Chapter 6
Applications of Definite Integrals
(定積分的應用)

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Fall 2022

本章預定授課範圍

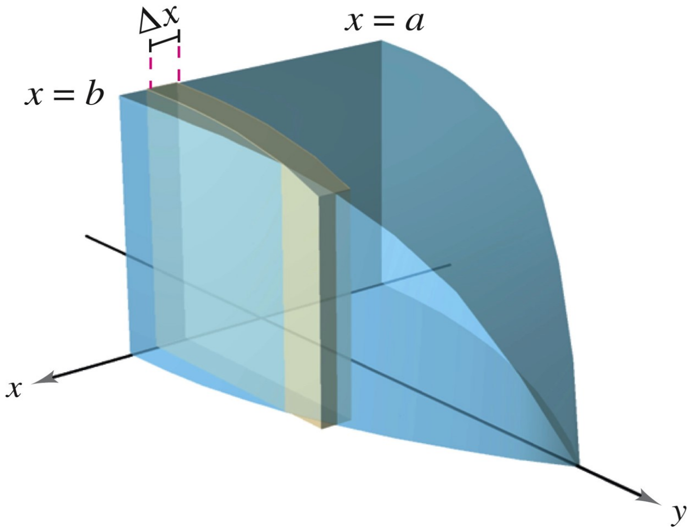
Section 6.1
Volumes Using Cross-Sections
(利用橫截面求體積)

Def (已知截面積的固體體積)

Let $A(x)$ be the cross-sectional area (截面積) of a solid taken perpendicular to (垂直於) the x-axis at each $x \in [a, b]$. If $A(x)$ is integrable on $[a, b]$, then the volume of the solid is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(c_i)\Delta x = \int_a^b A(x)\, dx \geq 0.$$

示意圖 (承上頁)



Find the volume of the pyramid (金字塔), where each cross-section of the pyramid perpendicular to the x-axis at $x \in [0, 3]$ is a square x meters on a side.

Example 1 的示意圖

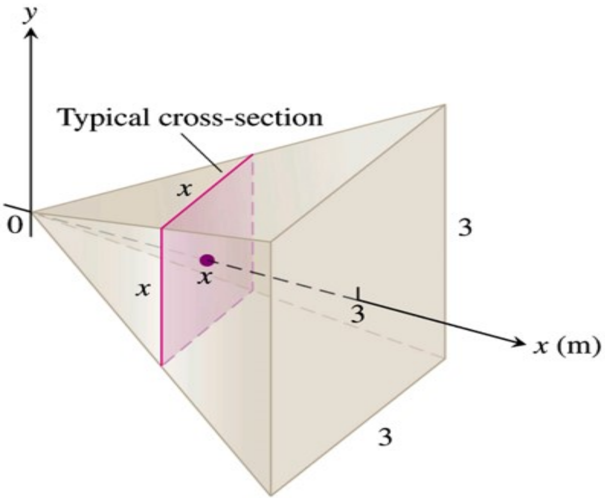


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

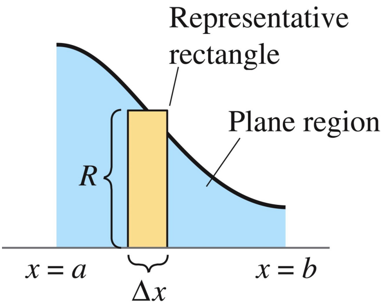
Solution of Example 1

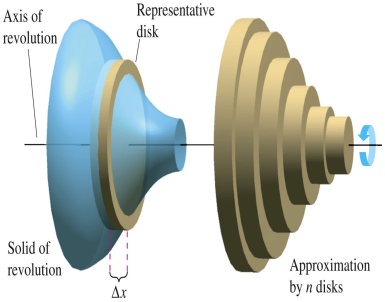
Since the cross-sectional area of the pyramid is

$$A(x) = x^2 \quad \forall\, x \in [0, 3],$$

the volume of the pyramid is given by

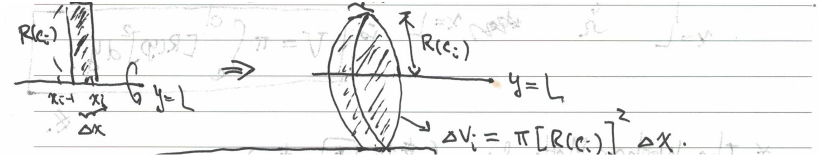
$$V = \int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = 9\ (m^3).$$


Representative
rectangle
Plane region
R
$x = a$
Δx
$x = b$


Axis of
revolution
Representative
disk
Solid of
revolution
Δx
Approximation
by n disks

The Volume of a Representative Disk

- Given a partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with equal width $\Delta x = \dfrac{b-a}{n}$, where $x_0 = a$ and $x_n = b$.
- For each $i = 1, 2, \ldots, n$, choose $c_i \in [x_{i-1}, x_i]$, the volume of a disk of width Δx is $\Delta V_i = \pi[R(c_i)]^2 \Delta x$.



Type I: Horizontal Axis of Revolution (水平旋轉軸)

Given the plane region Ω and a horizontal axis of revolution $y = L$. If the radius function $R(x) \geq 0$ is conti. on $[a, b]$, then the volume of a solid formed by rotating (or revolving) Ω about $y = L$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi [R(c_i)]^2 \Delta x = \int_a^b \pi [R(x)]^2 \, dx \geq 0.$$

Example 5 (Disk Method 的 Type I 例子)

Find the volume of a sphere generated by rotating the circle

$$x^2 + y^2 = a^2$$

of radius $a > 0$ about the x-axis ($y = 0$).

Example 5 的示意圖

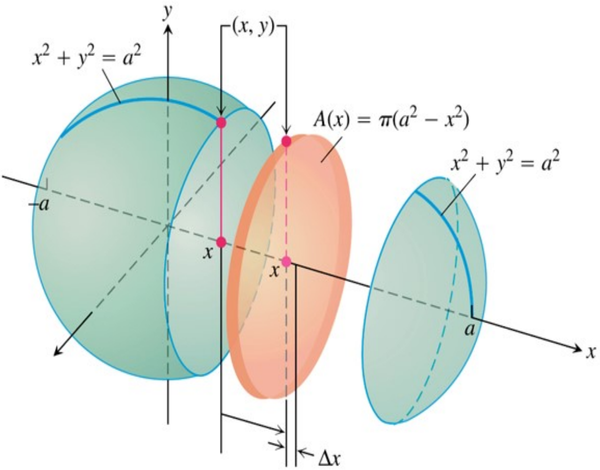


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

Solution of Example 5

Let $R(x) = \sqrt{a^2 - x^2} \quad \forall\, x \in [-a, a]$. Since the sphere is generated by rotating the region

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid -a \le x \le a, 0 \le y \le R(x)\}$$

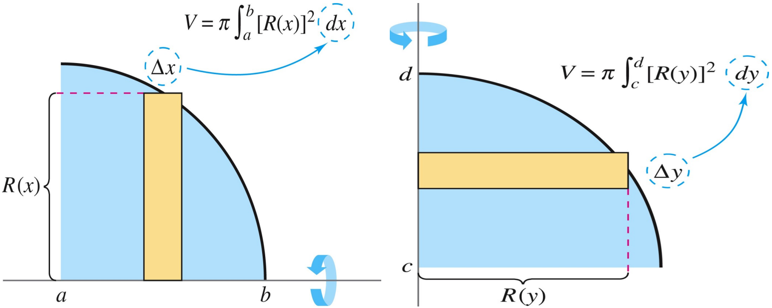
about the x-axis, its volume is given by

$$V = \int_{-a}^{a} \pi [R(x)]^2 \, dx = \int_{-a}^{a} \pi (a^2 - x^2) \, dx$$
$$= \pi \left(a^2 x - \frac{x^3}{3} \right) \Big|_{-a}^{a} = \pi \left(\frac{2}{3} a^3 + \frac{2}{3} a^3 \right) = \frac{4}{3} \pi a^3.$$

Type II: Vertical Axis of Revolution (垂直旋轉軸)

Given the plane region Ω and a vertical axis of revolution $x = L$. If the radius function $R(y) \geq 0$ is conti. on $[c, d]$, then the volume of a solid formed by revolving Ω about $x = L$ is

$$V = \int_c^d \pi [R(y)]^2 \, dy \geq 0.$$


$V = \pi \int_a^b [R(x)]^2 dx$
Δx
$R(x)$
a
b
$V = \pi \int_c^d [R(y)]^2 dy$
Δy
d
c
$R(y)$

Find the volume of the solid generated by revolving the region

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 1 \leq y \leq 4, 0 \leq x \leq 2/y\}$$

about the y-axis ($x = 0$).

Example 7 的示意圖

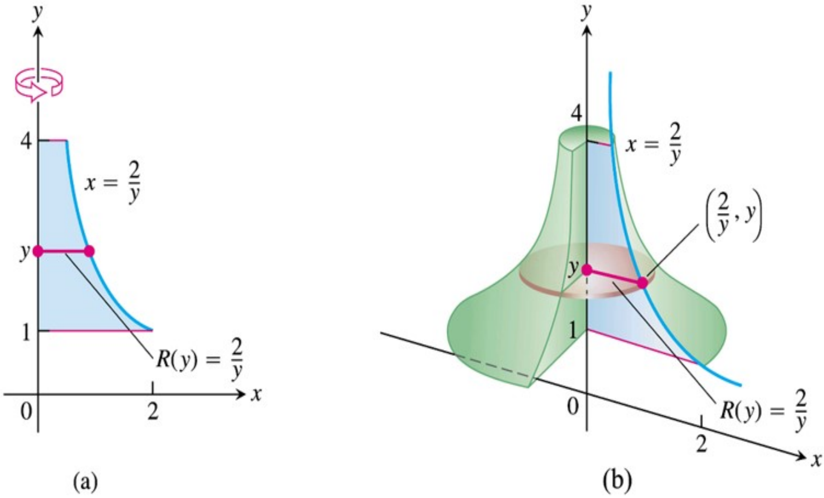


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7

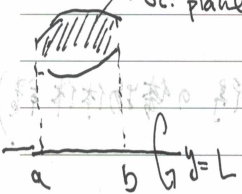
Solution of Example 7

Consider $R(y) = \frac{2}{y}$ for $1 \le y \le 4$. Then the volume of the solid by revolving the region Ω about the y-axis is

$$V = \int_1^4 \pi [R(y)]^2 \, dy = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = 4\pi \int_1^4 y^{-2} \, dy$$

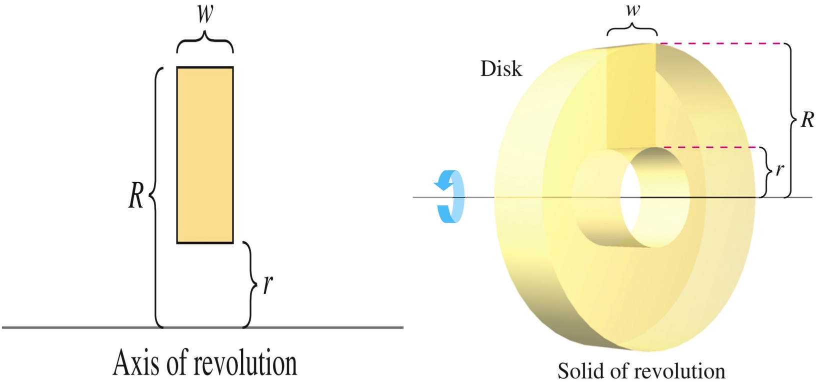
$$= 4\pi(-y^{-1})\Big|_1^4 = 4\pi(3/4) = 3\pi.$$

* **The Washer Method:** (墊圈法)



[Q]: What is the volume of a solid formed by revolving Ω about $y=L$?

墊圈 (washer) 的示意圖



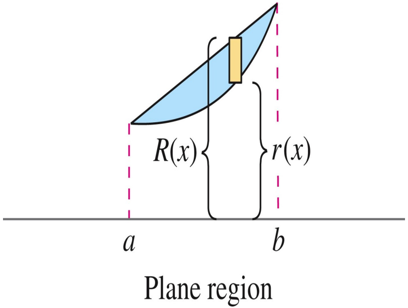
- Given a partition $\{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ with equal width $\Delta x = \dfrac{b-a}{n}$, where $x_0 = a$ and $x_n = b$.
- For each $i = 1, 2, \ldots, n$, choose $c_i \in [x_{i-1}, x_i]$, the volume of a washer of width Δx is

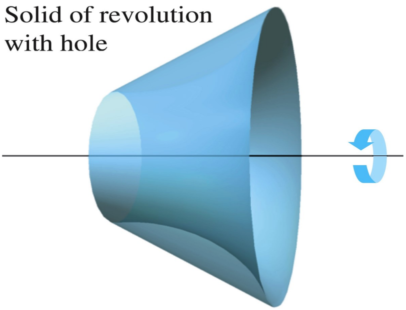
$$\begin{aligned} \Delta V_i &= \pi[R(c_i)]^2 \Delta x - \pi[r(c_i)]^2 \Delta x \\ &= \pi\Big([R(c_i)]^2 - [r(c_i)]^2\Big)\Delta x. \end{aligned}$$

Type I: Horizontal Axis of Revolution $y = L$

If the outer radius $R(x) \geq 0$ and inner radius $r(x) \geq 0$ are conti. on $[a, b]$, then the volume of a solid formed by revolving Ω about $y = L$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\Big([R(c_i)]^2 - [r(c_i)]^2\Big)\Delta x$$
$$= \pi \int_a^b \Big([R(x)]^2 - [r(x)]^2\Big)\, dx.$$


R(x)
r(x)
a
b
Plane region


Solid of revolution
with hole

A region Ω is bounded by the following curves

$$y = x^2 + 1 \quad \text{and} \quad y = -x + 3.$$

Find the volume of a solid formed by revolving the plane region Ω about the x-axis.

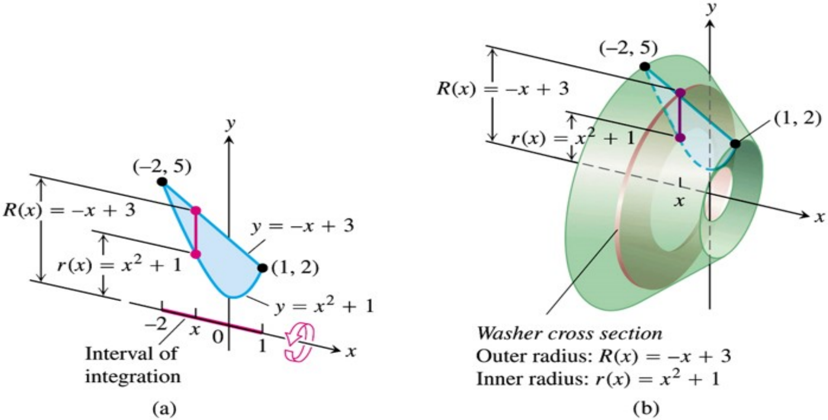


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

Note that the given curves intersect at $x = -2$ and $x = 1$ because

$$x^2 + 1 = -x + 3 \Longrightarrow x^2 + x - 2 = (x+2)(x-1) = 0 \Longrightarrow x = -2, 1.$$

Since $R(x) = -x + 3$ and $r(x) = x^2 + 1$ are conti. on $I = [-2., 1]$ with $R(x) \geq r(x) \quad \forall x \in I$, the plane region is given by

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid -2 \leq x \leq 1, r(x) \leq y \leq R(x)\}.$$

Thus, the volume of the solid formed by revolving Ω about the x-axis is

$$V = \pi \int_{-2}^{1} [(-x+3)^2 - (x^2+1)^2]\, dx = \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\, dx$$
$$= \pi \left(8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right)\Big|_{-2}^{1} = \frac{117\pi}{5}.$$

Type II: Vertical Axis of Revolution $x = L$

If the outer radius $R(y) \geq 0$ and inner radius $r(y) \geq 0$ are conti. on $[c, d]$, then the volume of a solid formed by revolving Ω about $x = L$ is

$$V = \pi \int_c^d \left([R(y)]^2 - [r(y)]^2 \right) dy.$$

A region Ω is bounded by the following curves

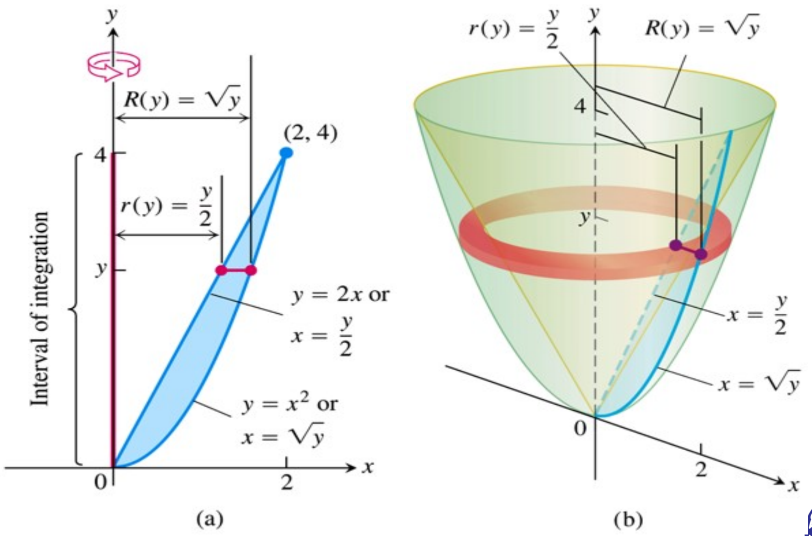
$$y = x^2 \quad \text{and} \quad y = 2x$$

or, equivalently, the following curves in terms of y

$$x = \sqrt{y} \quad \text{and} \quad x = \frac{y}{2}$$

in the first quadrant. Find the volume of a solid formed by revolving Ω about the y-axis.

Example 10 的示意圖



Note that the given curves intersect at $(0,0)$ and $(2,4)$ because

$$\sqrt{y} = \frac{y}{2} \Longrightarrow y = \frac{y^2}{4} \Longrightarrow y^2 - 4y = y(y-4) = 0 \Longrightarrow y = 0, 4.$$

So, the plane region can be written as

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 4, r(y) \le x \le R(y)\},$$

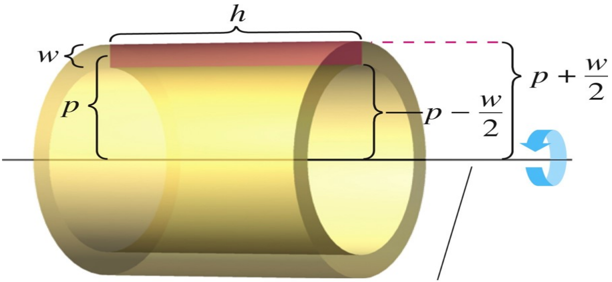
where $r(y) = y/2$ and $R(y) = \sqrt{y}$ with $r(y) \le R(y) \quad \forall\, y \in [0,4]$. Thus, the volume of the solid of revolution is given by

$$V = \pi \int_0^4 [(\sqrt{y})^2 - (y/2)^2]\, dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{12} \right) \Big|_0^4 = \frac{8\pi}{3}.$$

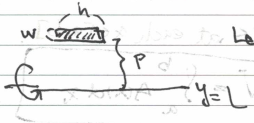
Section 6.2
Volumes Using Cylindrical Shells
(利用柱狀殼層求體積)

The volume of a shell of revolution is given by

$$\Delta V = \pi(p + \frac{w}{2})^2 h - \pi(p - \frac{w}{2})^2 h = 2\pi phw.$$



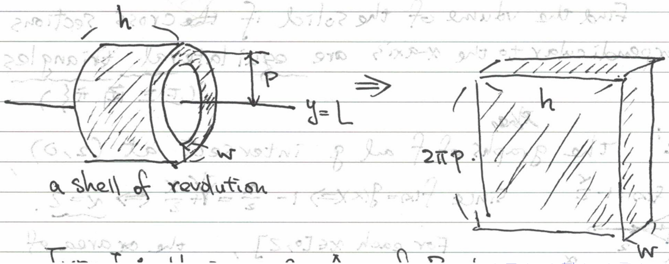
* Observation :



Let Ω be a rectangle in $\mathbb{R}^2$;
$w=$ width of Ω and
$h=$ height of Ω.

$\Rightarrow$ The volume of the shell is

$$\Delta V = \pi\left(p+\frac{w}{2}\right)^2 h - \pi\left(p-\frac{w}{2}\right)^2 h$$
$$= 2\pi p h w.$$



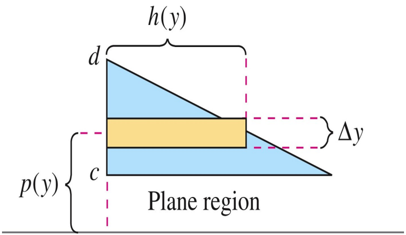
a shell of revolution

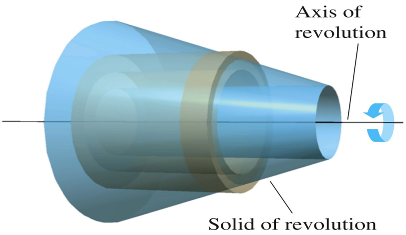
Type I: Horizontal Axis of Revolution $y = L$

If $p(y)$ and $h(y)$ are conti. functions of y on $[c, d]$, then the volume of the solid formed by revolving a plane region Ω about $y = L$ is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi p(c_i) h(c_i) \Delta y = 2\pi \int_c^d p(y) h(y)\, dy.$$

示意圖 (承上頁)


$h(y)$
d
Δy
c
$p(y)$
Plane region


Axis of revolution
Solid of revolution

Example 3 (Shell Method 的 Type I 例子)

A region Ω is bounded by $y = \sqrt{x}$, the x-axis and $x = 4$, i.e.,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 2, y^2 \le x \le 4\}.$$

Find the volume of the solid by revolving Ω about the x-axis.

Example 3 的示意圖

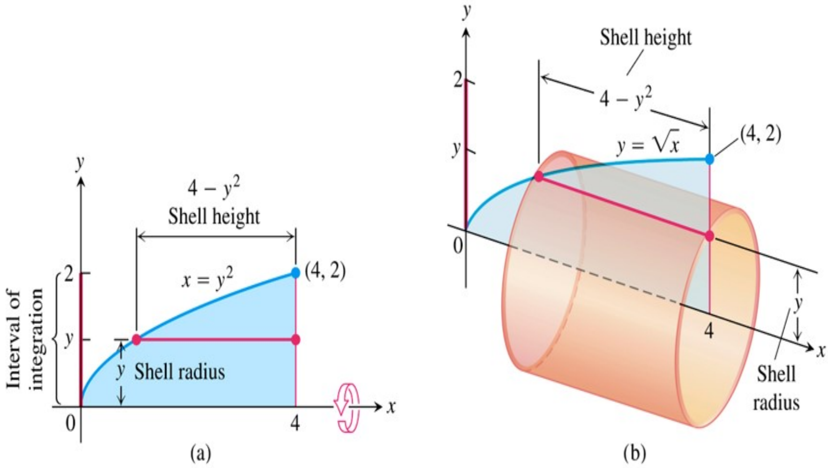


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

Solution of Example 3

Let $p(y) = y$ be the shell radius and $h(y) = 4 - y^2$ be the shell height for $0 \leq y \leq 2$. Then the volume of the solid formed by rotating Ω about the horizontal line $y = 0$ (the x-axis) is

$$V = \int_c^d 2\pi p(y) h(y)\, dy = 2\pi \int_0^2 y(4 - y^2)\, dy$$

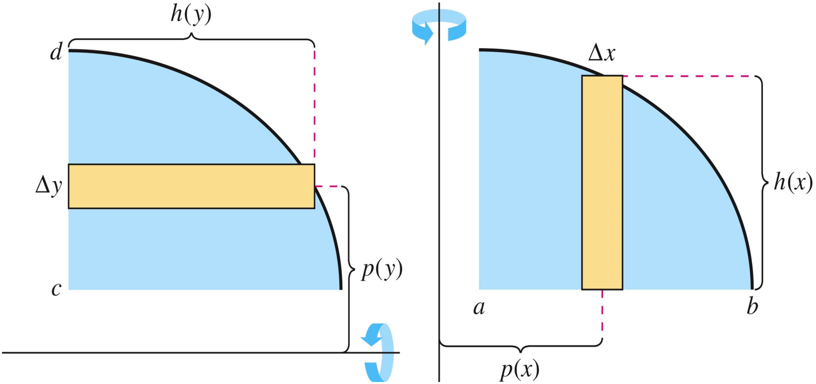
$$= 2\pi \int_0^2 (4y - y^3)\, dy = 2\pi \left(2y^2 - \frac{y^4}{4}\right)\Big|_0^2 = 8\pi.$$

Type II: Vertical Axis of Revolution $x = L$

If $p(x)$ and $h(x)$ are conti. functions of x on $[a, b]$, then the volume of the solid formed by revolving a plane region Ω about $x = L$ is

$$V = 2\pi \int_a^b p(x)h(x)\,dx.$$

如何選取函數 $p(\cdot)$ 和 $h(\cdot)$?



Example 2 (Shell Method 的 Type II 例子)

A region Ω is bounded by $y = \sqrt{x}$, the x-axis and $x = 4$, i.e.,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 4, 0 \le y \le \sqrt{x}\}.$$

Find the volume of the solid by revolving Ω about the y-axis.

Example 2 的示意圖

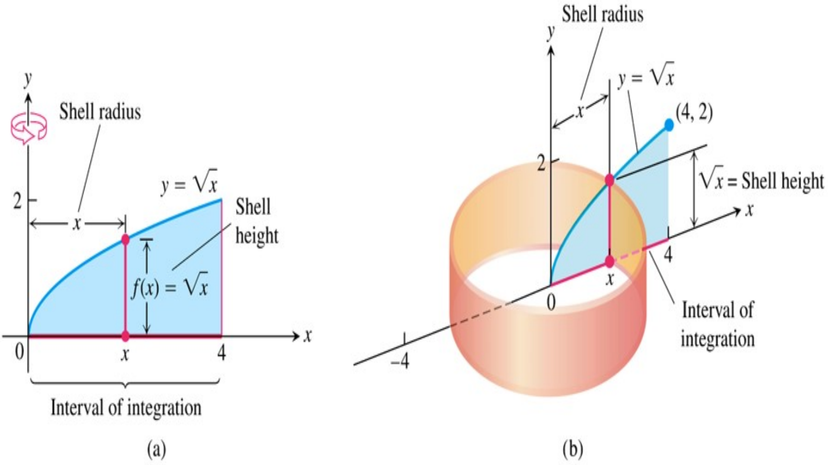


FIGURE 6.20 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

Solution of Example 2

Let $p(x) = x$ be the shell radius and $h(x) = \sqrt{x} = x^{1/2}$ be the shell height for $0 \le x \le 4$. Then the volume of the solid formed by rotating Ω about the vertical line $x = 0$ (the y-axis) is

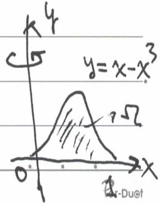
$$V = \int_a^b 2\pi p(x) h(x)\, dx = 2\pi \int_0^4 x \cdot \sqrt{x}\, dx$$
$$= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi \left(\frac{2}{5} x^{5/2} \right) \Big|_0^4 = \frac{128\pi}{5}.$$

Example 1: (Washer Method 幾乎不可行的例子)

Find the volume of the solid formed by revolving

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le x - x^3\}$$

about the y-axis.



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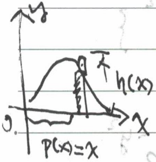
Sol: Note that the Washer Method fails for this example!!



$\because$ $p(x) = x$ and $h(x) = y = x - x^3$ are conti. on $[0,1]$

$$\therefore V = 2\pi \int_0^1 x(x - x^3)\,dx = 2\pi \int_0^1 (x^2 - x^4)\,dx$$

$$= 2\pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1 = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{4\pi}{15} \quad \#$$

Section 6.3
Arc Length
(弧長)

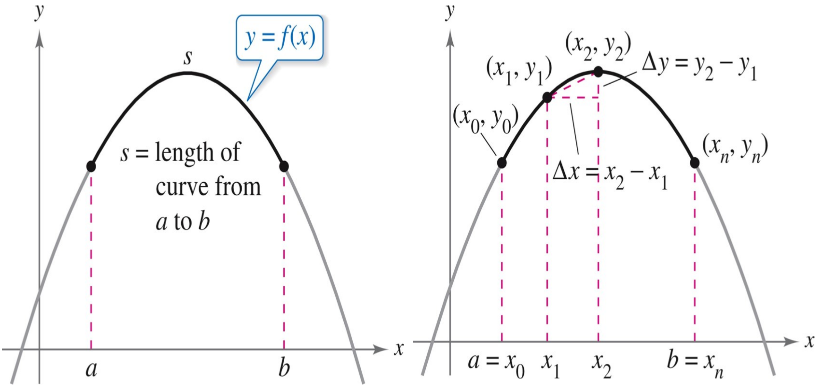
Def (平滑曲線的定義)

Let f be a real-valued function defined on $[a, b]$.

(1) f is continuously differentiable (連續可微) if its first derivative f' is conti. on $[a, b]$. In this case, we denote $f \in C^1[a, b]$.

(2) The graph of $f \in C^1[a, b]$ is called a smooth curve (平滑曲線).

弧長的示意圖


y
y = f(x)
s
s = length of
curve from
a to b
a
b
x
y
(x_2, y_2)
(x_1, y_1)
Δy = y_2 − y_1
(x_0, y_0)
(x_n, y_n)
Δx = x_2 − x_1
a = x_0
x_1
x_2
b = x_n
x

For a smooth curve $y = f(x)$ with $f \in C^1[a, b]$, the arc length of f on the subinterval $[x_{i-1}, x_i]$ is given by

$$\Delta L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}(\Delta x_i)$$

for each $i = 1, 2, \ldots, n$. Moreover, it follows from M.V.T. that $\exists\, c_i \in (x_{i-1}, x_i)$ s.t. $f'(c_i) = \dfrac{\Delta y_i}{\Delta x_i}$. Thus, we see that

$$\Delta L_i = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}(\Delta x_i) = \sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$

for each $i = 1, 2, \ldots, n$.

Type I: 第一型弧長公式

The (arc) length of a smooth curve $y = f(x)$ on $[a, b]$ is

$$L = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta L_i = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$

$$= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Find the (arc) length of the smooth curve

$$y = f(x) = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

from $x = 0$ to $x = 1$.

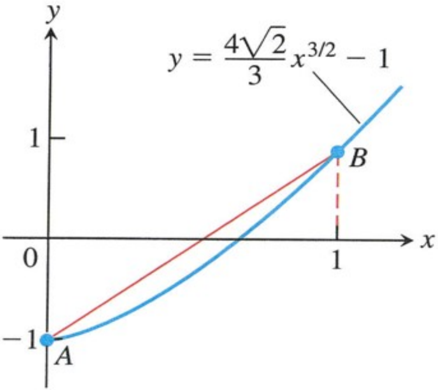


FIGURE 6.24 The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

Solution of Example 1

Since $f'(x) = 2\sqrt{2}x^{1/2}$ for $x \geq 0$, we see that

$$\sqrt{1 + [f'(x)]^2} = \sqrt{1 + (2\sqrt{2}x^{1/2})^2} = \sqrt{1 + 8x}.$$

So, the arc length of the given curve on $[0, 1]$ is

$$L = \int_0^1 \sqrt{1 + 8x}\, dx = \frac{1}{8} \cdot \frac{2}{3}(1 + 8x)^{3/2}\Big|_0^1 = \frac{13}{6} \approx 2.17.$$

Type II: 第二型弧長公式

The (arc) length of a smooth curve $x = g(y)$ on $[c, d]$ is

$$L = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \Delta L_i = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} \sqrt{1 + [g'(c_i)]^2}(\Delta y_i)$$

$$= \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Find the (arc) length of the smooth curve

$$y = f(x) = \left(\frac{x}{2}\right)^{2/3}$$

from $x = 0$ to $x = 2$. Note that $f'(0)$ does not exist in this case.

Example 4 的示意圖

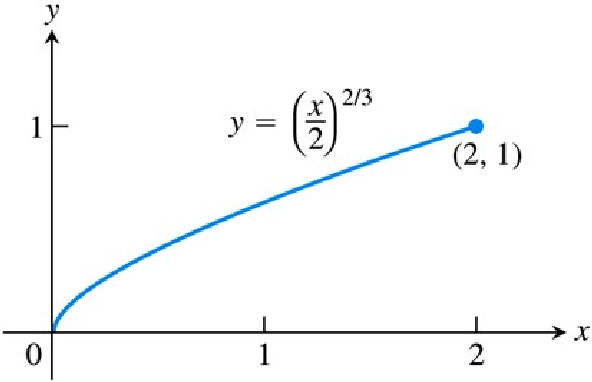


FIGURE 6.26 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

Note that the arc length formula of Type I is NOT applicable here! Thus, we may rewrite x in terms of y, namely,

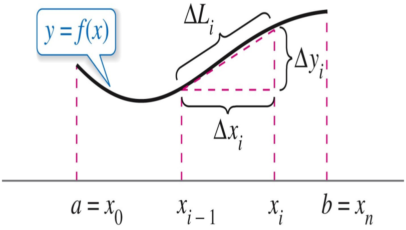
$$x = g(y) = 2y^{3/2}, \quad 0 \leq y \leq 1.$$

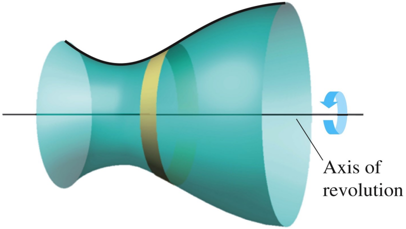
Then the length of $x = g(y)$ from $y = 0$ to $y = 1$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_0^1 \sqrt{1 + (3y^{1/2})^2}\, dy$$
$$= \int_0^1 \sqrt{1 + 9y}\, dy = \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1 = \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27.$$

Section 6.4
Areas of Surfaces of Revolution
(旋轉體的表面積)

Surfaces of Revolution (旋轉曲面)





Main Question

What is the surface area S formed by revolving a smooth curve $y = f(x)$ about the horizontal axis of revolution $y = L$?

On the ith subinterval $I_i = [x_{i-1}, x_i]$, choose $c_i, d_i \in I_i$, we let

$$\begin{aligned} r(d_i) &= \text{the redius of revolution at } d_i \in I_i, \\ \Delta L_i &= \text{the arc length of the smooth curve on } I_i \\ &= \sqrt{1 + [f'(c_i)]^2}(\Delta x_i), \end{aligned}$$

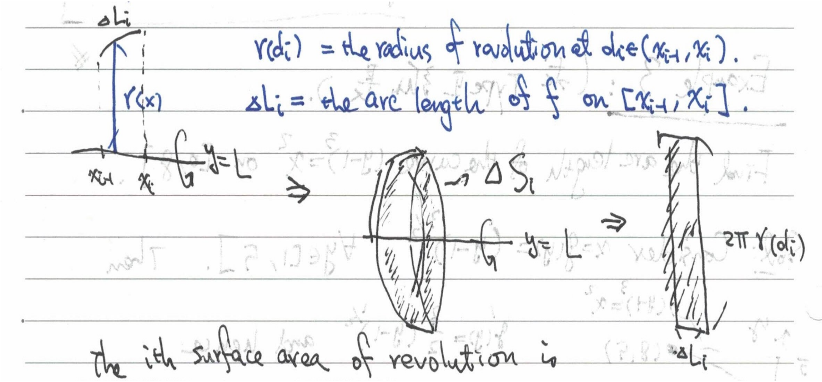
where $r(x) \geq 0$ is conti. on $[a, b]$.

Note (第 i 個旋轉體表面積)

If $d_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, it follows from M.V.T. that $\exists\, c_i \in (x_{i-1}, x_i)$ s.t. the ith surface area of revolution is

$$\Delta S_i \approx 2\pi r(d_i) \cdot \Delta L_i = 2\pi r(d_i)\sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$

for $i = 1, 2, \ldots, n$.


ΔLi
r(x)
xi-1 xi
y=L
r(di) = the radius of revolution at di∈(xi-1, xi).
ΔLi = the arc length of f on [xi-1, xi].
⇒
ΔSi
y= L
⇒
2π r(di)
ΔLi
The ith surface area of revolution is

Type I: Horizontal Axis of Revolution $y = L$

If $r(x) \geq 0$ is conti. on $[a, b]$ and $f \in C^1[a, b]$, then the area of the surface formed by revolving a curve $y = f(x)$ about $y = L$ is

$$S = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi r(d_i)\sqrt{1 + [f'(c_i)]^2}(\Delta x_i)$$
$$= 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

Find the area of the surface formed by revolving the smooth curve

$$y = f(x) = 2\sqrt{x}, \quad 1 \le x \le 2,$$

about the x-axis.

Example 1 的示意圖

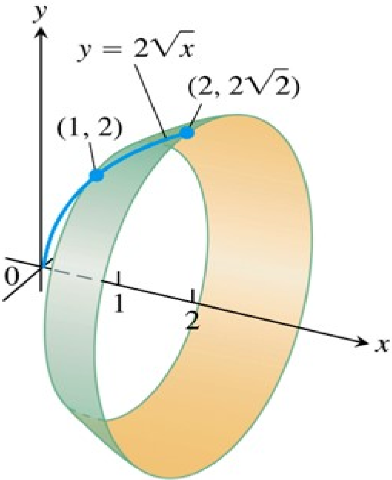


FIGURE 6.34 In Example 1 we calculate the area of this surface.

Let $r(x) = y = f(x) = 2\sqrt{x} \quad \forall\, x \in [1, 2]$. Then the area of the surface of revolution is given by

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\, dx = 2\pi \int_1^2 (2\sqrt{x})\sqrt{1 + (x^{-1/2})^2}\, dx$$

$$= 4\pi \int_1^2 \sqrt{x+1}\, dx = 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big|_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2}).$$

Type II: Vertical Axis of Revolution $x = L$

If $r(y) \geq 0$ is conti. on $[c, d]$ and $g \in C^1[c, d]$, then the area of the surface formed by revolving a curve $x = g(y)$ about $x = L$ is

$$S = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} 2\pi r(d_i) \sqrt{1 + [g'(c_i)]^2} (\Delta y_i)$$

$$= 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} \, dy.$$

Find the area of the surface formed by revolving the smooth curve

$$x = g(y) = 1 - y, \quad 0 \le y \le 1,$$

about the y-axis.

Example 2 的示意圖

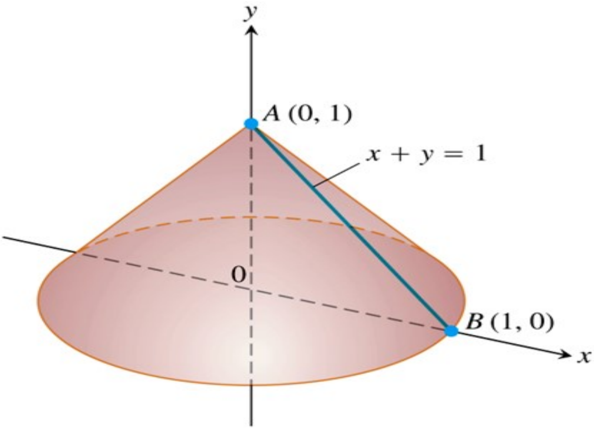


FIGURE 6.35 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

Let $r(y) = x = g(y) = 1 - y \quad \forall\, y \in [0,1]$. Then the area of the surface of revolution is given by

$$S = 2\pi \int_c^d r(y)\sqrt{1+[g'(y)]^2}\,dy = 2\pi \int_0^1 (1-y)\sqrt{1+(-1)^2}\,dy$$
$$= 2\sqrt{2}\pi\left(y - \frac{y^2}{2}\right)\Big|_0^1 = 2\sqrt{2}\pi(1-\frac{1}{2}) = \sqrt{2}\pi.$$

Thank you for your attention!