Chapter 7
Integrals and Transcendental Functions
(積分與超越函數)

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Fall 2022

本章預定授課範圍

7.1 The Logarithm Defined as an Integral

7.2 Exponential Change and Seperable Differential Equations

Section 7.1
The Logarithm Defined as an Integral
(以積分形式定義的對數函數)

Formal Definition of $\ln x$

Def (自然對數函數的正式定義)

The natural logarithm is defined by

$$\ln x := \int_1^x \frac{1}{t}\, dt, \quad x > 0.$$

Note: From the viewpoint of the area between $f(x) = 1/x$ and the x-axis, we see that

$$\ln x = \begin{cases} \int_1^x \frac{1}{t}\, dt \geq 0, & x \geq 1 \\ -\int_x^1 \frac{1}{t}\, dt < 0, & 0 < x < 1. \end{cases}$$

Def (歐拉數 e 的定義)

Euler's number $e \approx 2.71828\cdots$ is an irrational number satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

Recall (自然對數函數的微分與積分)

Let $u(x)$ be a diff. function of x. Then

(1) $\frac{d}{dx}\Big(\ln|u(x)|\Big) = \frac{u'(x)}{u(x)}$ for $u(x) \neq 0$.

(2) $\int \frac{u'(x)}{u(x)}\,dx = \int \frac{du}{u} = \ln|u(x)| + C.$

Example 1 (與 $\ln|u|$ 有關的積分)

If we let $u(x) = 3 + 2\sin\theta$, then $du = 2\cos\theta\, d\theta$ and

$$u(\frac{-\pi}{2}) = 3 - 2 = 1, \quad u(\frac{\pi}{2}) = 3 + 2 = 5.$$

Thus, from above u-substitution, we immediately obtain

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{4\cos\theta}{3 + 2\sin\theta}\, d\theta = \int_1^5 \frac{2\, du}{u} = 2\ln|u|\Big|_1^5$$
$$= 2(\ln 5 - \ln 1) = 2\ln 5.$$

Since $\ln : (0, \infty) \to (-\infty, \infty)$ is increasing on $(0, \infty)$, it must be one-to-one and hence has an inverse $\ln^{-1} : (-\infty, \infty) \to (0, \infty)$.

Def (自然指數函數的定義)

The natural exponential function is defined by

$$e^x := \ln^{-1} x \quad \forall x \in (-\infty, \infty).$$

Note: $e = \ln^{-1}(1)$ because $\ln(e) = 1$.

Recall (e^u 的微分與積分)

Let $u(x)$ be a diff. function of x. Then

(1) $\dfrac{d}{dx}(e^u) = e^u \cdot u\,'$.

(2) $\displaystyle\int e^u\, du = e^u + C$.

Def (以 a 為底指數函數的定義)

The exponential function with base $a > 0$ is defined by

$$a^x := e^{x \ln a} > 0 \quad \forall\, x \in (-\infty, \infty).$$

Note: From the Chain Rule, we see that

$$\frac{d}{dx}(a^x) = a^x \ln a = \begin{cases} > 0, & a > 1 \\ < 0, & 0 < a < 1. \end{cases}$$

Since $f(x) = a^x$ is a monotonic function on $(-\infty, \infty)$, it is one-to-one and hence has an inverse $f^{-1} : (0, \infty) \to (-\infty, \infty)$.

Def (以 a 為底對數函數的定義)

For $0 < a \neq 1$, the logarithm with base a, denoted by $\log_a x$, is defined to be the inverse of a^x. Moreover, we have

$$\log_a x = \frac{\ln x}{\ln a} \quad \forall\, x > 0.$$

(a) For $3x + 1 > 0$ or $x > -1/3$, we obtain

$$\frac{d}{dx}\log_{10}(3x+1) = \frac{d}{dx}\left[\frac{\ln(3x+1)}{\ln 10}\right] = \frac{3}{(\ln 10)(3x+1)}.$$

(b) Rewriting $\log_2 x = \dfrac{\ln x}{\ln 2}$, we see that

$$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx = \frac{1}{\ln 2}\int \ln x\,d(\ln x)$$
$$= \frac{1}{\ln 2}\cdot\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C.$$

Section 7.2
Exponential Change and Seperable Differential Equations
(指數型變化與可分離微分方程式)

For $k \neq 0$, the solution of the initial value problem (I.V.P.)

$$\frac{dy}{,dt} = ky, \quad y(0) = y_0$$

is $y(t) = y_0 e^{kt}$ for $t \geq 0$.

Def (指數型成長或衰減)

The change of e^{kt} is said to undergo exponential growth (指數型成長) if $k > 0$ and exponential decay (指數型衰減) if $k < 0$. The number k is called the rate constant of the change (變化率常數).

Separable Differential Equations

The general solution (通解) $y(x)$ of a separable (可分離的) differential equation of the form

$$\frac{d}{dx}[y(x)] = g(x)H(y(x)) \Longrightarrow \frac{1}{H(y)}\,dy = g(x)\,dx$$

is given by integrating on both sides of the equation, i.e.,

$$\int \frac{1}{H(y)}\,dy = \int g(x)\,dx,$$

which usually defines y as an implicit function of x.

solve $\dfrac{dy}{dx} = (1+y)e^x$, where $y > -1$.

Sol: The given differential equation is separable, since

$$\frac{dy}{1+y} = e^x\,dx.$$

Thus, the general solution $y(x)$ is defined implicitly by the equation

$$\ln(1+y) = e^x + C.$$

Solve $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

Sol: The given differential equation is separable because

$$\frac{y}{y^2+1}\,dy = \frac{x}{x+1}\,dx.$$

Taking integration on both sides of the equation gives

$$\int \frac{y}{y^2+1}\,dy = \int \frac{x}{x+1}\,dx = \int \left(1 - \frac{1}{x+1}\right) dx$$
$$\Longrightarrow \frac{1}{2}\ln(y^2+1) = x - \ln|x+1| + C,$$

which gives y as an implicit function of x.

Thank you for your attention!