Chapter 4
Applications of Derivatives
(導數的應用)

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Fall 2022

本章預定授課範圍

Section 4.1
Extreme Values of Functions on Closed Intervals
(函數在閉區間上的極值)

Def (局部極值)

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$ with $c \in D$.

(1) f has a local maximum value (局部極大值) at c if $\exists$ open interval I containing c s.t. $f(x) \leq f(c) \quad \forall x \in D \cap I$.

(2) f has a local minimum value (局部極小值) at c if $\exists$ open interval I containing c s.t. $f(x) \geq f(c) \quad \forall x \in D \cap I$.

(3) Local maximum and minimum values are called the local extrema (局部極值) of f.

Note: Local extrema of f are also called the relative extrema (相對極值) of f.

Some Questions

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$.

- Does f *always* have a local extreme value on D?
- How to find the local extrema of f?
- What is $f'(c)$ if $f(c)$ is a local extreme value?

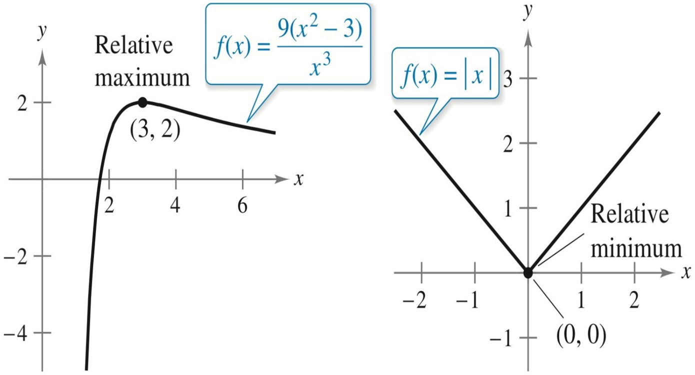
Example (極值發生處的導數)

(a) The rational function

$$f(x) = \frac{9(x^2 - 3)}{x^3} \quad \text{with} \quad f'(x) = \frac{9(9 - x^2)}{x^4}$$

has a rel. max. at the point $(3, 2)$, and $f'(3) = 0$ in this case.

(b) The function $f(x) = |x|$ has a rel. min. value $f(0) = 0$ at the origin $(0, 0)$, but $f'(0) \nexists$. (Why?)



y
Relative maximum
$f(x) = \frac{9(x^2-3)}{x^3}$
(3, 2)
2
−2
−4
4
6
x
$f(x) = |x|$
3
2
1
−1
Relative minimum
(0, 0)

Def (臨界點的定義)

If the first derivative of f at c satisfies

$$f'(c) = 0 \quad \text{or} \quad f'(c) \nexists$$

for some interior point c of $D = \text{dom}(f)$, then the value c is called a critical point (臨界點) of f.

Thm 2 (發生局部極值的必要條件)

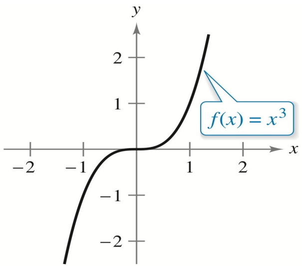
If f has a local extremum at an interior point c of $D = \text{dom}(f)$, then

$$f'(c) = 0 \quad \text{or} \quad f'(c) \nexists,$$

i.e. $x = c$ must be a critical point of f.

Example (Thm 2 的反例)

For $f(x) = x^3$, $x = 0$ is the only critical point of f, since $f'(x) = 3x^2 = 0 \iff x = 0$. But, $f(0) = 0$ is NOT a local extremum of f.



Cubing function

Suppose that $f(c)$ is a local extremum with $f'(c) \neq 0$ $\exists$.
Without loss of generality, we may assume that $f'(c) > 0$.

For $\varepsilon = \dfrac{f'(c)}{2} > 0$, $\exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \frac{f'(c)}{2} \Longrightarrow \frac{f(x) - f(c)}{x - c} > \frac{f'(c)}{2} > 0.$$

Thus, we know that

$$f(x) > f(c) \quad \forall\, x \in (c, c+\delta) \quad \text{and} \quad f(x) < f(c) \quad \forall\, x \in (c-\delta, c).$$

This contradicts to the assumption and hence completes the proof.

Def of Absolute Extrema (絕對極值)

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$ with $c \in D$.

(1) $f(c)$ is an absolute maximum value (絕對極大值) of f on D if $f(x) \leq f(c) \quad \forall\, x \in D$.

(2) $f(c)$ is an absolute minimum value (絕對極小值) of f on D if $f(x) \geq f(c) \quad \forall\, x \in D$.

(3) Absolute maximum and minimum values are called the absolute extrema (絕對極值) of f on D.

Thm 1 (Extreme Value Theorem; E.V.T. 極值定理)

If f is conti. on $I = [a, b]$, then $\exists\, c_1, c_2 \in I$ s.t.

$$f(c_1) \le f(x) \le f(c_2) \quad \forall\, x \in I,$$

i.e., $f(c_1)$ is the absolute minimum value of f and $f(c_2)$ is the absolute maximum value of f on I, respectively.

How to find the points c_1 and c_2 in Thm 1?

Step 1 find all critical numbers $c_1, c_2, \ldots, c_k$ of f in the open interval (a, b), where $k \in \mathbb{N}$.

Step 2 evaluate $f(a)$, $f(b)$ and $f(c_i)$ for $i = 1, 2, \ldots, k$.

Step 3 compare the function values obtained in Step 2.

Example 3 (Thm 1 的例子)

Find the absolute max. and min. values of the function

$$f(x) = 10x(2 - \ln x)$$

on the closed interval $[1, e^2]$.

Applying the Product Rule, we see that

$$f'(x) = 10(2 - \ln x) + 10x\left(\frac{-1}{x}\right) = 10(1 - \ln x), \quad x > 0,$$

and hence $x = e$ is the only critical point of f in $(1, e^2)$.

Since $f(1) = 10(2 - 0) = 20$, $f(e) = 10e \approx 27.2$ and $f(e^2) = 0$, we know that $f(e^2) = 0$ is the absolute min. vale and $f(e) = 10e$ is the absolute max. value, respectively.

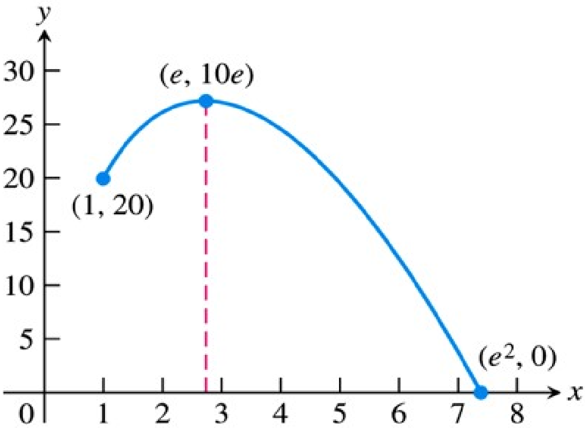


FIGURE 4.8 The extreme values of $f(x) = 10x(2 - \ln x)$ on $[1, e^2]$ occur at $x = e$ and $x = e^2$ (Example 3).

Find the absolute max. and min. values of the function

$$f(x) = x^{2/3} = \sqrt[3]{x^2}$$

on the closed interval $[-2, 3]$. Note that $D = \mathrm{dom}(f) = \mathbb{R}$.

Applying the Power Rule, we see that

$$f'(x) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}, \quad x \neq 0,$$

and $f'(0) \nexists$. Hence, $x = 0$ is the only critical point of f in $(-2, 3)$.

Since $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$, $f(0) = 0$ and $f(3) = \sqrt[3]{9} \approx 2.08$, we know that $f(0) = 0$ is the absolute min. vale and $f(3) = \sqrt[3]{9}$ is the absolute max. value, respectively.

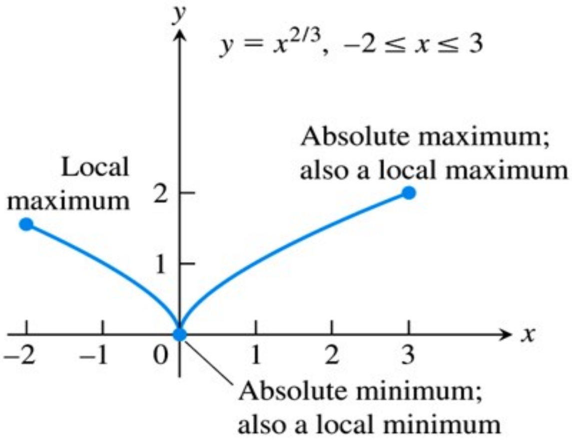
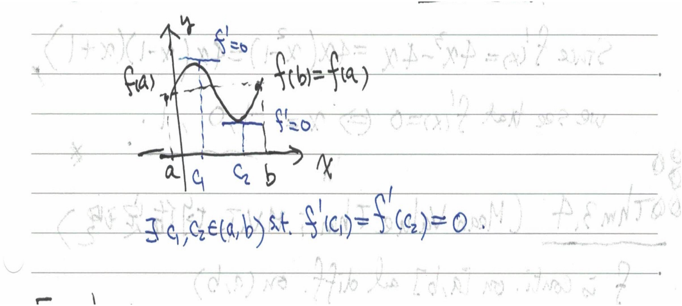


FIGURE 4.9 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

Section 4.2
The Mean Value Theorem
(均值定理)

Thm 3 (Rolle's Theorem; 羅爾定里)

Suppose that f is conti. on $[a, b]$ and diff. on (a, b). If $f(a) = f(b)$, then $\exists\, c \in (a, b)$ s.t. $f'(c) = 0$.



Proof of Thm 3

- Since f is conti. on $[a, b]$ and $f(a) = f(b)$, it follows from E.V.T. that $\exists\, c \in (a, b)$ s.t. $f(c)$ is a relative extremum. Otherwise, f must be a constant function on $[a, b]$ and hence $f'(x) = 0 \quad \forall\, x \in (a, b)$.
- Next, we shall claim that $f'(c) = 0$. If not, say $f'(c) > 0$, then it fowws from the ε-δ Def of a limit that $\exists\, \delta > 0$ s.t. $f(x) < f(c)$ for $x \in (c - \delta, c)$ and $f(c) < f(x)$ for $x \in (c, c + \delta)$. Thus, $f(c)$ is NOT a relative extremum and this gives a contradiction!
- Similarly, we can deduce that $f'(c) < 0 \Longrightarrow f(c)$ is NOT a relative extremum. Consequently, we must have $f'(c) = 0$ for some $c \in (a, b)$.

Example 1 (Rolle's Thm 的例子)

Show that the nonlinear equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Sice $f(x) = x^3 + 3x + 1$ is conti. on the interval $[-1, 0]$ and $f(-1) = -3 < 0 < 1 = f(0)$, it follows from I.V.T. that $\exists\, c \in (-1, 0)$ s.t. $f(c) = 0$.

Suppose that $\exists\, a, b \in \mathbb{R}$ with $a < b$ s.t. $f(a) = f(b) = 0$. Since f is conti. on $[a, b]$ and diff. on (a, b), it follows from Rolle's Thm that $\exists\, d \in (a, b)$ s.t. $f'(d) = 0$, but we see that

$$f'(x) = 3x^2 + 3 > 0 \quad \forall\, x \in \mathbb{R},$$

which gives a contradiction. So, the equation $f(x) = 0$ has exactly one zero in $(-\infty, \infty) = \mathbb{R}$.

Example 1 的示意圖

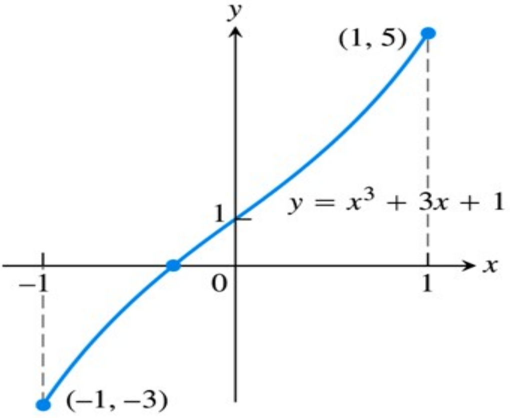
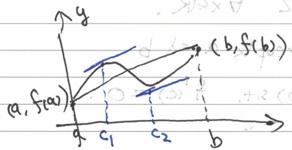


FIGURE 4.12 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 1).

Thm 4 (Mean Value Theorem; M.V.T. 均值定理)

If f is conti. on $[a, b]$ and diff. on (a, b), then $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a).$$



$\exists\, c_1, c_2 \in (a, b)$ s.t. $f'(c_1) = f'(c_2) = \dfrac{f(b) - f(a)}{b - a}$

切線斜率　　端点連線の割線斜率.

- Let $g : [a, b] \to \mathbb{R}$ be a function defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \quad \forall\, x \in [a, b].$$

  Since f is conti. on $[a, b]$ and diff. on (a, b), we know that g is conti. on $[a, b]$, diff. on (a, b) and $g(a) = 0 = g(b)$.

- Since $g'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a} \quad \forall\, x \in (a, b)$, it follows from Thm 3 (Rolle's Thm) that $\exists\, c \in (a, b)$ s.t.

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

  or, equivalently, we prove that $\exists\, c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For the following real-valued function

$$y = f(x) = x^2,$$

find a number $c \in (0, 2)$ s.t. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0}$.

Solution of Example 2

Since f is conti. on $[0,2]$ and diff. on $(0,2)$, it follows from M.V.T. that $\exists\, c \in (0,2)$ s.t.

$$2c = f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - 0}{2 - 0} = 2 \Longrightarrow c = 1.$$

Note that $c = 1 \in (0,2)$ is the only number satisfying the conclusion of Mean Value Theorem.

Example 2 的示意圖

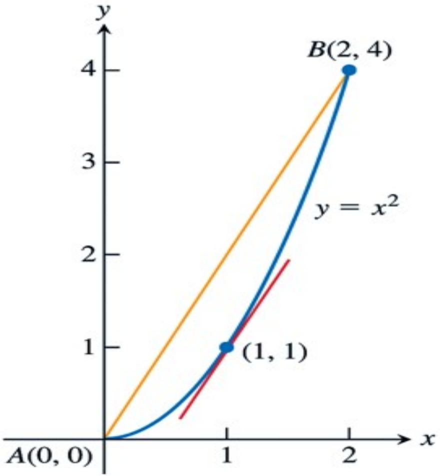


FIGURE 4.17 As we find in Example 2, $c = 1$ is where the tangent is parallel to the secant line.

For the function $y = f(x) = \dfrac{1}{x}$, fined a number $c \in (1,2)$ s.t.

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}.$$

Solution of above Example

Since $f'(x) = \dfrac{-1}{x^2}$ for $x \neq 0$, we know that f is conti. on $[1, 2]$ and diff. on $(1, 2)$, respectively.

From the M.V.T., $\exists\, c \in (1, 2)$ s.t.

$$\frac{-1}{c^2} = f'(c) = \frac{f(2) - f(1)}{2 - 1} = \frac{1}{2} - 1 = \frac{-1}{2},$$

and thus $c^2 = 2$ or $c = \pm\sqrt{2}$. Since $c = -\sqrt{2} \notin (1, 2)$, we see that $c = \sqrt{2} \in (1, 2)$ satisfies the conclusion of Mean Value Theorem.

Example (M.V.T. 的補充題)

For any $a, b \in \mathbb{R}$, prove the following inequality

$$|\sin a - \sin b| \leq |a - b|.$$

Proof: Let $a, b \in \mathbb{R}$. Without loss of generality, we may assume that $a < b$. Since $f(x) = \sin x$ is conti. on $[a, b]$ and diff. on (a, b), it follows from M.V.T. that $\exists\, c \in (a, b)$ s.t.

$$\sin b - \sin a = f'(c) \cdot (b - a) = (\cos c) \cdot (b - a).$$

So, we immediately see that

$$|\sin a - \sin b| = |\cos c| \cdot |a - b| \leq |a - b|$$

because $|\cos c| \leq 1$, and hence this completes the proof.

Cor 1 (均值定理的第一個推論)

If $f'(x) = 0 \quad \forall x \in (a, b)$, then $f(x) = C \quad \forall x \in (a, b)$, where C is a constant.

pf: For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, f is conti. on $[x_1, x_2]$ and diff. on (x_1, x_2). Thus, from M.V.T., we see that

$$f(x_2) - f(x_1) = f'(d)(x_2 - x_1) = 0 \quad \text{or} \quad f(x_1) = f(x_2)$$

for some $d \in (x_1, x_2)$. We then conclude that $f(x) = C \quad \forall x \in (a, b)$, where C is a constant.

Cor 2 (均值定理的第二個推論)

If $f'(x) = g'(x) \quad \forall x \in (a, b)$, then $\exists$ a cnstant C s.t. $f(x) = g(x) + C \quad \forall x \in (a, b)$.

pf: Let $h(x) := f(x) - g(x) \quad \forall x \in (a, b)$, then $h'(x) = f'(x) - g'(x) = 0 \quad \forall x \in (a, b)$. From Cor. 1, $\exists$ a cnstant C s.t. $h(x) = C \quad \forall x \in (a, b)$ or $f(x) = g(x) + C \quad \forall x \in (a, b)$.

Example 4 (Cor 2 的例子)

Find a real-valued function $y = f(x)$ satisfying the first-order (autonomous) differential equation

$$y' = f'(x) = \sin x$$

and the initial condition $f(0) = 2$.

Let $g(x) = -\cos x$. Then we see that

$$g'(x) = (-\cos x)' = \sin x = f'(x) \quad \forall x \in \mathbb{R}.$$

It follows from Cor. 2 that $f(x) = -\cos x + C$ for some constant C.

Moreover, since the graph of f passes through $(0, 2)$, we have $2 = f(0) = -1 + C$ or $C = 3$. So, the desired function is $f(x) = -\cos x + 3$.

Section 4.3
Monotonic Functions and the First Derivative Test
(單調函數與一階導數測試)

Def (嚴格單調函數的定義)

Let f be a real-valued function defined on an interval I.

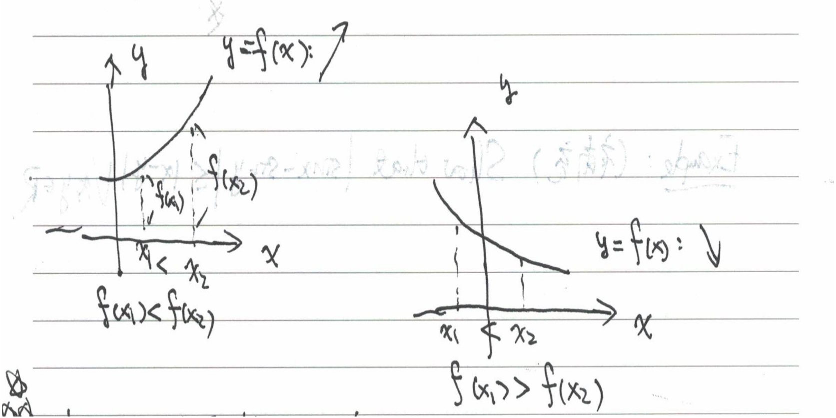
(1) f is (strictly) increasing (嚴格遞增; $\nearrow$) on I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

(2) f is (strictly) decreasing (嚴格遞減; $\searrow$) on I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

(3) The increasing or decreasing functions are called (strictly) monotonic functions (嚴格單調函數).

Note: strictly monotonic functions are one-to-one, but one-to-one functions are NOT necessarily monotonic!

嚴格單調函數的示意圖



Non-strictly Monotonic Functions

Def (非嚴格單調函數)

Let f be a real-valued function defined on an interval I.

(1) f is non-strictly increasing or nondecreasing (非遞減) on I if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

(2) f is non-strictly decreasing or nonincreasing (非遞增) on I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ with $x_1 < x_2$.

Cor 3 (單調函數的充分條件)

Suppose that f is conti. on $[a, b]$ and diff. on (a, b).

(1) $f'(x) > 0 \quad \forall x \in (a, b) \Longrightarrow f$ is increasing ($\nearrow$) on $[a, b]$.

(2) $f'(x) < 0 \quad \forall x \in (a, b) \Longrightarrow f$ is decreasing ($\searrow$) on $[a, b]$.

Example (Cor 3 的反例)

The function $f(x) = x^{1/3}$ is increasing on $\mathbb{R}$, but its first derivative satisfies $f'(x) = \dfrac{1}{3x^{2/3}} > 0 \quad \forall x \in \mathbb{R}\backslash\{0\}$.

For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, since f is conti. on $[x_1, x_2]$ and diff. on (x_1, x_2), it follows from M.V.T. that $\exists\, c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

(1) If $f'(x) > 0 \quad \forall\, x \in (a, b)$, then $f'(c) > 0$ and hence $f(x_2) - f(x_1) > 0$ or $f(x_2) > f(x_1)$. This implies that f is increasing ($\nearrow$) on (a, b).

(2) If $f'(x) < 0 \quad \forall\, x \in (a, b)$, then $f'(c) < 0$ and hence $f(x_2) - f(x_1) < 0$ or $f(x_2) < f(x_1)$. This implies that f is decreasing ($\searrow$) on (a, b).

Find the critical points of the real-valued function

$$f(x) = x^3 - 12x - 5$$

and identify the open intervals on which f is increasing or decreasing, respectively.

Solution of Example 1 (1/2)

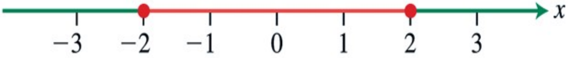
Since the first derivative of f is given by

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2),$$

the critical points of f are $x = -2$ and $x = 2$, respectively.

Hence we know that $f'(x) > 0$ for $x \in (-\infty, -2) \cup (2, \infty)$ and $f'(x) < 0$ for $x \in (-2, 2)$. From Cor 3, f is increasing on the open intervals $(-\infty, -2)$ and $(2, \infty)$, and decreasing on $(-2, 2)$.

Interval	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



Example 1 的示意圖

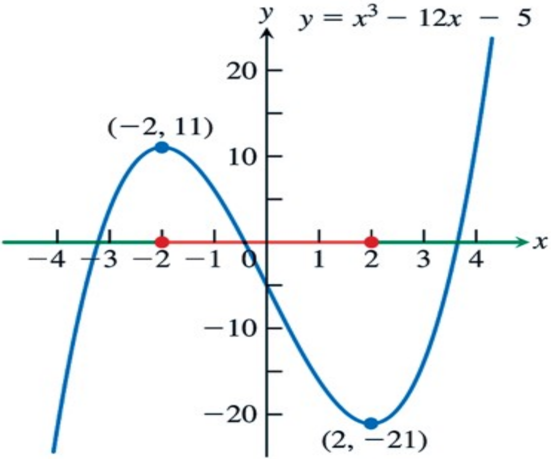


FIGURE 4.20 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

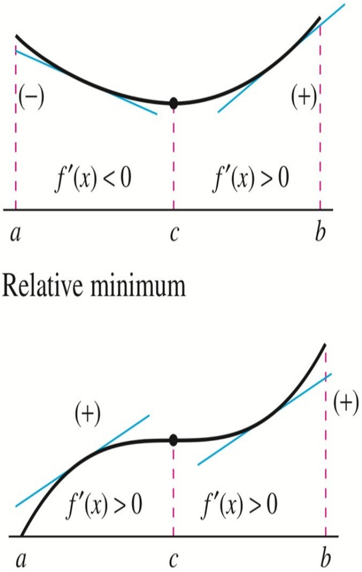
Thm (First Derivative Test; 一階導數測試)

Let f be diff. on an open interval containing c except possibly at c. If $x = c$ is a critical proint of f, then

(1) sign of f' changes from $(+)$ to $(-)$ at $c \Longrightarrow f(c)$ is a local max. value of f.

(2) sign of f' changes from $(-)$ to $(+)$ at $c \Longrightarrow f(c)$ is a local min. value of f.

(3) sign of f' does not change on both sides of $c \Longrightarrow f(c)$ is not a local extremum.


(−)
(+)
$f'(x) < 0$
$f'(x) > 0$
a
c
b
Relative minimum
(+)
(−)
$f'(x) > 0$
$f'(x) < 0$
a
c
b
Relative maximum
(+)
(+)
$f'(x) > 0$
$f'(x) > 0$
a
c
b
(−)
(−)
$f'(x) < 0$
$f'(x) < 0$
a
c
b

Example 3 (一階導數測試法的例子)

Let $f(x) = (x^2 - 3)e^x$ for all $x \in \mathbb{R}$.

(a) Find the critical points of f.

(b) Identify the open intervals on which f is increasing or decreasing.

(c) Find the local and absolute extreme values of f.

(a) From the first derivative of f given by

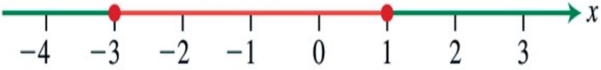
$$f'(x) = 2xe^x + (x^2 - 3)e^x = e^x(x^2 + 2x - 3) = e^x(x+3)(x-1),$$

we see that $x = -3$ and $x = 1$ are critical points of f.

(b) Since $f'(x) > 0$ for $x \in (-\infty, -3) \cup (1, \infty)$ and $f'(x) < 0$ for $x \in (-3, 1)$, it follows from Cor 3 that f is increasing on $(-\infty, -3)$ and $(1, \infty)$, and decreasing on $(-3, 1)$, respectively.

Solution of Example 3 (2/3)

Interval	$x < -3$	$-3 < x < 1$	$1 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing



(c) From the First Derivative Test, since the sign of f' changes from $(+)$ to $(-)$ at $x = -3$, $f(-3) = 6e^{-3} \approx 0.299$ is a local max. value. Similarly, $f(1) = -2e \approx -5.437$ is a local min. value because the sign of f' changes from $(-)$ to $(+)$ at $x = 1$.

In fact, since $f(x) > 0$ for $|x| > \sqrt{3}$, we see that $f(1) = -2e$ is the absolute min. value and there is no absolute maxi. value because $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^2 - 3)e^x = \infty$.

Example 3 的示意圖

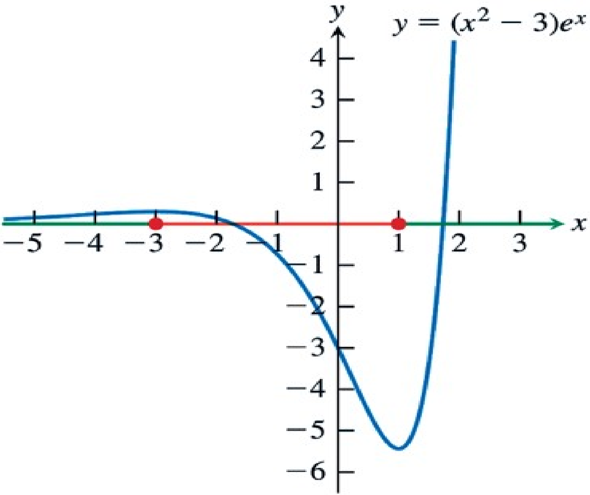


FIGURE 4.23 The graph of $f(x) = (x^2 - 3)e^x$ (Example 3).

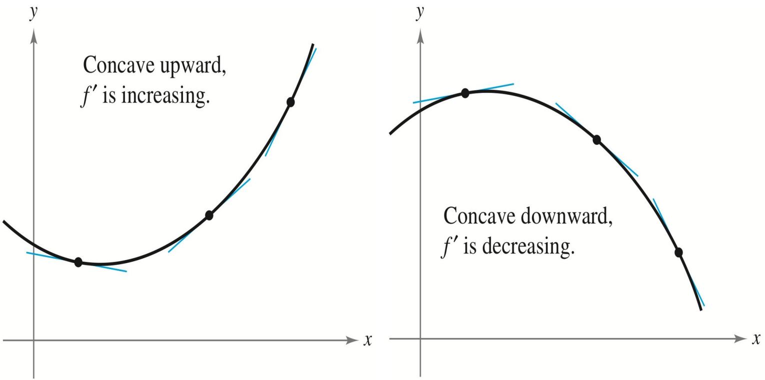
Section 4.4
Concavity and Curve Sketching
(凹性與曲線描繪)

Concavity (函數圖形的凹性)

Let f be diff. on an open interval $I = (a, b)$.

(1) The graph of f is concave up (凹向上; C.U.) on I if its first derivative f' is $\nearrow$ on I.

(2) The graph of f is concave down (凹向下; C.D.) on I if its first derivative f' is $\searrow$ on I.


y
Concave upward,
f′ is increasing.
x
y
Concave downward,
f′ is decreasing.
x

Thm (Test for Concavity; 凹性測試法)

Suppose that $f''(x)$ exists on an open interval I.

(1) $f''(x) > 0 \quad \forall x \in I \implies$ the graph of f is C.U. on I.

(2) $f''(x) < 0 \quad \forall x \in I \implies$ the graph of f is C.D. on I.

pf: It follows immediately from the Def of Concavity and $f''(x) = \frac{d}{dx}[f'(x)]$ that

(1) $f''(x) > 0 \quad \forall x \in I \implies f'$ is increasing on $I \implies$ the graph of f is C.U. on I.

(2) $f''(x) < 0 \quad \forall x \in I \implies f'$ is decreasing on $I \implies$ the graph of f is C.D. on I.

Example 1 (判斷凹性的例子)

(a) The graph of $f(x) = x^3$ is C.D. on the open interval $(-\infty, 0)$ because $f''(x) = 6x < 0$ for $x < 0$, and its graph is C.U. on $(0, \infty)$ because $f''(x) = 6x > 0$ for $x > 0$.

(b) The graph of $f(x) = x^2$ is C.U. on $\mathbb{R} = (-\infty, \infty)$ because $f''(x) = 2 > 0$ for all $x \in \mathbb{R}$.

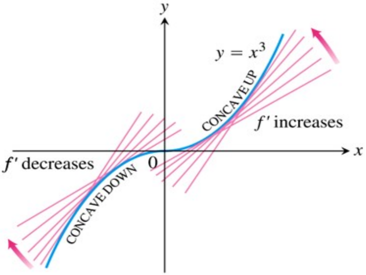


FIGURE 4.24 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

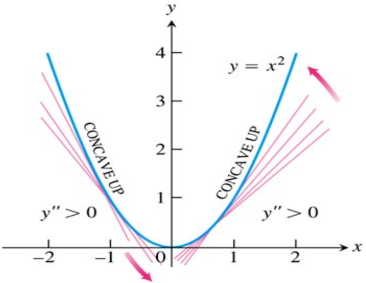


FIGURE 4.25 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

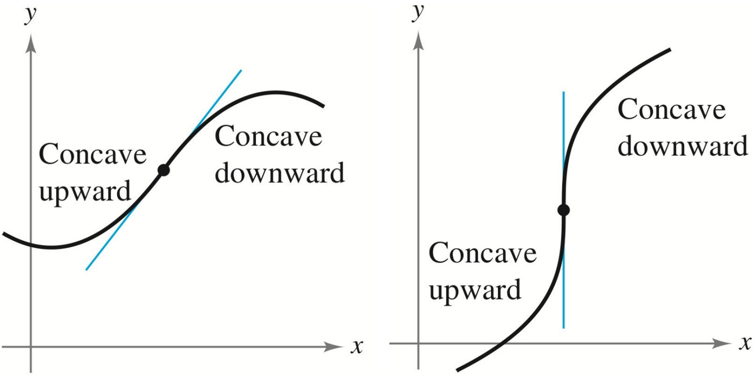
Points of Inflection (反曲點的定義; P.I.)

Let f be conti. on an open interval containing c. If the graph of f

1. has a (vertical) tangent line at $(c, f(c))$, and
2. its concavity changes on both sides of c,

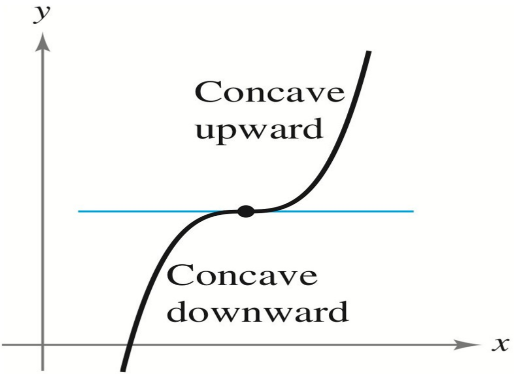
then $(c, f(c))$ is called a point of inflection of the graph of f.
(函數圖形凹性改變的轉折點即為反曲點!)

反曲點的示意圖 (1/2)



y
Concave
upward
Concave
downward
x
y
Concave
downward
Concave
upward
x

反曲點的示意圖 (2/2)



Thm P.I. (反曲點的必要條件)

Suppose that $f''(x)$ exists on an open interval containing c. If $(c, f(c))$ is a point of inflection of the graph of f, then

$$f''(c) = 0 \quad \text{or} \quad f''(c) \nexists.$$

Without loss of generality, we assume that $f''(c) > 0$ $\exists$. Since f'' exists at c, we know that,

$$\lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c).$$

Thus, for $\varepsilon = \frac{f''(c)}{2} > 0$, $\exists\, \delta > 0$ s.t. if $0 < |x - c| < \delta$, then

$$\left| \frac{f'(x) - f'(c)}{x - c} - f''(c) \right| < \frac{f''(c)}{2} \quad \text{or} \quad \frac{f'(x) - f'(c)}{x - c} > \frac{f''(c)}{2} > 0.$$

Then $f'(x) < f'(c)$ for $x \in (c - \delta, c)$ and $f'(c) < f'(x)$ for $x \in (c, c + \delta)$. Thus, it follows from the Def of concavity that the graph of f is C.U. on $I = (c - \delta, c + \delta)$. This contradicts to our assumption that $(c, f(c))$ is a point of inflection of f, and hence we complete the proof.

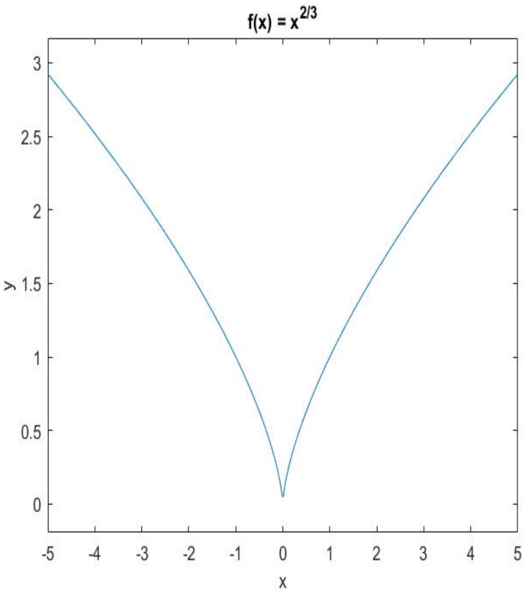
Example (Thm P.I. 的反例)

Consider $f(x) = \sqrt[3]{x^2} = x^{2/3} \quad \forall\, x \in \mathbb{R}$. Then its first and second derivatives are given by

$$f'(x) = \frac{2}{3} x^{-1/3} \quad \text{and} \quad f''(x) = \frac{-2}{9} x^{-4/3} < 0$$

for all $x \neq 0$. In this case, we see that $f''(0) \nexists$ and the origin $(0,0)$ is NOT a point of inflection!

示意圖 (承上例)



Determine the concavity and find the inflection points of

$$f(x) = x^3 - 3x^2 + 2.$$

Since the first and second derivatives of f are given by

$$f'(x) = 3x^2 - 6x, \quad f''(x) = 6x - 6 = 6(x-1),$$

we see that $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.

So, it follows from Thm P.I. that the graph of f is C.D. on $(-\infty, 1)$ and is C.U. on $(1, \infty)$, respectively. Since the concavity of f changes on both sides of $x = 1$, $(1, f(1)) = (1, 0)$ is the only infection point of the graph of f.

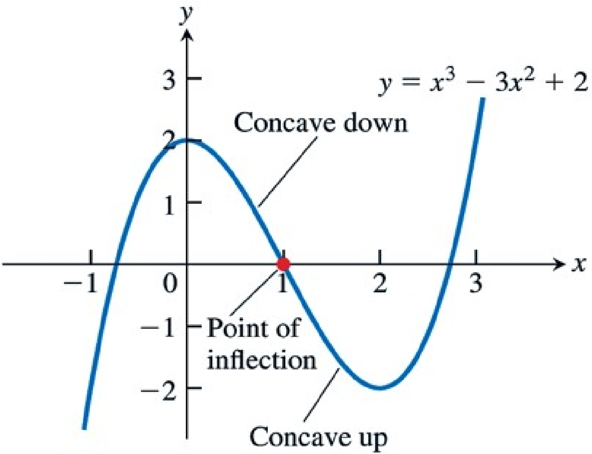


FIGURE 4.27 The concavity of the graph of f changes from concave down to concave up at the inflection point.

Thm 5 (Second Derivative Test; 二階導數測試法)

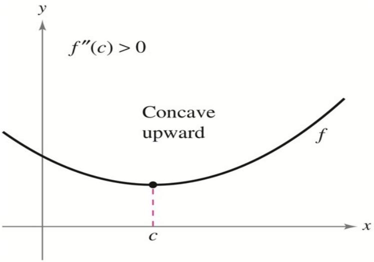
Suppose that $f'(c) = 0$ and f'' $\exists$ on an open interval containing c.

(1) $f''(c) > 0 \Longrightarrow f(c)$ is a local min. value.

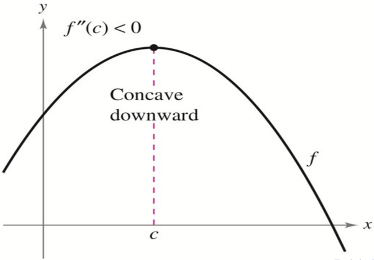
(2) $f''(c) < 0 \Longrightarrow f(c)$ is a local max. value.

(3) $f''(c) = 0 \Longrightarrow$ the test is inconclusive.

示意圖 (承上頁)



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.



Section 4.5
Indeterminate Forms and L'Hôpital's Rule
(不定型與羅必達法則)

Types of Indeterminate Forms

For the limit of a quotient $f(x)/g(x)$ as $x \to c$, we may have

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{0}{0}, \ \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad 1^{\infty}, \ \infty^{0}, \ 0^{0}, \quad \infty - \infty,$$

which are called the indeterminate forms (不定型).

Thm 6 (L'Hôpital's Rule; 羅必達法則)

Suppose that f and g are diff. on an open interval I containing c, and that $g'(x) \neq 0 \quad \forall x \in I \setminus \{c\}$. If the limit satisfies

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty},$$

then we have

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}, \qquad \text{(分子和分母各別微分喔!)}$$

assuming that the limit on the right exists.

Note: Thm 6 also holds for the one-sided limits. (羅必達法則也適用於求解單邊極限值!)

Remark (與 Sec. 2.4 的結果比較)

Some special limits of Section 2.4 can be derived immediately from the L'Hôpital's Rule, since we see that the limits are indeterminate forms of type $\frac{0}{0}$ and hence

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta}{1} = \cos(0) = 1,$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{1} = \sin(0) = 0.$$

Applying the L'Hôpital's Rule, we see that

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2} \quad (\text{Type } \frac{0}{0})$$
$$=\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1+0}=0.$$

(b) Applying the L'Hôpital's Rule, we immediately obtain

$$\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} \quad (\text{Type } \frac{\infty}{\infty})$$
$$= \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$$

(c) Applying the L'Hôpital's Rule twice, we have

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

HW: read the part (a) by yourself.

Indeterminate Forms of Type $0 \cdot \infty$

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = \pm\infty$, then consider

$$\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} \frac{f(x)}{1/g(x)} \quad (\text{Type } \frac{0}{0})$$

or

$$\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} \frac{g(x)}{1/fx)} \quad (\text{Type } \frac{\infty}{\infty}).$$

(b) Applying the L'Hôpital's Rule, we obtain

$$\lim_{x\to 0^+} \sqrt{x}\ln x \quad (\text{Type } 0\cdot\infty) = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \quad (\text{Type } \frac{\infty}{\infty})$$

$$= \lim_{x\to 0^+} \frac{1/x}{(-1/2)x^{-3/2}} = \lim_{x\to 0^+} (-2)x^{1/2} = 0.$$

Indeterminate Forms of Type $\infty - \infty$

The original limit will become an indeterminate form of type $\frac{0}{0}$, if we apply the technique of reduction to common denominator.

(使用通分技巧將原極限問題變成標準不定型!)

Evaluate the following limit

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right),$$

which is an indeterminate form of type $\infty - \infty$.

Applying the L'Hôpital's Rule twice, we see that

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}\quad (\text{Type }\frac{0}{0};\text{通分!})$$
$$=\lim_{x\to 1^+}\frac{1-1/x}{\ln x+(x-1)(1/x)}\quad (\text{使用 L'H Rule!})$$
$$=\lim_{x\to 1^+}\left(\frac{1-1/x}{\ln x+(x-1)(1/x)}\cdot\frac{x}{x}\right)$$
$$=\lim_{x\to 1^+}\frac{x-1}{x\ln x+x-1}\quad (\text{Type }\frac{0}{0})$$
$$=\lim_{x\to 1^+}\frac{1}{\ln x+x(1/x)+1}\quad (\text{再次使用 L'H Rule!})$$
$$=\frac{1}{0+1+1}=\frac{1}{2}.\quad (\text{直接代入求極限喔!})$$

Indeterminate Forms of Type 1^∞, ∞^0 or 0^0

Assume that $\lim_{x \to c} [f(x)]^{g(x)} = 1^\infty, \infty^0, 0^0$. If we know that

$$\lim_{x \to c} g(x) \ln[f(x)] = L \quad (\text{Type } \frac{0}{0} \text{ or } \frac{\infty}{\infty})$$

using the L'Hôpital's Rule, then

$$\lim_{x \to c} [f(x)]^{g(x)} = \lim_{x \to c} e^{g(x) \ln[f(x)]} = e^L.$$

Example 7 (Type 1^∞ 的不定型)

Show that the following limit

$$\lim_{x\to 0^+} (1+x)^{1/x} = e$$

holds, which is an indeterminant form of type 1^∞.

Solution of Example 7

Note that the given limit can be rewritten as

$$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} e^{\ln(1+x)/x}.$$

From the L'Hôpital's Rule, we see that

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = \lim_{x\to 0^+} \frac{1/(1+x)}{1} = \lim_{x\to 0^+} \frac{1}{1+x} = 1 := L,$$

and hence $\lim_{x\to 0^+} (1+x)^{1/x} = e^L = e$.

Evaluate the folloiwng limit

$$\lim_{x\to\infty} x^{1/x},$$

which is an indeterminant form of type ∞^0.

Solution of Example 8

From the L'Hôpital's Rule, we see that

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0 := L.$$

So, we immediately obtain

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\frac{\ln x}{x}} = e^L = e^0 = 1.$$

Example (補充題; Type 0^0 的不定型)

Evaluate the following limit

$$\lim_{x \to 0^+} x^x,$$

which is an indeterminant form of type 0^0.

The given limit can be rewritten as

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}.$$

It follows from the L'Hôpital's Rule that

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} (-x) = 0 := L.$$

Combining above results together, we see that

$$\lim_{x\to 0^+} x^x = e^L = e^0 = 1.$$

Section 4.8
Antiderivatives
(反導函數)

Def (反導函數的定義)

A function F is an antiderivative (反導函數) of f on the interval I if $F'(x) = f(x) \quad \forall x \in I$.

Find an antiderivative for each of the following fuctions.

(a) $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ on $\mathbb{R}$, since $F'(x) = f(x) \quad \forall x \in \mathbb{R}$.

(b) $G(x) = \sin x$ is an antiderivative of $g(x) = \cos x$ on $\mathbb{R}$, since $G'(x) = g(x) \quad \forall x \in \mathbb{R}$.

(c) $H(x) = \ln|x| + e^{2x}$ is an antiderivative of $h(x) = \dfrac{1}{x} + 2e^{2x}$ for $x \neq 0$, since $H'(x) = h(x)$ for $x \neq 0$.

Thm 8 (最廣反導函數的表示式)

If F is an antiderivative of f on the interval I, then

$$G \text{ is an antiderivative of } f \text{ on } I \Longleftrightarrow G(x) = F(x) + C,$$

where C is a constant. In this case, we say that G is the most general antiderivative (最廣反導函數) of f on I.

pf: It follows directly from Cor. 2 of Mean Value Theorem!

Example 2 (計算特定的反導函數)

Find the antiderivative $F(x)$ of the following function

$$f(x) = 3x^2$$

satisfying the (initial) condition $F(1) = -1$.

Since $F_0(x) = x^3$ is an antiderivative of $f(x) = 3x^2$ on $\mathbb{R}$, it follows from Thm 8 that the most general antiderivative of f is

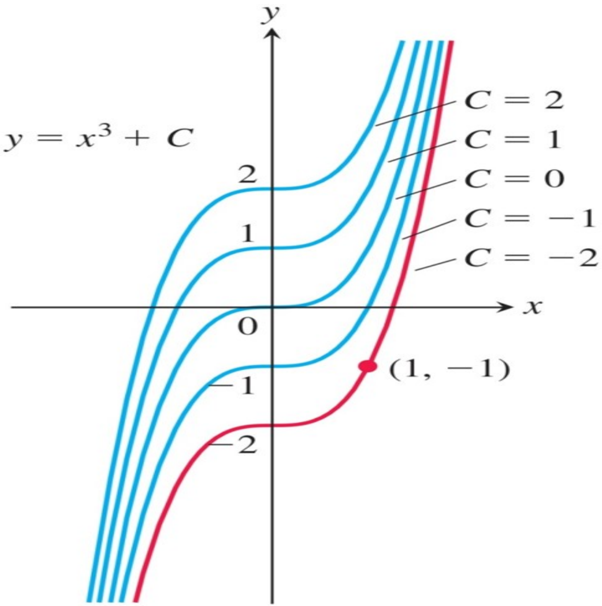
$$F(x) = F_0(x) + C = x^3 + C,$$

where is C is a constant. Moreover, from the initial condition $F(1) = -1$, we see that

$$-1 = F(1) = 1 + C \Longrightarrow C = -2.$$

Thus, the desired antiderivative is $F(x) = x^3 - 2$.

Example 2 的示意圖



Def (不定積分的定義)

Let $F(x)$ be an antiderivative of f on the interval I. The collection of all antiderivatives of f on I is called the indefinite integral of f w.r.t. x (函數 f 對 x 的不定積分), and it is denoted by

$$\int f(x)\,dx = F(x) + C,$$

where f is the integrand (被積函數) of the integral, x is the variable of integration (積分變數) and C is the constant of integration (積分常數).

Remark (積分與微分的關係)

If $F'(x) = f(x) \quad \forall x \in I$, then

$$\int F'(x)\,dx = \int f(x)\,dx = F(x) + C$$

and

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = \frac{d}{dx}\Big[F(x) + C\Big] = F'(x) = f(x),$$

i.e., the integration and differentiation are inverse operations to each other! (積分與微分互為反運算!)

Thm (Basic Integration Rules; 1/4)

Suppose that the antiderivatives of f and g exist and let $0 \neq k \in \mathbb{R}$.

(1) $\displaystyle \int k\,dx = kx + C.$

(2) $\displaystyle \int [k \cdot f(x)]\,dx = k \cdot \left[\int f(x)\,dx\right].$

(3) $\displaystyle \int [f(x) \pm g(x)]\,dx = \left[\int f(x)\,dx\right] \pm \left[\int g(x)\,dx\right].$

(4) $\displaystyle \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$.

Thm (Basic Integration Rules; 2/4)

(5) $$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C.$$

(6) $$\int \sin(kx)\,dx = \frac{-1}{k}cos(kx) + C.$$

(7) $$\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C.$$

(8) $$\int \sec(kx)\tan(kx)\,dx = \frac{1}{k}\sec(kx) + C.$$

(9) $$\int \csc^2(kx)\,dx = \frac{-1}{k}\cot(kx) + C.$$

(10) $$\int \csc(kx)\cot(kx)\,dx = \frac{-1}{k}\csc(kx) + C.$$

Thm (Basic Integration Rules; 3/4)

(11) $$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C.$$

(12) $$\int a^{kx}\,dx = \frac{1}{k(\ln a)}a^{kx} + C \text{ for } 0 < a \neq 1.$$

(13) $$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln|x| + C \text{ for } x \neq 0.$$

Thm (Basic Integration Rules; 4/4)

(14) $$\int \frac{1}{\sqrt{1-k^2x^2}}\,dx = \frac{1}{k}\sin^{-1}(kx) + C.$$

(15) $$\int \frac{1}{1+k^2x^2}\,dx = \frac{1}{k}\tan^{-1}(kx) + C.$$

(16) $$\int \frac{1}{x\sqrt{k^2x^2-1}}\,dx = \sec^{-1}(kx) + C \text{ for } kx > 1.$$

(a) $\int x^5\, dx = \frac{1}{5+1} x^{5+1} + C = \frac{1}{6} x^6 + C.$

(b) $\int \frac{1}{\sqrt{x}}\, dx = \int x^{-1/2}\, dx = \frac{1}{(-1/2)+1} x^{(-1/2)+1} + C =$
$2x^{1/2} + C = 2\sqrt{x} + C.$

(c) $\displaystyle\int \sin(2x)\,dx = \frac{-1}{2}\cos(2x) + C$, $\displaystyle\int \cos(x/2)\,dx = \frac{\sin(x/2)}{1/2} + C = 2\sin(x/2) + C$.

(d) $\displaystyle\int e^{-3x}\,dx = \frac{-1}{3}e^{-3x} + C$, $\displaystyle\int 2^x\,dx = \frac{2^x}{\ln 2} + C$.

Thank you for your attention!