Chapter 5
Integrals
(積分)

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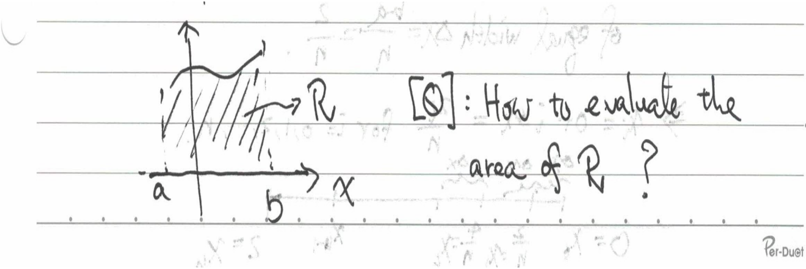
本章預定授課範圍

Section 5.1
Area and Estimating with Finite Sums
(面積與有限和估計)

Let $f \geq 0$ be conti. on $[a, b]$. Consider a plane region defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad 0 \leq y \leq f(x)\}.$$



Lower and Upper Sums (1/2)

Dividing $[a, b]$ into n subintervals

$$[x_0, x_1], \cdots, [x_{i-1}, x_i], \cdots, [x_{n-1}, x_n]$$

of equal width $\Delta x = \dfrac{b-a}{n}$, where $x_i = a + i(\Delta x)$ for $i = 0, 1, \ldots, n$. The endpoints of these subintervals satisfy

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

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$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

For each $1 \le i \le n$, since f is conti. on each $[x_{i-1}, x_i]$, it follows from E.V.T. that $\exists\, m_i, M_i \in [x_{i-1}, x_i]$ s.t.

$$f(m_i) \le f(x) \le f(M_i) \quad \forall\, x \in [x_{i-1}, x_i].$$

(1) Lower sum (下和) approximation:

$$s(n) = s_n = f(m_1)\Delta x + f(m_2)\Delta x + \cdots + f(m_n)\Delta x.$$

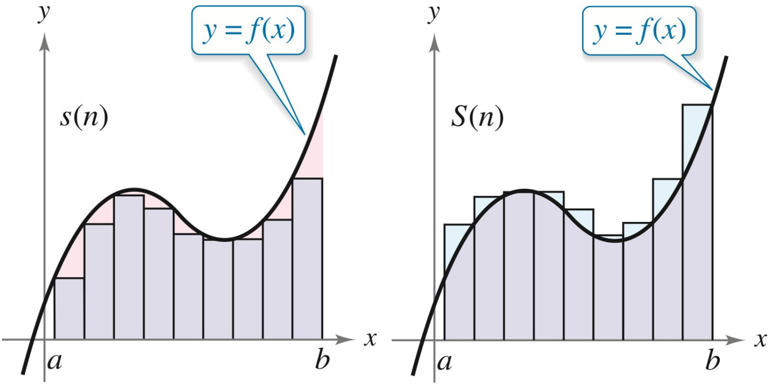
(2) Upper sum (上和) approximation:

$$S(n) = S_n = f(M_1)\Delta x + f(M_2)\Delta x + \cdots + f(M_n)\Delta x.$$

(3) Midpoint sum (中點和) approximation:

$$f\left(\frac{x_0 + x_1}{2}\right)\Delta x + f\left(\frac{x_1 + x_2}{2}\right)\Delta x + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\Delta x.$$

示意圖 (承上頁)



Remark

From the definitions of $s(n)$ and $S(n)$, we see that

$$s(n) \le \text{area}(\mathcal{R}) \le S(n) \quad \forall\, n \in \mathbb{N}.$$

Section 5.2
Sigma Notation and Limits of Finite Sums
(Σ 符號與有限和的極限)

Def (Σ Notation)

The sum of real numbers $a_1, a_2, \ldots, a_n$ is denoted by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n,$$

where i is the index (足碼) of the summation and a_i is the ith term (第 i 項) of the sum.

Thm (Summation Formulas)

(1) $\sum_{i=1}^{n} c = c + c + \cdots + c = cn$ for any constant c.

(2) $\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

(3) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

(4) $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{n^2(n+1)^2}{4}$.

Thm (下和與上和的極限值)

If $f \geq 0$ is conti. on $[a, b]$, then

$$\lim_{n\to\infty} s(n) = A = \lim_{n\to\infty} S(n) \quad \exists.$$

In this case, we know that area$(\mathcal{R}) = A \quad \exists$ by the Sandwich Thm.

Def (非負連續函數與 x-軸所夾面積)

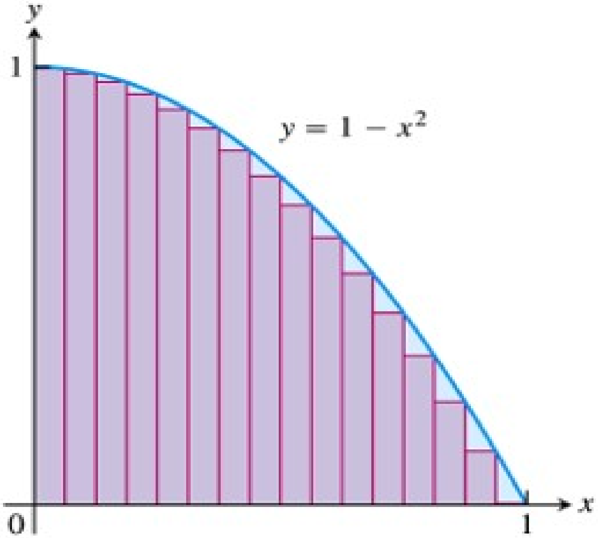
If $f \geq 0$ is conti. on $[a, b]$, then the area of the region $\mathcal{R}$ bounded by the graph of f, the x-axis, $x = a$ and $x = b$ is

$$A = \text{area}(\mathcal{R}) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$ and $\Delta x = \dfrac{b-a}{n}$.

Find the limiting value of lower sum approximations to the area of the plane region given by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le 1 - x^2\}.$$



(a)

Dividing $I = [0, 1]$ into n subintervals of equal width $\Delta x = \frac{1}{n}$ with the endpoints $x_i = 0 + i\Delta x = \frac{i}{n}$ for $i = 0, 1, 2, \ldots, n$. Since the graph of $f(x) = 1 - x^2$ is decreasing on I, $f(m_i) = f(\frac{i}{n}) = 1 - \frac{i^2}{n^2}$ is the min. value on the ith subinterval for $i = 1, 2, \ldots, n$.

So, the lower sum approximation to the area of $\mathcal{R}$ is

$$s(n) = \sum_{i=1}^{n} f(m_i)\Delta x = \Big(\sum_{i=1}^{n} \frac{1}{n}\Big) - \frac{1}{n^3}\Big(\sum_{i=1}^{n} i^2\Big) = 1 - \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6},$$

and hence its limiting value is given by

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \Big[1 - \frac{1}{6}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big)\Big] = 1 - \frac{2}{6} = \frac{2}{3}.$$

Def (黎曼和的定義)

Let f be defined on a closed interval $I = [a, b]$.

(1) The set $\Delta = \{x_0, x_1, \cdots, x_{n-1}, x_n\}$ is called a partition (分割) of I if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

(2) The width of the ith subinterval $[x_{i-1}, x_i]$ is $\Delta x_i = x_i - x_{i-1}$ for each $i = 1, 2, \ldots, n$.

(3) The norm (範數) of a partition Δ is defined by $\|\Delta\| = \max_{1 \le i \le n} \Delta x_i$.

(4) If $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann sum (黎曼和) of f for the partition Δ.

Remarks

(1) For a general partition Δ of $[a, b]$, we see that

$$\frac{b-a}{n} \leq \|\Delta\| \Longrightarrow \frac{b-a}{\|\Delta\|} \leq n.$$

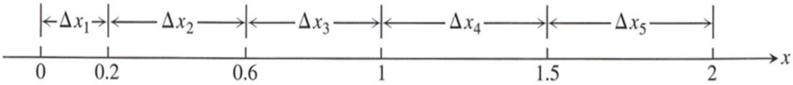
So, $\|\Delta\| \to 0 \Rightarrow n \to \infty$, but $n \to \infty \not\Rightarrow \|\Delta\| \to 0$!

(2) If $\Delta x_i = \frac{b-a}{n} \quad \forall\, i$, then $\|\Delta\| \to 0 \Longleftrightarrow n \to \infty$.

For the partition $\Delta = \{x_0, x_1, x_2, x_3, x_4, x_5\} = \{0, 0.2, 0.6, 1, 1.5, 2\}$ of the interval $[0, 2]$, the norm of Δ is given by

$$\|\Delta\| = \max_{1 \le i \le 5} \Delta x_i = \max\{0.2, 0.4, 0.4, 0.5, 0.5\} = 0.5,$$

where there are two subintervals of this length.



Section 5.3
The Definite Integral
(定積分)

Def (定積分的定義)

Let f be defined on a closed interval $I = [a, b]$.

(1) If the limit $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f(c_i)\Delta x_i$ $\exists$ for any partition Δ of I and any $c_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, then f is integrable (可積分的) over (or on) I. In this case, the limit

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called the definite integral (定積分) of f from a to b.

(2) The number a is called the lower limit of integration (積分下限) and b is called the upper limit of integration (積分上限).

Remarks

(1) $\displaystyle\int f(x)\,dx = F(x) + C$ denotes a family of functions, but $\displaystyle\int_a^b f(x)\,dx$ is a real number.

(2) In general, the following notations

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(w)\,dw = \cdots$$

denote the same definite integral of f from a to b.

Thm 1 (定積分的存在性)

If f is conti. on $I = [a, b]$, then f is integrable over I, i.e., the definite integral of f from a to b

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \quad \exists$$

for any partition Δ of I and any $c_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$.

General Version of Thm 1

f has at most finitely many jump discontinuities on $[a, b] \Longrightarrow f$ is (Riemann) integrable over $[a, b]$.

(函數 f 在 $[a, b]$ 上最多僅有限個跳躍不連續點 $\Longrightarrow f$ 在 $[a, b]$ 上必定是一個黎曼可積的函數!)

Example 1 (不可積函數的例子)

The real-valued function defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

is NOT integrable over $[0, 1]$, since the lower and upper sum approximations have different limits, i.e.,

$$\lim_{\|\Delta\| \to 0} s(n) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(m_i)\Delta x_i = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (0)\Delta x_i = 0,$$

$$\lim_{\|\Delta\| \to 0} S(n) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(M_i)\Delta x_i = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (1)\Delta x_i = 1.$$

Def (Two Special Definite Integrals)

(1) $\int_a^a f(x)\,dx = 0$ for any $a \in \mathbb{R}$.

(2) If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Thm (定積分的性質)

Suppose that f and g are integrable on $[a,b]$, and let $k \in \mathbb{R}$.

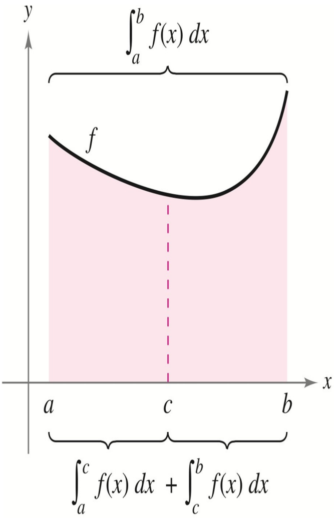
(1) $\displaystyle\int_a^b [k \cdot f(x)]\,dx = k \cdot \left(\int_a^b f(x)\,dx\right).$

(2) $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right).$

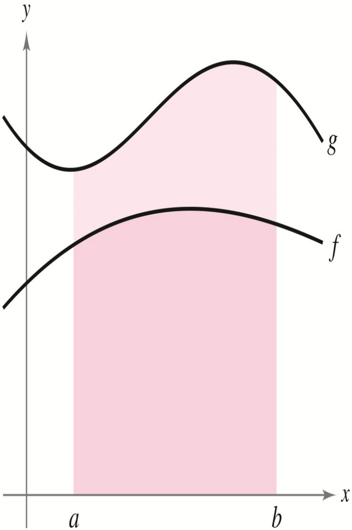
(3) Additivity (可加性): $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for $a < c < b$.

(4) Preservation of Inequality (不等號的維持):

$$f(x) \le g(x) \quad \forall\, x \in [a,b] \Longrightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$



y
$\int_a^b f(x)\,dx$
f
x
a
c
b
$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$



y
g
f
x
a
b

Note

In general, we know that

$$\int_a^b [f(x)g(x)]\,dx \neq \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right) \quad \text{and}$$

$$\int_a^b \frac{f(x)}{g(x)}\,dx \neq \frac{\int_a^b f(x)\,dx}{\int_a^b g(x)\,dx}.$$

Suppose that f and h are integrable functions satisfying

$$\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \int_{-1}^{1} h(x)\,dx = 7.$$

(a) $\displaystyle\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2.$

(b) $\displaystyle\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\left(\int_{-1}^{1} f(x)\,dx\right) + 3\left(\int_{-1}^{1} h(x)\,dx\right) = 2(5) + 3(7) = 31.$

(c) $\displaystyle\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3.$

Def (非負可積函數與區間 $[a, b]$ 所圍區域的面積)

If $f \geq 0$ is integrable over $[a, b]$, then the area of the region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$$

or the area under the curve $y = f(x)$ over $[a, b]$ is given by

$$A = \text{area}(\mathcal{R}) = \int_a^b f(x)\, dx \geq 0.$$

Find the area under the following curve

$$y = f(x) = x$$

over the closed interval $[0, b]$ with $b > 0$.

Example 4 的示意圖

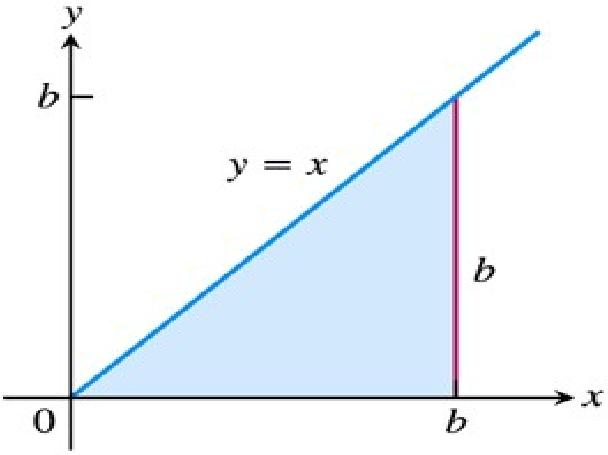


FIGURE 5.12 The region in Example 4 is a triangle.

Solution of Example 4

Since $f \geq 0$ is conti. on $[0, b]$, the area under $y = f(x)$ over $[0, b]$ is $A = \int_0^b f(x)\,dx = \int_0^b x\,dx$.

Method 1 From the upper sum approximation with $\Delta x = b/n$, we obtain

$$A = \int_0^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} (\frac{ib}{n})(\frac{b}{n})$$

$$= \lim_{n\to\infty} \frac{b^2}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \left[\frac{b^2}{n^2} \cdot \frac{n(n+1)}{2}\right] = \frac{b^2}{2}.$$

Method 2 $A = \int_0^b x\,dx = \frac{1}{2}(b)(b) = \frac{b^2}{2}$. (直角三角形的面積)

Def (函數在閉區間上的平均值)

If f is conti. on $I = [a, b]$, then the average value of f on I is given by

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Example 5 (計算函數在閉區間的平均值)

Find the average value of the real-valued function

$$f(x) = \sqrt{4 - x^2}$$

on the closed interval $[-2, 2]$.

Solution of Exmaple 5

Since $f \geq 0$ is conti. on $[-2, 2]$, it follows that

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} \sqrt{4 - x^2}\,dx = \frac{\pi}{2}(2)^2 = 2\pi. \quad \text{(半圓的面積)}$$

So, the average vale of f on $[-2, 2]$ is given by

$$\text{av}(f) = \frac{1}{2 - (-2)} \int_{-2}^{2} f(x)\,dx = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

Example 5 的示意圖

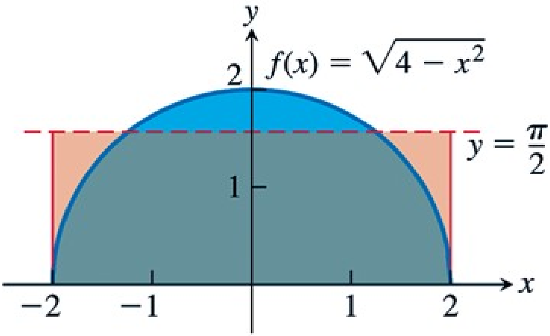


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5). The area of the rectangle shown here is $4 \cdot (\pi/2) = 2\pi$, which is also the area of the semicircle.

Section 5.4
The Fundamental Theorem of Calculus
(微積分基本定理; F.T.C.)

Thm 3 (定積分的均值定理)

If f is conti. on $I = [a, b]$, then $\exists\, c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \quad \text{or} \quad f(c)\cdot(b-a) = \int_a^b f(x)\,dx.$$

Proof of Thm 3

Since f is conti. on $I = [a, b]$, it follows that $\int_a^b f(x)\,dx \; \exists$ and $\exists\, m, M \in I$ s.t. $f(m) \le f(x) \le f(M) \quad \forall\, x \in I$. We thus obtain

$$f(m)(b-a) = \int_a^b f(m)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(M)\,dx = f(M)(b-a)$$

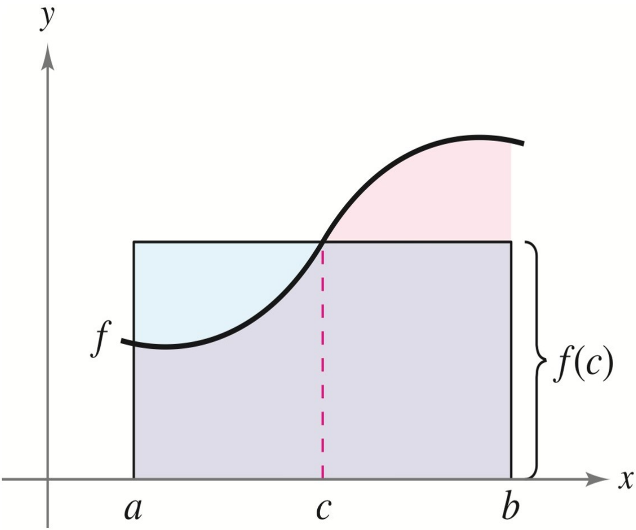
or, equivalently, we see that

$$f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M).$$

Now, from I.V.T., $\exists\, c \in [a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \equiv \mathrm{av}(f).$$

示意圖 (承上頁)



The Definite Integral as a Function

If f is conti. on an open interval I containing a, define a real-valued function by

$$F(x) = \int_a^x f(t)\,dt \qquad \forall\, x \in I.$$

Some Questions

- Is F differentiable on the open interval I?
- If yes, what is the derivative of F?
- Furthermore, is F an antiderivative of f on I?

Thm 4 (The F.T.C., Part 1)

If f is conti. on $I = [a, b]$ and define a real-valued function by

$$F(x) = \int_a^x f(t)\,dt \qquad \forall\, x \in I,$$

then F is conti. on I and diff. on (a, b) with the derivative

$$F'(x) = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \qquad \forall\, x \in (a, b),$$

i.e. F is an antiderivative of f on the open interval (a, b).

Note: we will show that $F'(x) = f(x) \quad \forall\, x \in [a, b]$.

For any $x \in (a, b)$, we shall prove that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = f(x).$$

By M.V.T., $\exists\, c_1 \in [x, x+h]$ s.t. $f(c_1) = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$ for $h > 0$,
and $\exists\, c_2 \in [x+h, x]$ s.t. $f(c_2) = \dfrac{1}{-h}\displaystyle\int_{x+h}^{x} f(t)\,dt$ for $h < 0$. Thus,

$$\lim_{h\to 0^+} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0^+} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0^+} f(c_1) = f(x)$$

and

$$\lim_{h\to 0^-} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0^-} \frac{\int_{x+h}^{x} f(t)\,dt}{-h} = \lim_{h\to 0^-} f(c_2) = f(x),$$

since f is conti. at $x \in (a, b)$.

Applying the similar arguments, the one-sided derivatives at the endpoints a and b are given by

$$F'_+(a) = \lim_{h\to 0^+} \frac{\int_a^{a+h} f(t)\,dt - \int_a^a f(t)\,dt}{h} = f(a),$$
$$F'_-(b) = \lim_{h\to 0^-} \frac{\int_a^{b+h} f(t)\,dt - \int_a^b f(t)\,dt}{h} = f(b).$$

Since differentiability implies continuity, it follows that F is also conti. at $x = a$ and $x = b$, respectively. Thus, we conclude that F is conti. on $I = [a, b]$ and diff. on (a, b). This completes the proof.

Remark (Thm 4–Part 1 的變形版本)

If f is conti. on an open interval I containing a, then it follows from Thm 4 that

$$\frac{d}{dx}\left[\int_x^a f(t)\,dt\right] = \frac{d}{dx}\left[-\int_a^x f(t)\,dt\right] = -f(x)$$

for all $x \in I$. (積分上下限顛倒會差了一個負號喔!)

General Version of Thm 4–Part 1

If f is conti. on an open interval I containing a, and $u(x), v(x)$ are diff. functions of x with $\text{range}(u), \text{range}(v) \subseteq I$, then

(1) $$\frac{d}{dx}\left[\int_a^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x).$$

(2) $$\frac{d}{dx}\left[\int_{v(x)}^{u(x)} f(t)\,dt\right] = f(u(x)) \cdot u'(x) - f(v(x)) \cdot v'(x).$$

From the F.T.C., we see that

(a) $$\frac{d}{dx}\Big[\int_a^x (t^3+1)\,dt\Big] = x^3+1.$$

(b) $$\frac{d}{dx}\Big[\int_x^5 (3t\sin t)\,dt\Big] = \frac{d}{dx}\Big[-\int_5^x (3t\sin t)\,dt\Big] = -3x\sin x.$$

(c) $\dfrac{d}{dx}\Big[\displaystyle\int_1^{x^2} \cos t\,dt\Big] = (\cos x^2)(x^2)' = 2x\cos x^2.$

(d) $\dfrac{d}{dx}\Big(\displaystyle\int_{1+3x^2}^{4} \frac{1}{2+e^t}\,dt\Big) = \dfrac{d}{dx}\Big(-\displaystyle\int_{4}^{1+3x^2} \frac{1}{2+e^t}\,dt\Big) = \dfrac{-1}{2+e^{(1+3x^2)}}\cdot(1+3x^2)' = \dfrac{-6x}{2+e^{(1+3x^2)}}.$

The Evaluation Theorem for Definite Integrals

Thm 4 (The F.T.C., Part 2)

If f is conti. on $I = [a, b]$ and $F'(x) = f(x) \quad \forall\, x \in I$, then

$$\int_a^b f(x)\, dx = F(b) - F(a) := F(x)\Big|_a^b \text{ or } F(x)\Big]_a^b.$$

Note: the definite integral of f from a to b can be evaluated by the function values of antiderivative $F(a)$ and $F(b)$ directly!

Proof of Thm 4–Part 2

From Thm 4–part 1, we see that the function defined by

$$G(x) := \int_a^x f(t)\,dt$$

is an antiderivative of f on I. Thus, $F(x) = G(x) + C$ for some constant C. Then it follows immediately that

$$\begin{aligned} F(b) - F(a) &= [G(b) + C] - [G(a) + C] = G(b) - G(a) \\ &= G(b) - 0 = G(b) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx. \end{aligned}$$

Example 3 (Thm 4–Part 2 的例子; 1/2)

(a) $\displaystyle\int_0^{\pi} \cos x\,dx = \sin x\Big|_0^{\pi} = \sin\pi - \sin 0 = 0.$

(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\,dx = \sec x\Big|_{-\pi/4}^{0} = 1 - \sqrt{2}.$

(c) $\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left(x^{3/2} + \frac{4}{x}\right)\Big|_1^4 = (8+1) - (1+4) = 4.$

(d) $\int_0^1 \frac{1}{x+1}\, dx = \ln|x+1|\Big|_0^1 = \ln 2 - \ln 1 = \ln 2.$

(e) $\int_0^1 \frac{1}{x^2+1}\, dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}.$

Thm (函數曲線與 x-軸所圍區域的面積)

If f has zeros (零根) at the points $c_1, c_2, \ldots, c_k$ for some $k \in \mathbb{N}$, then the area of the region between $y = f(x)$ and the x-axis is

$$A = \left| \int_{c_1}^{c_2} f(x)\, dx \right| + \left| \int_{c_2}^{c_3} f(x)\, dx \right| + \cdots + \left| \int_{c_{k-1}}^{c_k} f(x)\, dx \right|.$$

Example 8 (計算函數圖形所圍區域的面積)

Find the area of the region between the x-axis and the graph of

$$y = f(x) = x^3 - x^2 - 2x$$

for $-1 \leq x \leq 2$.

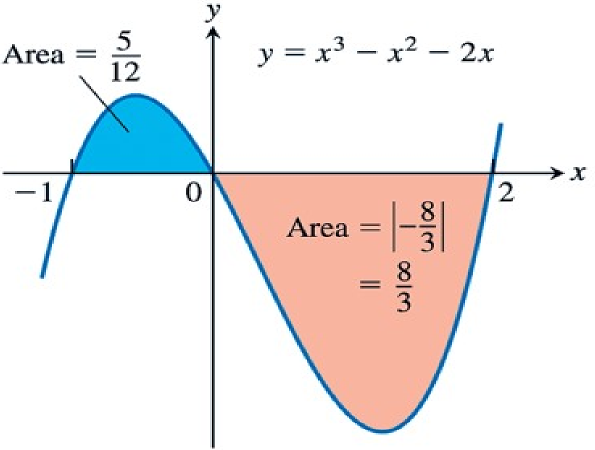


FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

Solution of Example 8

We first notice that

$$f(x) = x(x^2 - x - 2) = x(x+1)(x-2) = 0 \Longrightarrow x = -1, 0, 2.$$

Since $f(x) \geq 0$ for $-1 \leq x \leq 0$ and $f(x) \leq 0$ for $0 \leq x \leq 2$, the area of the region between the graph of $y = f(x)$ and the x-axis is

$$\begin{aligned} A &= \left| \int_{-1}^{0} (x^3 - x^2 - 2x)\, dx \right| + \left| \int_{0}^{2} (x^3 - x^2 - 2x)\, dx \right| \\ &= \left| \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_{-1}^{0} \right| + \left| \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \Big|_{0}^{2} \right| \\ &= \left| \frac{5}{12} \right| + \left| \frac{-8}{3} \right| = \frac{37}{12}. \end{aligned}$$

Section 5.5
Indefinite Integrals and the Substitution Method
(不定積分與代換法)

Main Goal

Applying a substitution $u = g(x)$ to deal with the integral

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du,$$

and its associated definite integrals.

Thm 6 (合成函數的反導函數)

Let $g : D \to I$ be a function with range$(g) = I$ being an interval, and let f be conti. on I. If g is diff. on D and $F'(x) = f(x) \quad \forall\, x \in I$, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C,$$

where C is a constant of integration.

Remark (The method of u-substitution; u-代換法)

Let $u = g(x)$. Then $du = g'(x)\,dx$ and it follows from Thm 6 that

$$\int f(u)\,du = \int f(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C,$$

where C is a constant of integration.

Note: in general, the indefinite integral $\int f(u)\,du$ is easily evaluated if a "good" substitution $u = g(x)$ is chosen.

Example 4 (Thm 6 的例子)

Find the following indefinite integral

$$\int \cos(7\theta + 3)\, d\theta$$

using the u-substitution method.

Solution of Example 4

Let $u = g(x) = 7\theta + 3$. Then $du = 7\,d\theta$ and hence we have

$$\int \cos(7\theta + 3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\cdot 7\,d\theta = \frac{1}{7}\int \cos u\,du$$
$$= \frac{1}{7}\sin u + C = \frac{\sin(7\theta+3)}{7} + C,$$

where C is a constant of integration.

Thm (General Power Rule for Integration)

If $u = g(x)$ is a diff. function of x and $n \neq -1$, then

$$\int [g(x)]^n g'(x)\, dx = \int u^n\, du = \frac{u^{n+1}}{n+1} + C = \frac{[g(x)]^{n+1}}{n+1} + C,$$

where C is a constant of integration.

Find the following indefinite integral

$$\int (x^3 + x)^5 (3x^2 + 1)\, dx$$

using the u-substitution method.

Let $u = g(x) = x^3 + x$. Then $du = (3x^2 + 1)\,dx$ and hence

$$\int (x^3 + x)^5 (3x^2 + 1)\,dx = \int u^5\,du = \frac{u^{5+1}}{5+1} + C$$
$$= \frac{u^6}{6} + C = \frac{(x^3 + x)^6}{6} + C,$$

where C is a constant of integration.

Example 6 (與課本解法不同)

Evaluate the following indefinite integral

$$\int x\sqrt{2x+1}\,dx$$

using the u-substitution method.

Solution of Example 6

Let $u = \sqrt{2x+1}$. Then $du = \dfrac{1}{\sqrt{2x+1}}\,dx = \dfrac{1}{u}\,dx$ and $x = \dfrac{u^2-1}{2}$.

Thus, we see that

$$\int x\sqrt{2x+1}\,dx = \int \left(\frac{u^2-1}{2}\right)u^2\,du = \int \left(\frac{u^4-u^2}{2}\right)du$$
$$= \frac{u^5}{10} - \frac{u^3}{6} + C = \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + C,$$

where C is a constant of integration.

Example 7 (利用三角函數等式求積分)

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

(c) $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C = \ln|\sec x| + C.$$

Example 8 (特殊 u-代換法的例子)

(a) If we let $u = e^x$, then $du = e^x\,dx$ and hence

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{(e^x)^2 + 1} = \int \frac{1}{u^2+1}\,du$$
$$= \tan^{-1} u + C = \tan^{-1}(e^x) + C.$$

(b) Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x)\,dx$ and thus we obtain

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\,dx$$
$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C.$$

Thm (三角函數的積分公式)

Let $u = u(x)$ be a diff. function of x. Then

(1) $\displaystyle\int \sin u\, du = -\cos u + C,$

(2) $\displaystyle\int \cos u\, du = \sin u + C,$

(3) $\displaystyle\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C,$

(4) $\displaystyle\int \cot u\, du = \ln|\sin u| + C,$

(5) $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C,$

(6) $\displaystyle\int \csc u\, du = -\ln|\csc u + \cot u| + C,$

where C is a constant of integration.

Section 5.6
Definite Integral Substitutions and the Area Between Curves
(定積分的代換法與曲線間的面積)

Thm 7 (定積分的 u-代換法)

If $u = g(x)$ has a conti. derivative on $I = [a, b]$ and f is conti. on range(g), then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

(將對 x 的定積分變數變換成對 u 的定積分!)

(a) Let $u = \cot x$. Then $du = -\csc^2 x\,dx$, $\cot(\pi/4) = 1$, $\cot(\pi/2) = 0$ and thus we see that

$$\int_{\pi/4}^{\pi/2} \cot x \csc^2 x\,dx = \int_1^0 u\,(-du) = \frac{-u^2}{2}\Big|_1^0 = \frac{1}{2}.$$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x\,dx = \ln|\sec x|\Big|_{-\pi/4}^{\pi/4} = 0.$ Note that the integrand $f(x) = \tan x$ is an odd function.

Thm 8 (奇偶函數在 $[-a, a]$ 上的定積分)

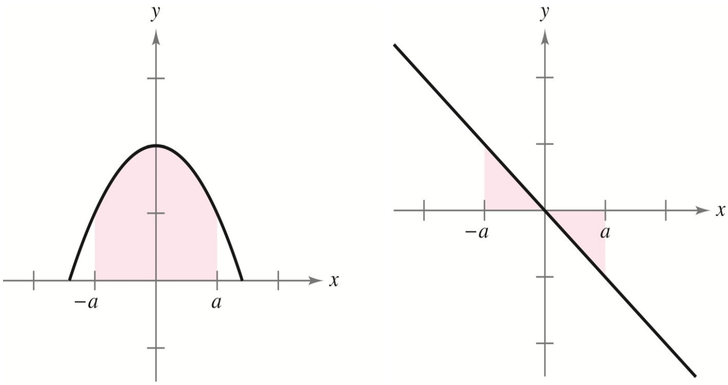
Let f be integrable on the closed interval $I = [-a, a]$ with $a > 0$.

(1) If f is even, i.e., $f(-x) = f(x) \quad \forall\, x \in I$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(2) If f is odd, i.e., $f(-x) = -f(x) \quad \forall\, x \in I$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

Thm 8 的示意圖 (承上頁)



(1) Since $f(x) = f(-x) \quad \forall\, x \in I$, itl follows from Thm 7 that

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{-a}^{0} f(-x)(-1)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad (\text{let } u = -x) \\ &= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.\end{aligned}$$

(2) Since $f(x) = -f(-x) \quad \forall\, x \in I$, it follows from Thm 7 that

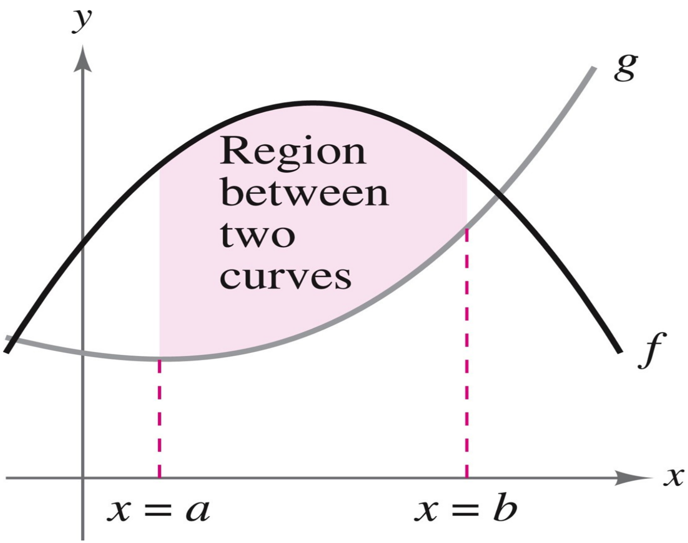
$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{-a}^{0} f(-x)(-1)\,dx + \int_{0}^{a} f(x)\,dx \\ &= \int_{a}^{0} f(u)\,du + \int_{0}^{a} f(x)\,dx \qquad (\text{let } u = -x) \\ &= -\int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx = 0.\end{aligned}$$

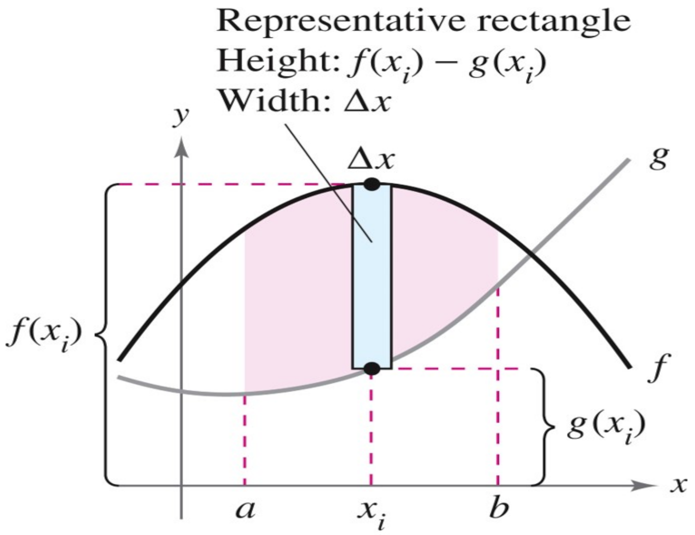
Example 3 (Thm 8 的例子)

Note that $f(x) = x^4 - 4x^2 + 6$ is an even function because $f(-x) = f(x) \quad \forall\, x \in \mathbb{R}$. Therefore, it follows from Thm 8 that

$$\int_{-2}^{2} f(x)\,dx = 2\int_{0}^{2} f(x)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\Big(\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\Big)\Big|_0^2 = \frac{232}{15}.$$

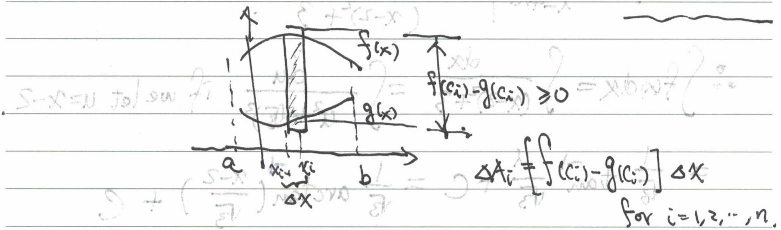

y
g
Region
between
two
curves
f
x
x = a
x = b


Representative rectangle
Height: $f(x_i) - g(x_i)$
Width: Δx
y
Δx
g
$f(x_i)$
f
$g(x_i)$
x
a
x_i
b

Type I (第一型面積公式)

If f and g are conti. on $I = [a, b]$ with $g(x) \leq f(x) \quad \forall x \in I$, then the area of $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g(x) \leq y \leq f(x)\}$ is given by

$$A = \text{area}(\mathcal{R}) = \lim_{n\to\infty} \sum_{i=1}^{n} [f(c_i) - g(c_i)]\Delta x = \int_a^b [f(x) - g(x)]\, dx \geq 0.$$



Find the area of the region in the first quadrant (第一象限) that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

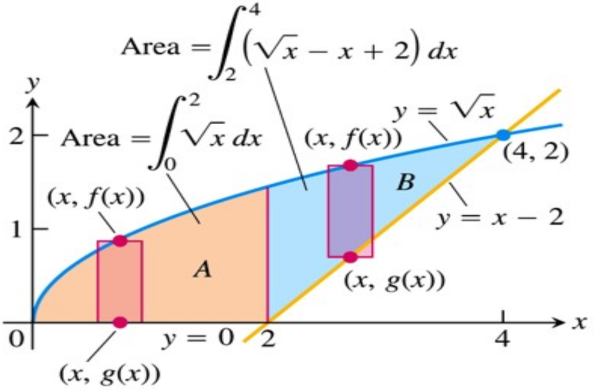


FIGURE 5.30 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

Solution of Example 6

Note that the curves $y = \sqrt{x}$ and $y = x - 2$ intersect at the point $(4, 2)$ in the first quadrant, since

$$\sqrt{x} = x - 2 \Longrightarrow x^2 - 5x + 4 = (x-1)(x-4) = 0 \Longrightarrow x = 4.$$

So, the area of the region is given by

$$\begin{aligned} A &= \int_0^2 \sqrt{x}\,dx + \int_2^4 (\sqrt{x} - x + 2)\,dx \\ &= \left(\frac{2}{3}x^{3/2}\right)\Big|_0^2 + \left(\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right)\Big|_2^4 \\ &= \cdots = \frac{10}{3}. \end{aligned}$$

Type II (第二型面積公式)

If f and g are conti. on $I = [c, d]$ with $g(y) \leq f(y) \quad \forall\, y \in I$, then the area of the region between the graphs of $x = f(y)$ and $x = g(y)$

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, g(y) \leq x \leq f(y)\}$$

is given by

$$A = \text{area}(\mathcal{R}) = \int_c^d [f(y) - g(y)]\, dy \geq 0.$$

Find the area of the region in Example 6 by integrating w.r.t. y.

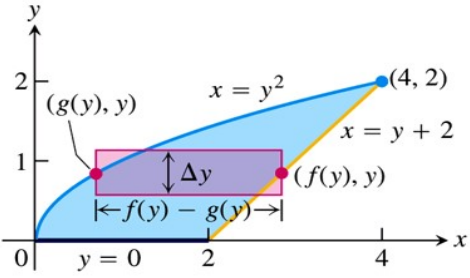


FIGURE 5.31 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 7).

Solution of Example 7

Since $x = f(y) = y + 2$ and $x = g(y) = y^2$ are conti. on the interval $I = [0, 2]$ in the y-axis, and $g(y) \leq f(y) \quad \forall\, y \in I$, it follows that

$$A = \int_0^2 (y + 2 - y^2)\, dy = \left(\frac{y^2}{2} + 2y - \frac{y^3}{3}\right)\Big|_0^2 = \frac{10}{3}.$$

Thank you for your attention!