Chapter 8
Techniques of Integration
(積分的技巧)

Hung-Yuan Fan (范洪源)

Department of Mathematics,
National Taiwan Normal University, Taiwan

Fall 2022

本章預定授課範圍

8.1 Integration by Parts

8.2 Trigonometric Integrals

8.3 Trigonometric Substitutions

8.4 Integration of Rational Functions by Partial Fractions

8.7 Improper Integrals

Section 8.1
Integration by Parts
(分部、分步或部分積分; 簡稱 I.B.P.)

Observation

If u and v are diff. functions of x, then

$$(uv)' = u'v + uv' = uv' + vu'.$$

Thus, we immediately obtain

$$\begin{aligned} \int (uv)'\,dx &= \int uv'\,dx + \int vu'\,dx \\ \Longrightarrow uv &= \int u\,dv + \int v\,du \\ \Longrightarrow \int u\,dv &= uv - \int v\,du. \end{aligned}$$

Integration by Parts Formula–Differential Version

Thm (I.B.P. 公式)

If $u = u(x)$ and $v = v(x)$ are functions of x havingg conti. derivatives, then we have

$$\int u\,dv = uv - \int v\,du.$$

How to choose u and dv?

- $u = u(x)$: easily differentiable function of x. (u: 好微分函數)
- $v = v(x)$: easily integrable function of x. (v: 好積分函數)

Type I of I.B.P.

The integrals of the form

$$\int x^n e^{ax}\,dx, \quad \int x^n \sin(ax)\,dx, \quad \int x^n \cos(ax)\,dx$$

with $n \in \mathbb{R}$ and $a \neq 0$.

How to select u and dv for this case?

- $u = x^n$.
- $dv = e^{ax}dx, \quad \sin(ax)dx, \quad \cos(ax)dx$.

Example 1 (Type I 的例子; 1/2)

Find the following indefinite integral

$$\int x \cos x \, dx$$

using the formula of integration by parts.

Solution of Example 1 (1/2)

Method 1: (使用 I.B.P. 公式)

Let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ and $v = \int 1\,dv = \int \cos x\,dx = \sin x$. From the I.B.P., we obtain

$$\int x\cos x\,dx = \int u\,dv = uv - \int v\,du$$
$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C,$$

where C is a constant of integration.

Method 2: (第二種更簡潔的寫法)

$$
\begin{aligned}
\int x \cos x \, dx &= \int x \, d(\sin x) \equiv \int u \, dv \\
&= u\,v - \int v \, du \qquad \text{(使用 I.B.P. 公式)} \\
&= x \sin x - \int \sin x \, dx \\
&= x \sin x + \cos x + C,
\end{aligned}
$$

where C is a constant of integration.

Example 3 (使用兩次 I.B.P. 的例子)

Evaluate the following indefinite integral

$$\int x^2 e^x \, dx$$

using the formula of integration by parts twice.

Method 1: (使用兩次 I.B.P. 公式)

Applying the I.B.P. formula twice, we see that

$$\begin{aligned}
\int x^2 \cdot e^x\,dx &= \int x^2\,d(e^x) \equiv \int u\,dv \\
&= x^2e^x - \int e^x\,d(x^2) = x^2e^x - \int 2xe^x\,dx \quad \text{(第一次 I.B.P.)} \\
&= x^2e^x - 2\int xe^x\,dx = x^2e^x - 2\int x\,d(e^x) \\
&= x^2e^x - 2\left(xe^x - \int e^x\,dx\right) \quad \text{(第二次 I.B.P.)} \\
&= x^2e^x - 2xe^x + 2e^x + C,
\end{aligned}$$

where C is a constant of integration.

Method 2: (使用表格法)

Applying the tabular method (表格法), we immediately obtain

$+$	x^2	$e^x\,dx$
$-$	$2x$	e^x
$+$	2	e^x

Thus the original integral satisfies

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + \int 2e^x\,dx,$$

and hence the indefinite integral is given by

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C,$$

where C is a constant of integration.

Example 6 (計算平面區域的面積)

Find the area of the region bounded by the curve

$$y = f(x) = xe^{-x} \geq 0$$

and the x-axis from $x = 0$ to $x = 4$.

Example 6 的示意圖

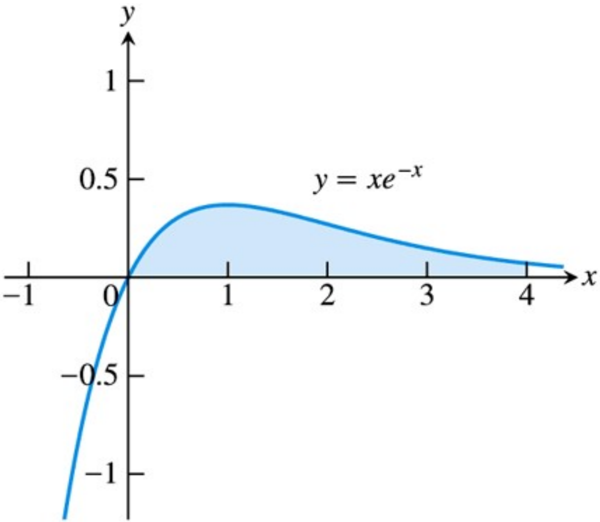


FIGURE 8.1 The region in Example 6.

Solution of Example 6

Since $f(x) = xe^{-x} \geq 0$ is conti. on $[0, 4]$, the area of the given region is $A = \int_0^4 xe^{-x}\,dx \geq 0$. Using the I.B.P. formula, we immediately obtain

$$\int xe^{-x}\,dx = \int x\,d(-e^{-x}) = -xe^{-x} - \int (-e^{-x})\,dx = -xe^{-x} - e^{-x} + C,$$

and hence $A = (-xe^{-x} - e^{-x})\Big|_0^4 = 1 - 5e^{-4} \approx 0.91.$

Type II of I.B.P.

The integrals of the form

$$\int x^n \ln x \, dx, \quad \int x^n \sin^{-1} x \, dx, \quad \int x^n \tan^{-1} x \, dx$$

with $n \neq -1$.

How to select u and dv for this case?

- $u = \ln x, \quad \sin^{-1} x, \quad \tan^{-1} x.$
- $dv = x^n dx.$

Example 2 (Type II 的例子)

Find the following indefinite integral

$$\int \ln x \, dx$$

using the formula of integration by parts.

Solution of Example 2

Let $u = \ln x$ and $dv = dx$. Then $du = \frac{1}{x}\,dx$ and $v = x$. Utilizing the I.B.P. formula, we thus have

$$\begin{aligned}\int \ln x\,dx &= \int u\,dv = uv - \int v\,du \\ &= x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int 1\,dx \\ &= x\ln x - x + C,\end{aligned}$$

where C is a constant of integration.

Type III of I.B.P.

The integrals of the form

$$\int e^{ax}\sin(bx)\,dx, \quad \int e^{ax}\cos(bx)\,dx$$

with $a, \neq 0$ and $b, \neq 0$.

How to select u and dv for this case?

- $u = e^{ax}$ and $dv = \sin(bx)dx, \quad \cos(bx)dx$.
- $u = \sin(bx), \quad \cos(bx)$ and $dv = e^{ax}dx$.
- 通常使用兩次 I.B.P. 公式!

Evaluate the folloing indefinite integral

$$\int e^x \cos x \, dx$$

using the formula of integration by parts twice.

Solution of Example 4 (1/2)

Method 1: (使用兩次 I.B.P. 公式)

Applying the I.B.P. formula twice, we see that

$$\int e^x \cos x\,dx = \int e^x\,d(\sin x) = e^x \sin x - \int \sin x d(e^x) \quad \text{(第一次 I.B.P)}$$

$$= e^x \sin x - \int e^x \sin x\,dx = e^x \sin x + \int e^x\,d(\cos x)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \quad \text{(第二次 I.B.P.)}$$

$$\Longrightarrow 2\int e^x \cos x\,dx = e^x(\sin x + \cos x).$$

So, $\int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C.$

Method 2: (使用表格法)

Applying the tabular method (表格法), we immediately obtain

$+$	$\cos x$	$e^x\,dx$
$-$	$-\sin x$	e^x
$+$	$-\cos x$	e^x

Thus the original integral satisfies

$$\int e^x \cos x\,dx = e^x \cos x + e^x \sin x + \int e^x(-\cos x)\,dx,$$

and hence the indefinite integral is given by

$$\int e^x \cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + C,$$

where C is a constant of integration.

Section 8.2
Trigonometric Integrals
(三角積分)

Main Goals

Try to find the integrals of the form

$$\text{(INT-1)} : \int \sin^m x \cos^n x \, dx, \text{ where } m, n \in \mathbb{Q}.$$

$$\text{(INT-2)} : \int \sec^m x \tan^n x \, dx, \text{ where } m, n \in \mathbb{Q}.$$

$$\text{(INT-3)} : \int \sin\alpha \cos\beta \, dx, \quad \int \sin\alpha \sin\beta \, dx, \quad \int \cos\alpha \cos\beta \, dx,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are functions of x.

Type I of (INT-1)

If $m = 2k+1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \cdot \sin x \, dx$$
$$= -\int (1 - \cos^2 x)^k \cos^n x \, d(\cos x) = -\int (1 - u^2)^k u^n \, du,$$

where we let $u = \cos x$.

Example 1 (Type I 的例子)

Evaluate the following indefinite integral

$$\int \sin^3 x \cos^2 x \, dx,$$

where $m = 3$ and $n = 2$ in this case.

If we let $u = \cos x$, then $du = -\sin x\, dx$ and thus

$$\int \sin^3 x \cos^2 x\, dx = \int \sin^2 x \cos^2 x \sin x\, dx$$
$$= \int (1 - \cos^2 x) \cos^2 x\, d(-\cos x)$$
$$= \int (1 - u^2) u^2\, d(-u) = \int (u^4 - u^2)\, du$$
$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.$$

Type II of (INT-1)

If $n = 2k + 1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x (\cos^2 x)^k \cdot \cos x \, dx$$
$$= \int \sin^m x (1 - \sin^2 x)^k \, d(\sin x) = \int u^m (1 - u^2)^k \, du,$$

where we let $u = \sin x$.

Example 2 (Type II 的例子)

Evaluate the following indefinite integral

$$\int \cos^5 x \, dx,$$

where $m = 0$ and $n = 5$ in this case.

Solution of Example 2

Let $u = \sin x$. Then $du = \cos x\,dx$ and thus

$$\int \cos^5 x\,dx = \int (\cos^2 x)^2 \cdot \cos x\,dx$$
$$= (1 - \sin^2 x)^2\,d(\sin x) = \int (1 - u^2)^2\,du$$
$$= \int (1 - 2u^2 + u^4)\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$
$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

Type III of (INT-1)

If m and n are even and nonnegative, try to use the identities

$$\sin^2 x = \frac{1-\cos(2x)}{2}, \quad \cos^2 x = \frac{1+\cos(2x)}{2}.$$

(當 m, n 為偶數或零，試用倍角公式!)

Example 3 (Type III 的例子)

Evaluate the following indefinite integral

$$\int \sin^2 x \cos^4 x \, dx,$$

where $m = 2$ and $n = 4$ in this case.

Solution of Example 3 (1/2)

$$\begin{aligned}\int \sin^2 x \cos^4 x\, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx \\ &= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx \\ &= \frac{1}{8}\left(x+\frac{\sin 2x}{2}-\int \cos^2 2x\, dx-\int \cos^3 2x\, dx\right).\end{aligned}$$

In addition, it is easily checked that

$$\int \cos^2 2x\,dx = \int \frac{1+\cos 4x}{2}\,dx = \frac{x}{2} + \frac{\sin 4x}{8} + C,$$
$$\int \cos^3 2x\,dx = \frac{1}{2}\int (1-\sin^2 2x)\,d(\sin 2x) = \frac{\sin 2x}{2} - \frac{\sin^3 2x}{6} + C.$$

Combining everything and simplifying, we get

$$\int \sin^2 x\cos^4 x\,dx = \frac{1}{8}\left(\frac{x}{2} - \frac{\sin 4x}{8} + \frac{\sin^3 2x}{6}\right) + C$$
$$= \frac{1}{16}\left(x - \frac{\sin 4x}{4} + \frac{\sin^3 2x}{3}\right) + C.$$

Example 4 (利用倍角公式化簡積分)

Evaluate the following definite integral

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$$

using the identity $1+\cos 4x = 2\cos^2 2x$.

Since it is true that

$$\cos^2 2x = \frac{1+\cos 4x}{2} \quad \text{or} \quad 1+\cos 4x = 2\cos^2 2x,$$

we immediately obtain

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx &= \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 2x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx \\ &= \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \quad (\because \ \cos 2x \ge 0 \quad \forall\, x \in [0, \frac{\pi}{4}]) \\ &= \frac{\sqrt{2}}{2} \sin 2x\Big|_0^{\pi/4} = \frac{\sqrt{2}}{2}.\end{aligned}$$

Type I of (INT-2)

If $m = 2k$ is even for some $k \in \mathbb{N}$, then

$$\int \sec^{2k} x \tan^n x \, dx = \int (\sec^2 x)^{k-1} \tan^n x \cdot \sec^2 x \, dx$$
$$= \int (1 + \tan^2 x)^{k-1} \tan^n x \, d(\tan x) = \int (1 + u^2)^{k-1} u^n \, du,$$

where we let $u = \tan x$.

Example 7 (Type I 的例子)

Evaluate the following indefinite integral

$$\int \sec^4 x \tan^4 x \, dx,$$

where $m = 4$ and $n = 4$ in this case.

Let $u = \tan x$. Then $du = \sec^2 x\,dx$ and thus

$$\begin{aligned}
\int \sec^4 x \tan^4 x\,dx &= \int (1+\tan^2 x)\tan^4 x \cdot \sec^2 x\,dx \\
&= \int (1+\tan^2 x)\tan^4 x\,d(\tan x) \\
&= \int (u^4 + u^6)\,du = \frac{u^5}{5} + \frac{u^7}{7} + C \\
&= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.
\end{aligned}$$

Type II of (INT-2)

If $n = 2k + 1$ is odd for some $k \in \mathbb{N}$, then

$$\int \sec^m x \tan^{2k+1} x\,dx = \int \sec^{m-1} x(\tan^2)^k \cdot \sec x \tan x\,dx$$

$$= \int \sec^{m-1} x(\sec^2 - 1)^k\,d(\sec x) = \int u^{m-1}(u^2 - 1)^k\,du,$$

where we let $u = \sec x$.

Example (補充題; Type II 的例子)

Find the following indefinite integral

$$\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx = \int (\sec x)^{-1/2} \tan^3 x \, dx,$$

where $m = -1/2$ and $n = 3$ in this case.

Solution of above Example

$$\begin{aligned}\int (\sec x)^{-1/2} \tan^3 x \, dx &= \int (\sec x)^{-3/2} \tan^2 x (\sec x \tan x) \, dx \\ &= \int (\sec x)^{-3/2} (\sec^2 x - 1) \, d(\sec x) \\ &= \int \left[(\sec x)^{1/2} - (\sec x)^{-3/2} \right] d(\sec x) \\ &= \frac{2}{3} (\sec x)^{3/2} + 2(\sec x)^{-1/2} + C.\end{aligned}$$

Type III of (INT-2)

If m is odd or n is even, try to use the identity

$$tan^2 x = \sec^2 x - 1.$$

(當 m 是奇數或 n 是偶數時，試用上述等式!)

Evaluate the following indefinite integral

$$\int \sec^3 x \, dx,$$

where $m = 3$ and $n = 0$ in this case.

Solution of Example 6 (1/2)

Since $\tan^2 x = \sec^2 x - 1$, we see that

$$\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx = \int \sec x\,d(\tan x)$$
$$= \sec x \tan x - \int \tan x\,d(\sec x) \quad \text{(使用 I.B.P.)}$$
$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx.$$

Solution of Example 6 (2/2)

Combining the integral of $\sec^3 x$, we get

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C,$$

and hence we obtain the following formula

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Integrals of Sine-Cosine Products

If $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are conti. functions of x, how to evaluate the integrals of the form

$$\text{(INT-3)} : \int \sin\alpha\cos\beta\, dx, \quad \int \sin\alpha\sin\beta\, dx, \quad \int \cos\alpha\cos\beta\, dx$$

using the product-to-sum identities (積化和差等式)?

Product-to-Sum Identities

(1) $\sin\alpha\sin\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big].$

(2) $\cos\alpha\cos\beta = \frac{1}{2}\Big[\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big].$

(3) $\sin\alpha\cos\beta = \frac{1}{2}\Big[\sin(\alpha-\beta) + \sin(\alpha+\beta)\Big].$

Example 8 (補充題; 積化和差的例子)

Use the product-to-sum identity to evaluate the integral

$$\int \sin 3x \cos 5x \, dx,$$

where $\alpha = \alpha(x) = 3x$ and $\beta = \beta(x) = 5x$.

Solution of Example 8

Applying the product-to-sum identity (3), we see that

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \int \frac{1}{2}\Big[\sin(3x-5x)+\sin(3x+5x)\Big]\, dx \\ &= \frac{1}{2}\int \Big[\sin(-2x)+\sin 8x\Big]\, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx \\ &= \frac{-\cos 8x}{16}+\frac{\cos 2x}{4}+C,\end{aligned}$$

where C is a constant of integration.

Section 8.3
Trigonometric Substitutions
(三角代換)

Main Goal

- To deal with the integrals involving

$$\sqrt{a^2 - x^2}, \quad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}$$

  with $a > 0$.
- Hhow to evaluate the three types of aforementioned integrals using the technique of trigonometric substitution?

Type I of Trigonometric Substitution

For integrals involving $\sqrt{a^2 - x^2}$ with $a > 0$, let

$$x = a\sin\theta.$$

Then $dx = a\cos\theta\, d\theta$ and we see that

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$$

for $\dfrac{-\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

Example 3 (Type I 的例子)

Evaluate the following indefinite integral

$$\int \frac{x^2}{\sqrt{9-x^2}}\, dx$$

using the trigonometric substitution $x = 3\sin\theta$.

Solution of Example 3 (1/2)

Let $x = 3\sin\theta$ for $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$. Then

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,dx &= \int \frac{(9\sin^2\theta)(3\cos\theta)}{\sqrt{9\cos^2\theta}}\,d\theta \\
&= 9\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\frac{\sqrt{9-x^2}}{3}\right) + C \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

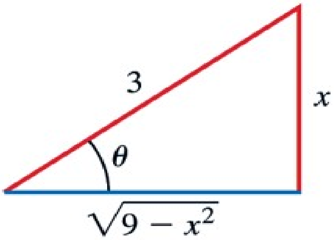


FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 3):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9 - x^2}}{3}.$$

Type II of Trigonometric Substitution

For integrals involving $\sqrt{a^2 + x^2}$ with $a > 0$, let

$$x = a \tan \theta.$$

Then $dx = a \sec^2 \theta \, d\theta$ and we see that

$$\sqrt{a^2 + x^2} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = a \sec \theta$$

for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$.

Example 1 (Type II 的例子)

Evaluate the following indefinite integral

$$\int \frac{dx}{\sqrt{4+x^2}}$$

using the trigonometric substitution $x = 2\tan\theta$.

Let $x = 2\tan\theta$ for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta}{\sqrt{4\sec^2\theta}}\,d\theta = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta \quad (\because\ \sec\theta = \frac{1}{\cos\theta} > 0)$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C = \ln\left|\sqrt{4+x^2} + x\right| + C.$$

Solution of Example 1 (2/2)

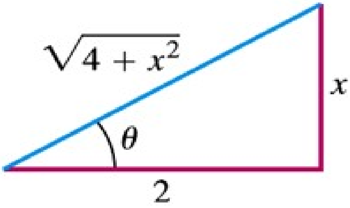


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Type III of Trigonometric Substitution

For integrals involving $\sqrt{x^2 - a^2}$ with $a > 0$, let

$$x = a \sec \theta.$$

Then $dx = a \sec \theta \tan \theta \, d\theta$ and we see that

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = a|\tan \theta|$$

for $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$.

Example 4 (Type III 的例子)

For all $x > \frac{2}{5}$, evaluate the indefinite integral

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}}$$

using the trigonometric substitution $x = \frac{2}{5}\sec\theta$.

Solution of Example 4 (1/2)

Let $x = \frac{2}{5}\sec\theta$ for $00 < \theta < \frac{\pi}{2}$. Then we see that

$$x > \frac{2}{5} \quad \text{and} \quad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta.$$

Therefore, the original integral becomes

$$\int \frac{dx}{5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}} = \int \frac{(2/5)\sec\theta\tan\theta}{5\cdot(2/5)\tan\theta}\, d\theta$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C,$$

where C is a constant of integration.

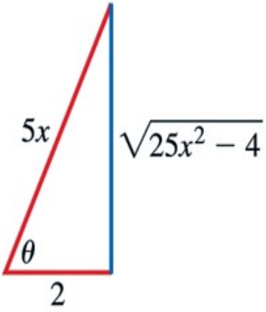


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 4).

Section 8.4
Integration of Rational Functions by Partial Fractions
(利用部分分式求有理函數的積分)

Long Division (長除法)

If $N(x)$ and $D(x)$ are polynomials with $\deg(N) \geq \deg(D)$, then

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N_1(x)}{D(x)},$$

where $Q(x)$ and $N_1(x)$ are polynomials with $\deg(N_1) < \deg(D)$. In this case, $\frac{N_1(x)}{D(x)}$ is called a proper rational function of x (真分式).

Note: integrating w.r.t. $x \Longrightarrow$

$$\int \frac{N(x)}{D(x)}\,dx = \int Q(x)\,dx + \int \frac{N_1(x)}{D(x)}\,dx.$$

Remarks

- $\int Q(x)\,dx$ can evaluated easily by the power rule.
- But, how to evaluate the integral $\int \frac{N_1(x)}{D(x)}\,dx$ using the technique of partial fraction decomposition?

如何設定部分分式的分解形式?

If $D(x) = (px+q)^m(ax^2+bx+c)^n$ with $p \neq 0$, $a \neq 0$ and $b^2-4ac<0$, try to write a sum of partial fractions of the form

$$\frac{N_1(x)}{D(x)} = \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_m}{(px+q)^m} + \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n},$$

where A_i $(i=1,\ldots,m)$ and B_j, C_j $(j=1,\ldots,n)$ are unknowns.

Example 6 (部分分式的例子)

Find the numbers A, B and C in the following equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Comparing the numerators of both sides of the equation, we have

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$
$$= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3}$$
$$= \frac{Ax^2 + (2A+B)x + (A+B+C)}{(x+1)^3}.$$

Thus, it follows that $A = 0$, $2A + B = 1$ and $A + B + C = -1$. Then we immediately obtain $B = 1$, $C = -2$ and hence

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$

Type I: Distinct or Repeated Linear Factors

The denominator (分母) $D(x)$ contains

- distinct linear factors (相異的一次因式)

$$(p_1x+q_1)(p_2x+q_2)\cdots(p_mx+q_m)$$

- repeated linear factors (重複的一次因式)

$$(px+q)^m,$$

where $p_j \neq 0$ $(j=1,2,\ldots,m)$ and $p \neq 0$ for some $m \in \mathbb{N}$.

Example 2 (重複的一次因式)

Evaluate the following indefinite integral

$$\int \frac{6x+7}{(x+2)^2}\, dx$$

using the technique of partial fraction decomposition.

Write the integrand as a sum of partial fractions

$$f(x) = \frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}.$$

From te numerators of above equation, we see that

$$6x+7 = A(x+2) + B = Ax + (2A+B),$$

and hence $A = 6$ and $2A + B = 7$. Then the last equation gives $B = -5$. So, the given integral is given by

$$\int f(x)\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx = 6\ln|x+2| + \frac{5}{x+2} + C.$$

Type II: Distinct or Repeated Quadratic Factors (二次因式)

The denominator $D(x)$ contains

- distinct quadratic factors

$$(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)\cdots(a_nx^2 + b_nx + c_n)$$

- repeated quadratic factors

$$(ax^2 + bx + c)^n,$$

where $b_j^2 - 4a_jc_j < 0$ $(j = 1, 2, \ldots, n)$ and $b^2 - 4ac < 0$ for some $n \in \mathbb{N}$.

Note: these quadratic factors are called irreducible (不可既約)!

Example 4 (二次因式與重複的一次因式)

Evaluate the following indefinite itegral

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

using the technique of partial fraction decomposition.

Solution of Example 4 (1/2)

Write the integrand as a sum of partial fractions

$$f(x) = \frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}.$$

Then we immediately have

$$\begin{aligned}
-2x+4 =& (Ax+B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1) \\
=& (A+C)x^3 + (-2A+B-C+D)x^2 \\
& + (A-2B+C)x + (B-C+D).
\end{aligned}$$

Solution of Example 4 (2/2)

Equating coefficients of powers of x gives

$$A+C=0, \quad -2A+B-C+D=0, \quad A-2B+C=-2, \quad B-C+D=4.$$

After solving the system of linear equations, we obtain

$$A=2, \quad B=1, \quad C=-2, \quad D=1, \qquad \text{(Check!)}$$

and thus the given integral shall be

$$\begin{aligned}\int f(x)\,dx &= \int \Big(\frac{2x}{x^2+1}+\frac{1}{x^2+1}-\frac{2}{x-1}+\frac{1}{(x-1)^2}\Big)\,dx\\ &= \ln(x^2+1)+\tan^{-1}x-2\ln|x-1|-\frac{1}{x-1}+K,\end{aligned}$$

where K is a constant of integration.

Evaluate the following indefinite integral

$$\int \frac{1}{x(x^2+1)^2}\,dx$$

using the technique of partial fraction decomposition.

Write the integrand as a sum of partial fractions

$$f(x) = \frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Then we immediately have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A. \end{aligned}$$

Solution of Example 5 (2/2)

Equating coefficients of powers of x gives

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

After solving the system of linear equations, we obtain

$$A = 1, \quad B = -1, \quad C = -0, \quad D = -1, \quad E = 0.$$

and thus the given integral shall be

$$\begin{aligned} \int f(x)\,dx &= \int \Big(\frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2}\Big)\,dx \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \end{aligned}$$

Section 8.7
Improper Integrals
(瑕積分)

Two Types of Improper Integrals

Type I : Infinite Limits of Integration (無窮積分上下限), e.g.,

$$\int_a^{\infty} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{b} f(x)\,dx.$$

Type II : Infinite Discontinuities (無窮不連續點), e.g., we say that $\displaystyle\int_a^b f(x)\,dx$ is an improper integral of Type II if

$$\lim_{x \to c^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty$$

for some $c \in [a, b]$.

Type I: Infinite Limits of Integration (1/2)

(1) If f is conti. on $[a, \infty)$, the improper integral

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\left[\int_a^b f(x)\,dx\right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

(2) If f is conti. on $(-\infty, b]$, the improper integral

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\left[\int_a^b f(x)\,dx\right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

Type I: Infinite Limits of Integration (2/2)

(3) If f is conti. on $(-\infty, \infty)$, the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

converges (收斂) whenever the improper integrals on the RHS both converge for all $c \in \mathbb{R}$.

Note: we say that the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ diverges if either of the improper integrals on the RHS diverges.

Find the area under the curve

$$y = f(x) = \frac{\ln x}{x^2}$$

from $x = 1$ to $x = \infty$.

Example 1 的示意圖

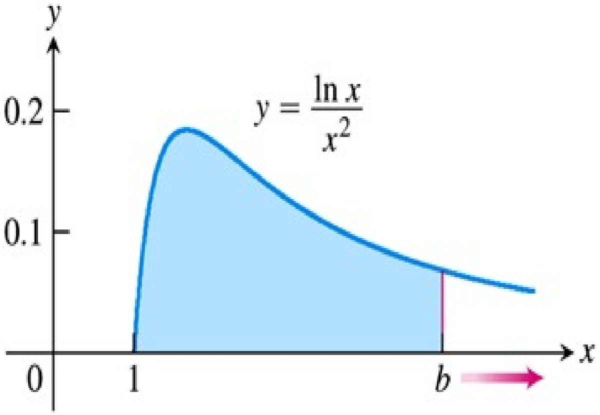


FIGURE 8.14 The area under this curve is an improper integral (Example 1).

Solution of Example 1

The area under $y = (\ln x)/x^2 \geq 0$ on $[1, \infty)$ is

$$\begin{aligned}
A &= \int_1^\infty \frac{\ln x}{x^2}\, dx = \lim_{b\to\infty} \int_1^b \ln x \, d(\frac{-1}{x}) \\
&= \lim_{b\to\infty} \Big(\frac{-\ln x}{x}\Big|_1^b - \int_1^b \frac{-1}{x^2}\, dx \Big) \\
&= \lim_{b\to\infty} \Big[\frac{-\ln b}{b} - \Big(\frac{1}{x}\Big|_1^b \Big) \Big] \\
&= \lim_{b\to\infty} \Big(\frac{-\ln b}{b} - \frac{1}{b} + 1 \Big) = 0 - 0 + 1 = 1,
\end{aligned}$$

since $\lim_{b\to\infty} \frac{\ln b}{b} = 0$ by L'Hôpital's Rule.

The improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges because

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \left(\int_a^0 \frac{dx}{1+x^2} \right) + \lim_{b\to\infty} \left(\int_0^b \frac{dx}{1+x^2} \right) \\
&= \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a) + \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) \\
&= \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\end{aligned}$$

Example 2 的示意圖

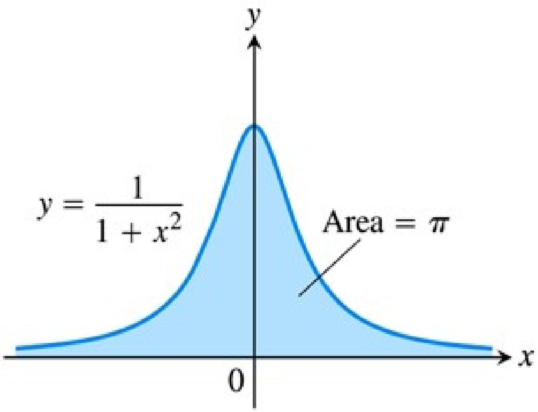


FIGURE 8.15 The area under this curve is finite (Example 2).

Type II: Infinite Discontinuities (1/2)

(1) If f is conti. on $[a, b)$ and $\lim_{x \to b^-} f(x) = \pm\infty$, the integral

$$\int_a^b f(x)\, dx = \lim_{c \to b^-} \left[\int_a^c f(x)\, dx \right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

(2) If f is conti. on $(a, b]$ and $\lim_{x \to a^+} f(x) = \pm\infty$, the integral

$$\int_a^b f(x)\, dx = \lim_{c \to a^+} \left[\int_c^b f(x)\, dx \right]$$

converges (收斂) whenever the limit exists. Otherwise, we say that the improper integral diverges (發散).

Type II: Infinite Discontinuities (2/2)

(3) If f is conti. on $[a, b]$ and has an infinite discontinuity at $c \in (a, b)$, the improper integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

converges (收斂) whenever the improper integrals on the RHS both converge.

Note: we say that the improper integral $\int_a^b f(x)\,dx$ diverges if either of the improper integrals on the RHS diverges.

Example 4 (第二型瑕積分的例子)

Determine the convergence of the improper integral

$$\int_0^1 \frac{1}{1-x}\,dx,$$

where $f(x) = \dfrac{1}{1-x}$ has an infinite discontinuity at $x = 1$.

Example 4 的示意圖

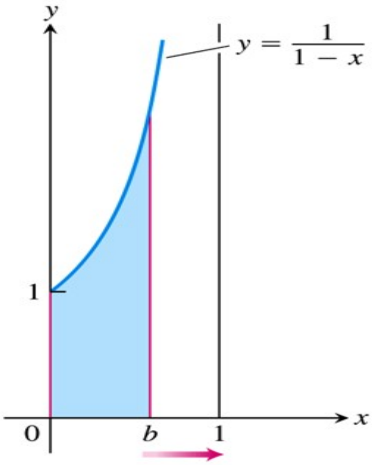


FIGURE 8.17 The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Note that it is an improper integral of Type II because $\lim_{x\to 1^-} \frac{1}{1-x} = \infty$ and we thus have

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \lim_{b\to 1^-} \left(-\ln|1-x| \Big|_0^b \right)$$
$$= \lim_{b\to 1^-} (-\ln|1-b| + \ln 1) = \infty.$$

So, the given integral diverges by Definition.

Evaluate the following improper integral

$$\int_0^3 \frac{dx}{(x-1)^{2/3}},$$

where $f(x) = \dfrac{1}{(x-1)^{2/3}}$ has an infinite discontinuity at $x = 1$.

Solution of Example 5

Since $\lim_{x\to 1} \frac{1}{(x-1)^{2/3}} = \infty$, it is an improper integral of Type II.

Then we see that

$$\begin{aligned}
\int_0^3 \frac{dx}{(x-1)^{2/3}} &= \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{2/3}} \\
&= \lim_{b\to 1^-} 3[(b-1)^{1/3}+1] + \lim_{a\to 1^+} 3[(2)^{1/3} - (a-1)^{1/3}] \\
&= 3 + 3\sqrt[3]{2},
\end{aligned}$$

and hence the improper integral converges by Definition.

The following improper integral of Type I satisfies

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \le 1. \end{cases}$$

Remark

This result will be used in the Integral Test (積分測試法) for determining the convergence of an infinite series (無窮級數的收斂性), see Section 9.3 for more details.

(1) If $p = 1$, then $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln|b| - \ln 1) = \infty$.

(2) If $p \neq 1$, it follows from Definition that

$$\int_1^\infty x^{-p}\,dx = \lim_{b\to\infty}\left(\frac{x^{-p+1}}{-p+1}\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{b^{-p+1}}{-p+1} + \frac{1}{p-1}\right)$$
$$= \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the desired result holds from (1) and (2).

Thm 2 (瑕積分的直接比較測試法)

Suppose that f and g are conti. on $[a, \infty)$ with $f(x) \le g(x) \quad \forall\, x \in [a, \infty)$.

(1) $\displaystyle\int_a^\infty g(x)\,dx$ converges $\Longrightarrow \displaystyle\int_a^\infty f(x)\,dx$ converges.

(2) $\displaystyle\int_a^\infty f(x)\,dx$ diverges $\Longrightarrow \displaystyle\int_a^\infty g(x)\,dx$ diverges.

Example (Thm 2 的例子; 補充題)

Does the following improper integral

$$\int_1^{\infty} e^{-x^2} \tan^{-1} x \, dx$$

converge or diverge? Give your reasons.

Since $x^2 \geq x$ for all $x \geq 1$, it follows that $-x^2 \leq -x$ for all $x \geq 1$ and hence we know that

$$f(x) \equiv e^{-x^2} \tan^{-1} x \leq \frac{\pi}{2} e^{-x} \equiv g(x) \qquad \forall\, x \in [1, \infty).$$

In addition, since the improper integral

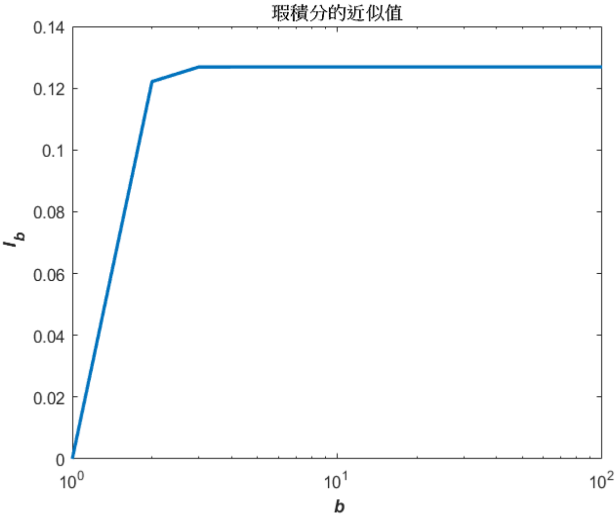
$$\int_1^\infty g(x)\, dx = \frac{\pi}{2} \int_1^\infty e^{-x}\, dx = \frac{\pi}{2} \lim_{b\to\infty} (-e^{-x})\Big|_1^b = \frac{\pi}{2e},$$

it follows from the Direct Comparison Theorem (Thm 2) that the improper integral $\int_1^\infty f(x)\, dx = \int_1^\infty e^{-x^2} \tan^{-1} x\, dx$ converges.

- What is the true value of $I = \int_1^\infty e^{-x^2} \tan^{-1} x \, dx$?
- Although it is usually difficult to get the true value of I, we may apply the technique of numerical integration (數值積分) to compute its approximate value.
- If $I_b \equiv \int_1^b e^{-x^2} \tan^{-1} x \, dx$ for $b \geq 1$, we see that

$$I = \lim_{b\to\infty} I_b \approx 0.1268694335685518.$$

Numerical Results (2/2)



Thank you for your attention!