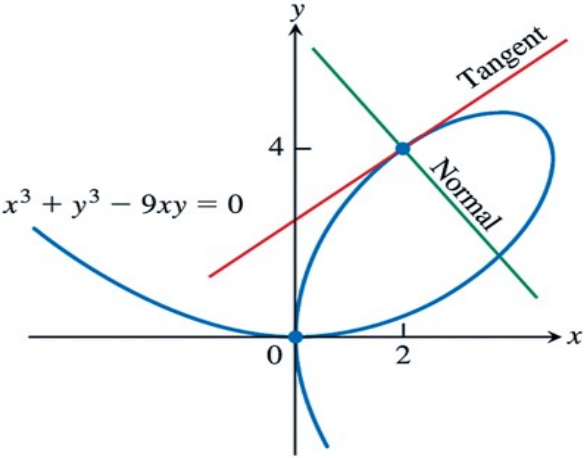


FIGURE 3.34 Example 5 shows how to find equations for the tangent and normal to the folium of Descartes at (2,4).

Section 3.8
Derivatives of Inverse Functions and Logarithms
(反函數與對數函數的微分法則)

Thm 3 (反函數的可微分性)

Let f be diff. on an open interval I. If f has an inverse function f^{-1}, then f^{-1} is diff. at any $x \in \text{range}(f)$ for which $f'(f^{-1}(x)) \neq 0$, with the derivative

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example 2 (Thm 3 的例子)

Let $f(x) = x^3 - 2$ for $x > 0$. Find the value of $\dfrac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding the formula for $f^{-1}(x)$.

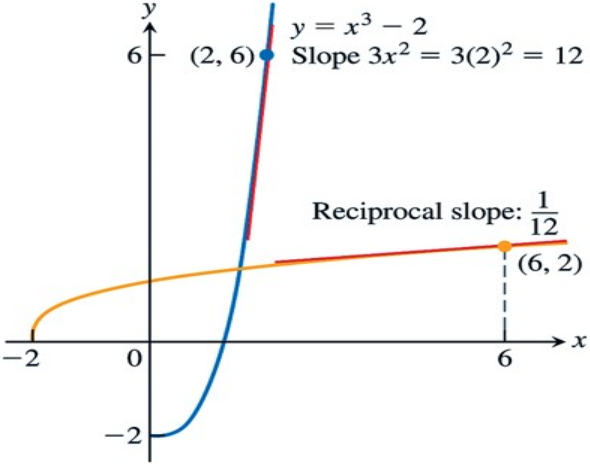


FIGURE 3.38 The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of f^{-1} at $x = 6$ (Example 2).

Solution of Example 2

Since $f\colon (0,\infty) \to (-2,\infty)$ is one-to-one (Check!), its inverse function $f^{-1} : (-2,\infty) \to (0,\infty) \quad \exists$.

Moreover, f^{-1} is diff. at $x = 6 = f(2)$ by Thm 3 and the derivative of f^{-1} at $x = 6$ is given by

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{12},$$

since $f'(2) = 3x^2\Big|_{x=2} = 12$ and $f^{-1}(6) = 2$.

Thm (Derivatives of $\ln x$ and $\ln u$)

If $u = u(x)$ is a diff. function of x, then

(1) $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ for $x > 0$, $\quad \dfrac{d}{dx} \ln u = \dfrac{u'}{u}$ for $u > 0$.

(2) $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$ for $x \neq 0$, $\quad \dfrac{d}{dx} \ln |u| = \dfrac{u'}{u}$ for $u \neq 0$.

Sketch of Proof

(1) For any $x > 0$, $y = \ln x$ can be rewritten as

$$e^y = x \Longrightarrow e^y y' = 1 \Longrightarrow y' = e^{-y} = e^{-\ln x} = x^{-\ln e} = x^{-1}.$$

So, $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ for $x > 0$.

(2) For any $x < 0$, applying the Chain Rule, we see that

$$\frac{d}{dx} \ln |x| = \frac{d}{dx} \ln(-x) = \frac{(-x)'}{-x} = \frac{-1}{-x} = \frac{1}{x}.$$

So, $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$ for $x \neq 0$.

Example 3 (計算 $\ln u$ 的微分)

(a) $\dfrac{d}{dx}\ln(2x) = \dfrac{(2x)'}{2x} = \dfrac{2}{2x} = \dfrac{1}{x}$ for $x > 0$.

(b) $\dfrac{d}{dx}\ln(x^2+3) = \dfrac{(x^2+3)'}{x^2+3} = \dfrac{2x}{x^2+3}$ for $x \in \mathbb{R}$.

Def (函數 a^x 和 $\log_a x$ 的定義)

Let $0 < a \neq 1$.

(1) The exponential function to the base a (以 a 為底的指數函數) is defined by

$$a^x := e^{x \ln a} \quad \forall x \in \mathbb{R}.$$

(2) The logarithmic function to the base a (以 a 為底的對數函數) is defined by

$$\log_a x := \frac{\ln x}{\ln a} \quad \forall x > 0.$$

Thm (函數 a^u 和 $\log_a u$ 的微分公式)

Let $u = u(x)$ be a diff. function of x and $0 < a \neq 1$.

(1) $\dfrac{d}{dx}a^x = (\ln a)a^x \quad \forall x \in \mathbb{R}$, $\quad \dfrac{d}{dx}a^u = (\ln a)a^u \cdot u'$.

(2) $\dfrac{d}{dx}\log_a x = \dfrac{1}{(\ln a)x}$ for $x > 0$, $\quad \dfrac{d}{dx}\log_a u = \dfrac{u'}{(\ln a)u}$ for $u > 0$.

(3) $\dfrac{d}{dx}\log_a |x| = \dfrac{1}{(\ln a)x}$ for $x \neq 0$, $\dfrac{d}{dx}\log_a |u| = \dfrac{u'}{(\ln a)u}$ for $u \neq 0$.

Example 5 (求 a^u 的導數)

(a) $\frac{d}{dx}3^x = 3^x \ln 3.$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3) \cdot (-x)' = -3^{-x} \ln 3.$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3) \cdot (\sin x)' = 3^{\sin x}(\ln 3) \cos x.$

Example 6 (Logarithmic Differentiation)

Find the firat derivative $\dfrac{dy}{dx}$ if the real-valued function is given by

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

for all $x > 1$.

Note that $y > 0$ if $x > 1$. Then we see that

$$\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1).$$

Differentiating this equation implicitly, we further obtain

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

$$\Longrightarrow y' = \frac{(x^2+1)(x+3)^{1/2}}{x-1}\Big(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\Big).$$

Example 7 (對數微分法的例子)

Find the derivative of the real-value function defined by

$$y = f(x) = x^x$$

for all $x > 0$.

Method 1: (換底數後求微分)

For $x > 0$, we can rewrite y as

$$y = x^x = e^{x \ln x}.$$

From the Chain Rule, we immediately obtain

$$\begin{aligned} y' &= e^{x \ln x} \cdot (x \ln x)' \\ &= e^{x \ln x}\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \\ &= x^x(\ln x + 1), \quad x > 0. \end{aligned}$$

Method 2: (利用對數微分法求導數)

Note that $y > 0$ if $x > 0$ and assume that $y = y(x)$ is a diff. function of x. Using the logarithmic differentiation for $\ln y = x \ln x$, we see that

$$\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$
$$\Longrightarrow y' = x^x(\ln x + 1), \quad x > 0.$$

Section 3.9
Inverse Trigonometric Functions
(反三角函數的微分法則)

Thm (反三角函數的微分公式)

Let $u = u(x)$ be a diff. function of x. Then

(1) $\dfrac{d}{dx}\sin^{-1} u = \dfrac{u'}{\sqrt{1-u^2}}$, $\dfrac{d}{dx}\cos^{-1} u = \dfrac{-u'}{\sqrt{1-u^2}}$ for $|u| < 1$.

(2) $\dfrac{d}{dx}\tan^{-1} u = \dfrac{u'}{1+u^2}$, $\dfrac{d}{dx}\cot^{-1} u = \dfrac{-u'}{1+u^2}$.

(3) $\dfrac{d}{dx}\sec^{-1} u = \dfrac{u'}{|u|\sqrt{u^2-1}}$, $\dfrac{d}{dx}\csc^{-1} u = \dfrac{-u'}{|u|\sqrt{u^2-1}}$ for $|u| > 1$.

If $y = \sec^{-1} x$ for $|x| > 1$, then $\sec y = x$ for $y \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Using the implicit differentiation, we see that

$$(\sec y \tan y)y' = 1 \Longrightarrow y' = \frac{1}{\sec y \tan y}.$$

Since $\sec y = x$, $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$ and hence

$$y' = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x > 1, \\ \frac{1}{(-x)\sqrt{x^2-1}}, & x < -1 \end{cases} \quad \text{or } y' = \frac{1}{|x|\sqrt{x^2-1}}$$

is always positive for $|x| > 1$, which is consistent with its graph.

Applying the Chain Rule, we obtain the following derivatives.

(a) $\frac{d}{dx}\sin^{-1}(x^2) = \frac{(x^2)'}{\sqrt{1-(x^2)^2}} = \frac{2x}{\sqrt{1-x^4}}$ for $x^2 = |x^2| < 1$ or $-1 < x < 1$.

(b) $\frac{d}{dx}\sec^{-1}(5x^4) = \frac{20x^3}{|5x^4|\sqrt{(5x^4)^2-1}} = \frac{4}{x\sqrt{25x^8-1}}$ for $5x^4 = |5x^4| > 1$ or $x > \frac{1}{\sqrt[4]{5}}$.

Differential 的定義

Def of Differentials

Let f be diff. on an open interval I with $x \in I$.

(1) The differential dx is an independent variable of any nonzero real number.

(2) The differential of $y = f(x)$ is defined by $dy = f'(x)dx$.

For example, if $y = f(x) = \tan(x^3)$, then the differential dy is

$$dy = f'(x)\,dx = 3x^2 \sec^2(x^3)\,dx.$$

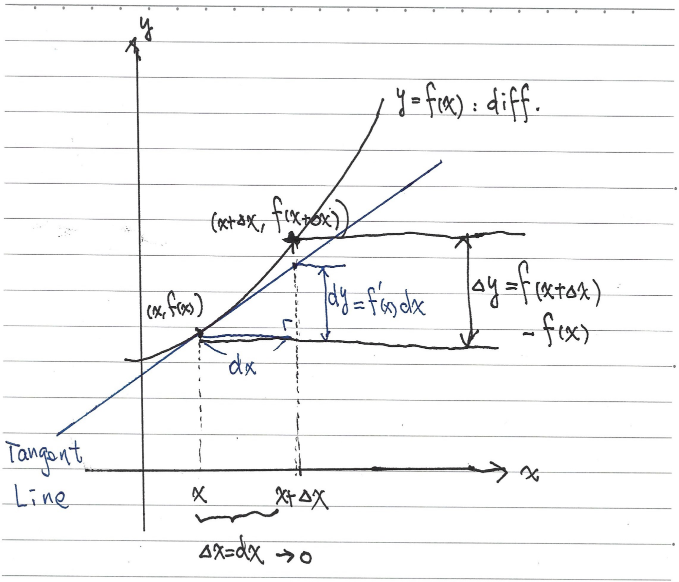
Thm (利用 dy 估計 Δy)

If $dx = \Delta x \approx 0$ is sufficiently small, then

(1) $f(x+\Delta x) - f(x) = \Delta y \approx dy = f'(x)dx.$

(2) $f(x+dx) = f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)dx.$

Δy 與 dy 的示意圖 (承上頁)



Thank you for your attention!