Chapter 9
Infinite Sequences and Series
(無窮序列與級數)

Hung-Yuan Fan (范洪源)

Department of Mathematics,
National Taiwan Normal University, Taiwan

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本章預定授課範圍

Section 9.1
Sequences
(序列)

Sequences (序列)

An infinite sequence of real numbers is denoted by

$$\{a_n\} = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \cdots, a_n, \cdots\},$$

where a_n is the nth term (第 n 項) of the sequence for $n \in \mathbb{N}$.

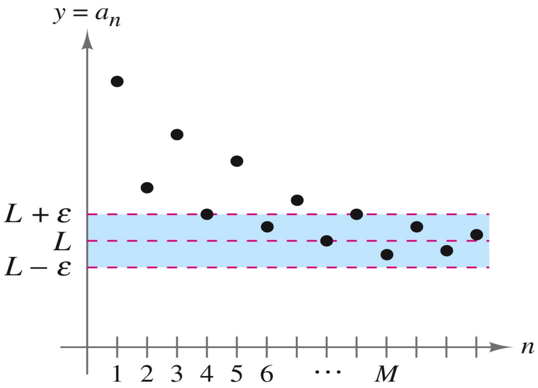
Def (序列的收斂性)

A sequence $\{a_n\}$ converges to a limit L, denoted by

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L,$$

if $\forall\, \varepsilon > 0$, $\exists\, M > 0$ s.t. $n > M \Longrightarrow |a_n - L| < \varepsilon$. Otherwise, we say that $\{a_n\}$ diverges if the limit does not exist.

示意圖 (承上頁)



Thm 4 (函數在 ∞ 處的極限 $\Rightarrow$ 序列的收斂性)

Let f be a real-valued function having the limit

$$\lim_{x\to\infty} f(x) = L.$$

If $\exists\, n_0 \in \mathbb{N}$ s.t. $a_n = f(n) \quad \forall\, n \geq n_0$, then

$$\lim_{n\to\infty} a_n = L.$$

Let $\varepsilon > 0$ be given arbitrarily. Since $\lim\limits_{x\to\infty} f(x) = L$, $\exists\, M > n_0$ s.t.

$$x > M \Longrightarrow |f(x) - L| < \varepsilon.$$

With $a_n = f(n)$ for all $n \geq n_0$, if $n > M$, then

$$|a_n - L| = |f(n) - L| < \varepsilon.$$

Thus, it follows from the Def. that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L.$$

Example (Thm 4 的反敘述不成立)

Consider $f(x) = \sin(\pi x)$ for all $x > 0$ and let $a_n = f(n)$ for $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = \lim_{n\to\infty} \sin(n\pi) = \lim_{n\to\infty} 0 = 0,$$

but we know that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin(\pi x) \quad \text{does not exist!}$$

Example 8 (Thm 4 的例子)

Does the sequence whose nth term is defined by

$$a_n = \left(\frac{n+1}{n-1}\right)^n, \quad n \geq 2$$

converge? If so, find its limit.

Let $f(x) = \left(\frac{x+1}{x-1}\right)^x$ for $x \geq 2$. Then $a_n = f(n) \quad \forall\, n \geq 2$.
Applying L'Hôpital's Rule, we see that

$$\begin{aligned}\lim_{x\to\infty} \ln f(x) &= \lim_{x\to\infty} x \ln \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{\ln(x+1) - \ln(x-1)}{1/x} \quad \text{(Type } 0\cdot\infty\text{)}\\ &= \lim_{x\to\infty} \frac{1/(x+1) - 1/(x-1)}{-1/x^2} = \lim_{x\to\infty} \frac{-2/(x^2-1)}{-1/x^2}\\ &= \lim_{x\to\infty} \frac{2x^2}{x^2-1} - \lim_{x\to\infty} \frac{2}{1-(1/x^2)} = 2,\end{aligned}$$

and hence $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^2$. It follows from Thm 4 that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = e^2.$$

Thm 1 (Limit Laws for Sequences)

Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$. Then

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$.
2. $\lim_{n\to\infty} (c \cdot a_n) = c \cdot L$ for all $c \in \mathbb{R}$.
3. $\lim_{n\to\infty} (a_n \cdot b_n) = L \cdot K$.
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$ if $b_n \neq 0 \quad \forall\, n \in \mathbb{N}$ and $K \neq 0$.

Note: $\lim_{n\to\infty} \frac{c}{n^r} = 0$ follows from Thm 4, where $c \in \mathbb{R}$ and $r > 0$.

(a) $\lim_{n\to\infty} \frac{-1}{n} = 0$ by Thm 4, since $\lim_{xx\to\infty} \frac{-1}{x} = 0$.

(b) $\lim_{n\to\infty} \frac{n-1}{n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1 - 0 = 1$.

(c) $\lim_{n\to\infty} \frac{5}{n^2} = 5\Big(\lim_{n\to\infty} \frac{1}{n}\Big)\Big(\lim_{n\to\infty} \frac{1}{n}\Big) = 5(0)()0) = 0.$

(d) $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{(4/n^6)-7}{1+(3/n^6)} = -7.$

Thm 2 (Sandwich Theorem for Sequences)

If $\exists\, M > 0$ s.t. $a_n \leq b_n \leq c_n \quad \forall\, n > M$, and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges to the same limit L, i.e., $\lim_{n\to\infty} b_n = L$.

Example 4 (Thm 2 的例子; 1/2)

(a) $\lim_{n\to\infty} \frac{\cos n}{n} = 0$ because $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$.

(b) $\lim_{n\to\infty} \frac{1}{2^n} = 0$ because $0 \le \frac{1}{2^n} \le \frac{1}{n}$.

(c) $\displaystyle\lim_{n\to\infty}\frac{(-1)^n}{n}=0$ because $\displaystyle\frac{-1}{n}\leq\frac{(-1)^n}{n}\leq\frac{1}{n}$.

(d) $\displaystyle\lim_{n\to\infty}|a_n|=0\Longrightarrow\lim_{n\to\infty}a_n=0$ because $-|a_n|\leq a_n\leq|a_n|$.

Thm (Absolute Value Thoerem)

Let $\{a_n\}$ be a sequence of real numbers. Then

$$\lim_{n\to\infty} |a_n| = 0 \iff \lim_{n\to\infty} a_n = 0.$$

Thm 5 (某些特殊序列的極限)

The following limits hold:

(1) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(2) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n} = 1$

(3) $\lim_{n\to\infty} x^{1/n} = 1 \quad \forall\, x > 0$

(4) $\lim_{n\to\infty} x^n = 0 \iff |x| < 1$

(5) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad \forall\, x \in \mathbb{R}$

(6) $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall\, x \in \mathbb{R}$

Note: Items (1)–(5) can be shown by Thm 4 and L'Hôpital's Rule.

Proof of Item (6)

Let $x \in \mathbb{R}$ be given arbitrarily. It suffices to show $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$.

(*) For $|x| \le 1$, it follows from Thm 2 that $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ because

$$0 \le \frac{|x|^n}{n!} \le \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

(**) For $|x| > 1$, choose the smallest $N \in \mathbb{N}$ s.t. $|x| \le N$. If $n > N$,

$$0 \le \frac{|x|^n}{n!} = \frac{|x|}{n}\left[\frac{|x|}{n-1} \cdot \frac{|x|}{n-2} \cdots \frac{|x|}{N}\right]\frac{|x|^{N-1}}{(N-1)!} \le \frac{|x|^N}{n(N-1)!}.$$

Thus, $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$ by Thm 2 with $\lim_{n\to\infty} \frac{|x|^N}{n(N-1)!} = 0$.

From (*) and (**), it follows that

$$\lim_{n\to\infty}\frac{|x|^n}{n!}=0 \Longrightarrow \lim_{n\to\infty}\frac{x^n}{n!}=0 \quad \forall\, x\in\mathbb{R}$$

by the Absolute Value Theorem for Sequences.

(a) $\lim_{n\to\infty} \frac{\ln n^2}{n} = \lim_{n\to\infty} \frac{2\ln n}{n} = 2 \cdot 0 = 0.$

(b) $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} (n^2)^{1/n} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} (n^{1/n})^2 = 1.$

(c) $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} (3^{1/n} \cdot n^{1/n}) = (1)(1) = 1.$

(d) $\lim_{n\to\infty} (\frac{-1}{2})^n = 0$ because $|-1/2| = 1/2 < 1$.

(e) $\lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2}$.

(f) $\lim_{n\to\infty} \frac{(100)^n}{n!} = 0$ by the item (6) of Thm 5.

Def (單調序列的定義)

A sequence $\{a_n\}$ is said to be monotonic (單調的) if either its terms are nondecreasing (非遞減的 or 非嚴格遞增的), i.e.,

$$a_n \leq a_{n+1} \quad \forall\, n \in \mathbb{N},$$

or its terms are nonincreasing (非遞增的 or 非嚴格遞減的), i.e.,

$$a_n \geq a_{n+1} \quad \forall\, n \in \mathbb{N}.$$

Def (有界序列的定義)

(1) The sequence $\{a_n\}$ is bounded from above (於上有界) if $\exists\, M \in \mathbb{R}$ s.t. $a_n \le M \quad \forall\, n \in \mathbb{N}$.

(2) The sequence $\{a_n\}$ is bounded from below (於下有界) if $\exists\, N \in \mathbb{R}$ s.t. $a_n \ge N \quad \forall\, n \in \mathbb{N}$.

(3) The sequence $\{a_n\}$ is bounded (有界的) if it is bounded from above and below, i.e. $\exists\, M > 0$ s.t. $|a_n| \le M \quad \forall\, n \in \mathbb{N}$.

Thm 6 (Bounded Monotonic Sequences)

If the sequence $\{a_n\}$ is bounded and monotonic, then it converges, i.e., $\exists$ unique $L \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = L$.

Thm 6 (Bounded Monotonic Sequences)

If the sequence $\{a_n\}$ is bounded and monotonic, then it converges, i.e., $\exists$ unique $L \in \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = L$.

Example (Thm 6 的例子)

The sequence $\{a_n\} = \{1/n\}$ is bounded and nonincreasing, since

$$|a_n| \leq 1 \quad \text{and} \quad a_n = \frac{1}{n} \geq \frac{1}{n+1} = a_{n+1} \quad \forall\, n \in \mathbb{N}.$$

So, it must converge by Thm 6 with $\lim_{n\to\infty} a_n = 0$.

Section 9.2
Infinite Series
(無窮級數)

Def (Partial Sums of a Series)

(a) An infinite series (無窮級數) of real numbers is denoted by

$$\sum a_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots ,$$

where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$.

(b) For each $n \in \mathbb{N}$, the nth partial sum (第 n 個部分和) of $\sum a_n$ is defined by $S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

Def (無窮級數的收斂性)

(a) We say that $\sum a_n$ converges if the sequence $\{S_n\}$ converges with $\lim_{n\to\infty} S_n = S$. In this case, S is called the sum of the series and write $S = \sum a_n$.

(b) We say that $\sum a_n$ diverges if the sequence $\{S_n\}$ diverges.

Two Questions

1. Does a given series converge or diverge?
2. What is the sum of a *convergent* series?

Note: these questions are not always easy to answer, especially the second one. (通常需要藉助數值方法求得近似和!)

Example 5 (利用定義求級數和)

Find the sum of the telescoping series given by

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right),$$

using the Definition directly.

Solution of Example 5

Since the nth partial sum of the given series is

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \Big(\frac{1}{i} - \frac{1}{i+1}\Big) \\ &= \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\ &= 1 - \frac{1}{n+1} \quad \forall\, n \in \mathbb{N}, \end{aligned}$$

it follows from Definition that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \Big(1 - \frac{1}{n+1}\Big) = 1.$$

For $a \neq 0$ and ratio (公比) $r \neq 0$, the geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots .$$

Thm (幾何級數的收斂性)

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1, \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

Example 2 (幾何級數的例子)

The series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \sum_{n=0}^{\infty} 5\left(\frac{-1}{4}\right)^n$ is a geometric series with $a = 5$ and $r = \frac{-1}{4}$. Since $|r| = 1/4 < 1$, the series coverges to

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{5}{1 - (\frac{-1}{4})} = \frac{5}{1 + \frac{1}{4}} = 4.$$

Example (補充題; 幾何級數的例子)

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n = \frac{3}{1-(1/2)} = 6$ by above Thm.

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ diverges because the ratio $|r| = |3/2| = 3/2 \geq 1$.

Thm 8 (無窮級數的基本性質)

Suppose that $\sum a_n = A$ and $\sum b_n = B$ are convergent series.

1. $\sum (c \cdot a_n) = c \cdot \left(\sum a_n\right) = c \cdot A$ for all $c \in \mathbb{R}$.
2. $\sum (a_n \pm b_n) = \left(\sum a_n\right) \pm \left(\sum b_n\right) = A \pm B$.
3. In general, it is also true that

$$\sum (a_n b_n) \neq \left(\sum a_n\right)\left(\sum b_n\right) \text{ and } \sum \left(\frac{a_n}{b_n}\right) \neq \frac{\sum a_n}{\sum b_n}.$$

Example 9 (Thm 8 的例子)

(a) $$\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = \frac{1}{1-(1/2)} - \frac{1}{1-(1/6)} = 2 - \frac{6}{5} = \frac{4}{5}.$$

(b) $$\sum_{n=0}^{\infty} \frac{4}{2^n} = 4\sum_{n=0}^{\infty} \frac{1}{2^n} = 4 \cdot \frac{1}{1-(1/2)} = 4 \cdot 2 = 8.$$

Thm 7 (收斂級數的必要條件)

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

(收斂級數第 n 項所形成的序列必定趨近 0!)

If the series $\sum a_n = S$ converges, then we know that

$$\lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{n-1},$$

where $S_n = \sum_{i=1}^{n} a_i$ is the nth partial sum of $\sum a_n$. So, we immediately obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$$

Thm (The *n*th Term Test; 第 n 項測試法)

If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum a_n$ diverges.

(若第 n 項不趨近到 0, 則級數必定發散!)

Note: 此定理為 Thm 7 的反敘述，常用於判斷級數的發散性。

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $\lim_{n\to\infty} n^2 = \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$.

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges because $\displaystyle\lim_{n\to\infty}(-1)^{n+1} \nexists$.

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{-n}{2n+5}$ diverges because $\displaystyle\lim_{n\to\infty}\frac{-n}{2n+5} = \frac{-1}{2} \neq 0$.

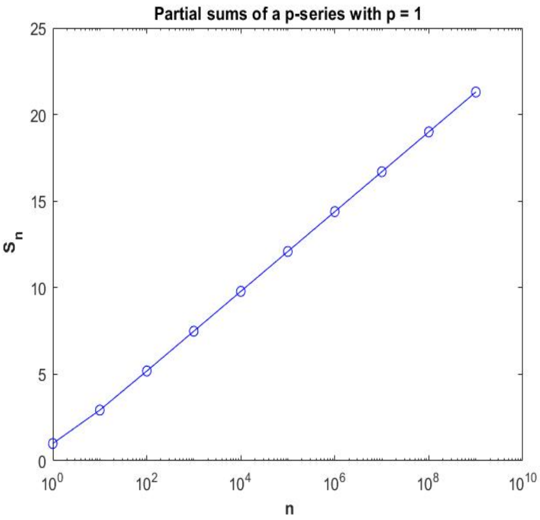
Example (Thm 7 的反例)

The harmonic series (調和級數) of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

satisfies $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$, but it is a divergent series!

Divergence of Harmonic Series (示意圖)



Section 9.3
The Integral Test
(積分測試法)

Thm 9 (The Integral Test; 積分測試法)

If $\exists N \in \mathbb{N}$ s.t. f is positive, conti. and decreasing ($\searrow$) on $[N, \infty)$, and $a_n = f(n) \quad \forall n \geq N$, then $\displaystyle\sum_{n=N}^{\infty} a_n$ and $\displaystyle\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Note: 上述定理只說明級數與瑕積分是同收同發，但並沒有說明兩者相等喔!

Proof of Thm 9 (1/2)

For any positive integer $n \geq N+1$, from the definitions of lower and upper sums, we see that

$$\text{Lower Sum} = \sum_{i=N+1}^{n} f(i) \leq \int_N^n f(x)\,dx \leq \sum_{i=N}^{n-1} f(i) = \text{Upper Sum}$$

$$\Longrightarrow S_n - S_N = \sum_{i=N+1}^{n} a_i \leq \int_N^n f(x)\,dx \leq \sum_{i=N}^{n-1} a_i = S_{n-1} - S_{N-1}.$$

Furthermore, from M.V.T. for definite integrals, $\exists\, c_n \in [N, n]$ s.t.

$$f(c_n) = \frac{1}{n-N}\int_N^n f(x)\,dx \Longrightarrow \int_N^n f(x)\,dx = (n-N)f(c_n) > 0.$$

Therefore, the following statements hold:

- $\int_N^\infty f(x)\,dx$ converges $\Longrightarrow \lim_{n\to\infty}(S_n - S_N)\ \exists \Longrightarrow \lim_{n\to\infty} S_n\ \exists$ or, equivalently, $\sum a_n$ converges.
- $\int_N^\infty f(x)\,dx$ diverges $\Longrightarrow \lim_{n\to\infty}\int_N^n f(x)\,dx = \lim_{n\to\infty}(n-N)f(c_n) = \infty \Longrightarrow$ $\lim_{n\to\infty}(S_{n-1} - S_{N-1}) = \infty \Longrightarrow \lim_{n\to\infty} S_{n-1} = \infty$ or, equivalently, $\sum a_n$ diverges.

Example 5 (Thm 9 的例子)

Determine the convergence or divergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(a) Let $f(x) = xe^{-x^2}$ for $x \geq 1$. Then $a_n := ne^{-n^2} = f(n) \quad \forall\, n \geq 1$, and f is positive and conti. on $[1, \infty)$. Moreover, f is ↘ on that interval because

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = (1 - 2x^2)e^{-x^2} < 0 \quad \forall\, x \in [1, \infty).$$

Since the improper integral of Type I

$$\int_1^\infty f(x)\, dx = \frac{1}{2}\int_1^\infty \frac{du}{e^u} = \lim_{b\to\infty} \frac{-1}{2}(e^{-b} - e^{-1}) = \frac{1}{2e},$$

it follows that $\sum_{n=1}^{\infty} a_n$ converges by the Integral Test.

(b) Let $f(x) = \frac{1}{2^{\ln x}}$ for $x \geq 1$. Then $a_n := 2^{-\ln n} = f(n) \quad \forall\, n \geq 1$, and f is positive, conti. and decreasing on $[1, \infty)$. Since the improper integral of Type I

$$\int_1^\infty f(x)\,dx = \int_0^\infty \frac{e^u\,du}{2^u} = \lim_{b\to\infty} \int_0^b \left(\frac{e}{2}\right)^u du$$
$$= \lim_{b\to\infty} \frac{1}{\ln(e/2)}\left[\left(\frac{e}{2}\right)^b - 1\right] = \infty,$$

it follows from the Integral Test that $\sum_{n=1}^{\infty} a_n$ is divergent.

Def (p-級數的定義)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series (p-級數) with $p \in \mathbb{R}$.

(b) If $p = 1$, then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is called a harmonic series (調和級數).

Example 3 (p-級數的收斂與發散)

Show that the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$, respectively.

Note

The example does not give the sum of a p-series with $p > 1$, but this sum can be evaluated numerically!

Solution of Example 3

- The series $\sum \frac{1}{n^p}$ diverges for $p \leq 0$ by the nth Term Test.
- For $p > 0$, the function $f(x) = \dfrac{1}{x^p}$ is positive, conti. and $\searrow$ on $[1, \infty)$. From Example 3 of Section 8.7, we know that

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \text{diverges}, & p \leq 1. \end{cases}$$

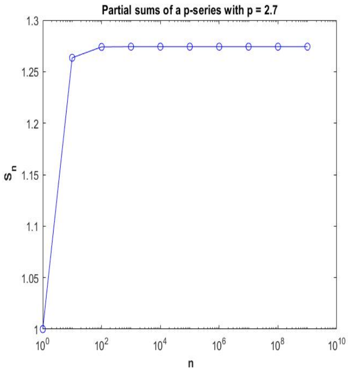
Thus, it follows from the Integral Test that the p-series $\sum \dfrac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$.

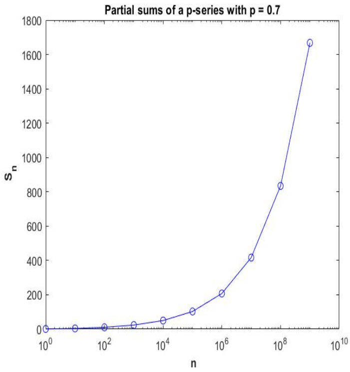
計算 p-級數的和 (檔名: p_series.m)

```
p = 2.7;  data = [];
for k = 0:8
  N = 10^k;
  n = 1:N;
  S_N = sum(1./(n.^p));
  data = [data;N S_N];
end
semilogx(data(:,1),data(:,2),'bo-');
title('Partial sums of a p-series with p = 2.7');
xlabel('\bf n');
ylabel('\bf S_n');
```

Note: $\sum_{n=1}^{\infty} \frac{1}{n^{2.7}} \approx \mathbf{1.274265}$.

下列圖形顯示 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ 的收斂與發散，其中 $p = 2.7$ 和 $p = 0.7$:





Section 9.4
Comparison Tests
比較測試法)

Thm 10 (Direct Comparison Test; 直接比較法)

If $\exists\, N \in \mathbb{N}$ s.t. $0 \leq a_n \leq b_n \quad \forall\, n \geq N$, then

1. $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges, and
2. $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges.

口訣:

(1) 大的級數收斂保證小的級數也收斂!

(2) 小的級數發散保證大的級數也發散!

Example 1 (Thm 10 的例子)

(a) The series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges by the Direct Comparison Test, since $\frac{5}{5n-1} \geq \frac{5}{5n} = \frac{1}{n} \quad \forall\, n \geq 1$ and $\sum \frac{1}{n}$ diverges.

(b) The seres $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges by the Direct Comparison Test, since $\frac{1}{n!} \leq \frac{1}{2^n} \quad \forall\, n \geq 4$ and $\sum \frac{1}{2^n}$ converges.

(c) The seres $\sum_{n=0}^{\infty} \frac{1}{2^n + \sqrt{n}}$ converges by the Direct Comparison Test, since $\frac{1}{2^n + \sqrt{n}} \leq \frac{1}{2^n} \quad \forall\, n \geq 1$ and $\sum \frac{1}{2^n}$ converges.

Thm 11 (Limit Comparison Test; 極限比較法)

$\exists\, N \in \mathbb{N}$ s.t. $a_n, b_n > 0 \quad \forall\, n \geq N$ and let $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$.

1. If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $L = 0$, then $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges, and $\sum a_n$ diverges $\Longrightarrow \sum b_n$ diverges.
3. If $L = \infty$, then $\sum a_n$ converges $\Longrightarrow \sum b_n$ converges, and $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

Example 2 (Thm 11 的例子)

Determine the convergence or divergence of the series.

(a) $$\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{2^n-1}$$

(c) $$\sum_{n=2}^{\infty} \frac{1+n\ln n}{n^2+5}$$

(a) Let $a_n = \dfrac{2n+1}{n^2+2n+1}$ and $b_n = \dfrac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n, b_n > 0 \quad \forall\, n \in \mathbb{N}$, and we also see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 > 0.$$

So, it follows from Thm 11 that $\sum a_n$ diverges because $\sum b_n = \sum \frac{1}{n}$ is a divergent harmonic series with $p = 1$.

(b) Let $a_n = \dfrac{1}{2^n - 1}$ and $b_n = \dfrac{1}{2^n}$ for $n \in \mathbb{N}$. Then $a_n, b_n > 0 \quad \forall\, n \in \mathbb{N}$, and we also see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1 > 0.$$

So, it follows from Thm 11 that $\sum a_n$ converges because $\sum b_n = \sum \frac{1}{2^n}$ is a convergent geometric series.

(c) Let $a_n = \dfrac{1 + n \ln n}{n^2 + 5}$ and $b_n = \dfrac{1}{n}$ for $n \in \mathbb{N}$. Then $a_n, b_n > 0 \quad \forall\, n \in \mathbb{N}$, and we also see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n + n^2 \ln n}{n^2 + 5} = \infty.$$

So, it follows from Thm 11 that $\sum a_n$ diverges because $\sum b_n = \sum \frac{1}{n}$ is divergent series. In fact, it is easily seen that

$$a_n \geq \frac{n \ln n}{2n^2} = \frac{\ln n}{2n} \geq \frac{1}{2n} = \frac{1}{2} b_n, \quad n \geq 3.$$

Does the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Observation

From Thm 4, $\lim_{n\to\infty} \frac{\ln n}{n^c} = 0$ holds for any $c > 0$. Therefore, $\exists\, N \in \mathbb{N}$ s.t. if $n > N$, then $\left|\frac{\ln n}{n^c}\right| < \frac{1}{2} \Longrightarrow \ln n < \frac{n^c}{2} < n^c$.

Solution of Example 3

Since $\ln n \leq n^c$ for $c > 0$ and n sufficiently large, we see that

$$a_n = \frac{\ln n}{n^{3/2}} \leq \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}} = b_n$$

for n sufficiently large. Then $a_n, b_n > 0$ for $n \geq 1$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{5/4} \ln n}{n^{3/2}} = \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} = 0$$

by Thm 4 and the L'Hôpital's Rule. Thus, $\sum a_n$ coverges because $\sum b_n$ is a convergent series with $p = 5/4 > 1$.

Section 9.5
Absolute Convergence: The Ratio and Root Tests
(絕對收斂: 比值法與根式法)

Def (無窮級數的收斂型態)

(1) $\sum a_n$ is absolutely convergent (絕對收斂) or converges absolutely if $\sum |a_n|$ converges.

(2) $\sum a_n$ is conditionally convergent (條件收斂) or coverges conditionally if $\sum a_n$ converges, but $\sum |a_n|$ diverges.

Thm 12 (The Absolute Convergence Test)

$\sum |a_n|$ converges $\Longrightarrow \sum a_n$ converges.

(絕對收斂保證原級數收斂)

Note: All conditionally convergent series are counterexamples of Thm 12.

Proof of Thm 12

Since $-|a_n| \leq a_n \leq |a_n|$ for all $n \in \mathbb{N}$, we see that

$$0 \leq a_n + |a_n| \leq 2|a_n| \quad \forall\, n \in \mathbb{N}.$$

Then $\sum(a_n + |a_n|)$ converges by the Direct Comparison Test because $\sum |a_n|$ is a convergent series. Therefore, we conclude that

$$\sum a_n = \sum \Big[(a_n + |a_n|) - |a_n|\Big] = \sum(a_n + |a_n|) - \sum |a_n|$$

converges by the Difference Rule for series.

Example 1 (Thm 12 的例子)

(a) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges (absolutely) by Thm 12, since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series.

(b) The series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges (absolutely) by the Direct Comparison Tst, since $\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series with $p = 2 > 1$.

Thm 13 (The Ratio Test; 比值法)

Let $a_n \neq 0 \quad \forall\, n \in \mathbb{N}$ with $\rho = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$.

1. $\rho < 1 \implies \sum a_n$ converges absolutely.
2. $\rho > 1$ or $\rho = \infty \implies \sum a_n$ diverges.
3. $\rho = 1 \implies$ the test is inconclusive.

Example 2 (Thm 13 的例子)

Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+5}{3^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

(a) Let $a_n = \dfrac{2^n + 5}{3^n}$ for $n \geq 0$. Then we see that

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1} + 5}{3^{n+1}} \cdot \frac{3^n}{2^n + 5} \right|$$
$$= \lim_{n\to\infty} \frac{2 + 5 \cdot 2^{-n}}{3(1 + 5 \cdot 2^{-n})} = \frac{2}{3} < 1.$$

Thus, from the Ratio Test that, $\sum a_n$ converges (absolutely).

Solution of Example 2 (2/3)

(b) Let $a_n = \dfrac{(2n)!}{n!n!}$ for $n \geq 1$. Then we see that

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!} \right|$$
$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty} \frac{2(2n+1)}{n+1} = 4 > 1.$$

Thus, $\sum a_n$ diverges by the Ratio Test.

(c) Let $a_n = \dfrac{4^n n! n!}{(2n)!}$ for $n \geq 1$. Then we see that

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{4^n n! n!} \right|$$
$$= \lim_{n\to\infty} \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{2(n+1)}{2n+1} = 1.$$

So, the Ratio Test fails! Since $\dfrac{a_{n+1}}{a_n} \geq 1$ or $a_{n+1} \geq a_n$ for $n \geq 1$, we have $a_n \geq a_1 = 2 \quad \forall\, n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} a_n \geq 2$ and thus $\lim\limits_{n\to\infty} a_n \neq 0$. So, $\sum a_n$ diverges by the nth Term Test.

Thm 14 (The Root Test; 根式法)

Let $a_n \neq 0 \quad \forall\, n \in \mathbb{N}$ with $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(|a_n|\right)^{\frac{1}{n}}$.

1. $\rho < 1 \implies \sum a_n$ converges absolutely.
2. $\rho > 1$ or $\rho = \infty \implies \sum a_n$ diverges.
3. $\rho = 1 \implies$ the test is inconclusive.

Example 3 (Thm 14 的例子)

Investigate the convergence of the series $\sum a_n$, where $a_n = \dfrac{n}{2^n}$ if n is odd and $a_n = \dfrac{1}{2^n}$ if n is even.

Solution of Example 3

We first note that

$$\frac{1}{2} = \left(\frac{1}{2^n}\right)^{1/n} \le |a_n|^{1/n} \le \left(\frac{n}{2^n}\right)^{1/n} = \frac{n^{1/n}}{2}$$

for all $n \ge 1$. Since $n^{1/n} \to 1$ as $n \to \infty$, it follows from the Sandwich Thm that $\rho = \lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{2} < 1$. Thus, $\sum a_n$ converges (absolutely) by the Root Test.

Section 9.6
Alternating Series and Conditional Convergence
(交錯級數與條件收斂)

An alternating series (交錯級數) is of the form

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots,$$

where the nth term $u_n > 0$ for all $n \in \mathbb{N}$.

Questions

- Does the alternating series *always* converge?
- How to estimate the sum S of a *convergent* alternating series?

Thm 15 (The Alternating Series Test; 交錯級數測試法)

If the sequence of positive terms $\{u_n\}$ satisfies

1. $\exists\, N \in \mathbb{N}$ s.t. $u_n \geq u_{n+1}$ for all $n \geq N$, and
2. $\lim_{n\to\infty} u_n = 0$,

then the alternating series $S = \sum (-1)^{n+1} u_n$ converges and its sum satisfies the following property

$$|S_n - S| \leq u_{n+1} \quad \forall\, n \in \mathbb{N}.$$

Example 1 (Thm 15 的例子)

The alternating harmonic series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

converges (to $\ln 2$) by the Alternating Series Test, since we see that

$$u_n \geq u_{n+1} \ \forall\, n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

But, the harmonic series of the form

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} u_n \right| = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

is a divergent p-series with $p = 1$.

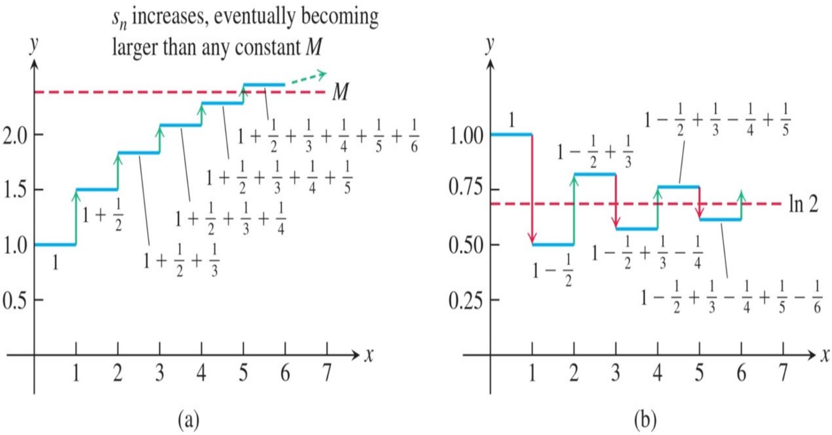


FIGURE 9.16 (a) The harmonic series diverges, with partial sums that eventually exceed any constant. (b) The alternating harmonic series converges to $\ln 2 \approx .693$.

Example 2 (證明序列的遞減性)

Show that the following sequence of real numbers

$$\{u_n\} = \left\{\frac{10n}{n^2+16}\right\}$$

is eventually nonincreasing (最終非遞增的), i.e., $\exists\, N \in \mathbb{N}$ s.t. $u_n \geq u_{n+1}$ for all $n \geq N$.

Let $f(x) = \dfrac{10x}{x^2+16}$ for $x \geq 1$. Then the derivative of f satisfies

$$f'(x) = \frac{10(x^2+16) - 10x(2x)}{(x^2+16)^2} = \frac{10(16-x^2)}{(x^2+16)^2} \leq 0 \quad \text{whenever } x \geq 4.$$

Since $u_n = f(n) \quad \forall\, n \in \mathbb{N}$, we immediately conclude that

$$u_n = f(n) \geq f(n+1) = u_{n+1} \quad \forall\, n \geq 4.$$

Def (條件收斂的定義)

A series $\sum a_n$ is conditionally convergent (條件收斂) if $\sum a_n$ converges, but $\sum |a_n|$ diverges, i.e., $\sum a_n$ is the convergent series that does not converge absolutely.

Example 4 (條件收斂的例子)

For $p > 0$, the sequence $\{u_n\} = \{\frac{1}{n^p}\}$ is decreasing with $\lim_{n\to\infty} u_n = 0$. Thus, the alternating p-series

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$$

converges by the Alternating Series Test. But, it converges absolutely for $p > 1$ and converges conditionally for $0 < p \leq 1$, respectively, since it is well known that

$$\sum_{n=1}^{\infty} |(-1)^{n-1} u_n| = \sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges for } p > 1, \\ \text{diverges for } 0 < p \leq 1. \end{cases}$$

Section 9.7
Power Series
(冪級數)

Def (以 c 點為中心的冪級數)

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

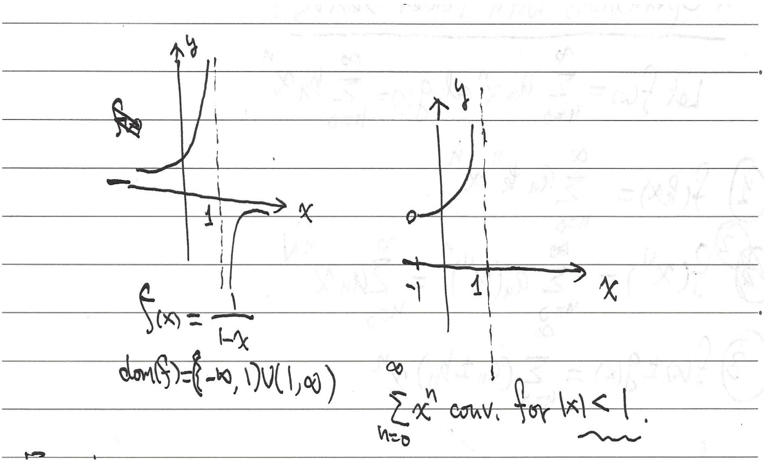
is called a power series (冪級數) centered at $c \in \mathbb{R}$ or a power series about $x = c$.

Example 1 (Geometric Power Series; 幾何冪級數)

- The function $f(x) = \dfrac{1}{1-x}$ is well-defined for $x \neq 1$.
- f has a geometric power series centered at $x = 0$, i.e.,

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$ or $-1 < x < 1$.


y
x
1
f(x) = 1/(1-x)
dom(f) = (-∞, 1) ∪ (1, ∞)
y
0
-1
1
x
∑_{n=0}^{∞} x^n conv. for |x| < 1.

The power series about $x = 2$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots$$

is a geometric series with first term $a = 1$ and ratio $r = \dfrac{-(x-2)}{2}$.
Thus, we know that

$$\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n = \frac{a}{1-r} = \frac{1}{1+\frac{x-2}{2}} = \frac{2}{x}$$

converges for $|r| = \dfrac{|x-2|}{2} < 1$ or $0 < x < 4$.

Some (Taylor) polynomial approximations of $y = f(x) = \dfrac{2}{x}$ for values of x near 2 are given by

$$\begin{aligned} y_0 &= P_0(x) := 1, \\ y_1 &= P_1(x) := 1 - \frac{x-2}{2} = 2 - \frac{x}{2}, \\ y_2 &= P_2(x) := 1 - \frac{x-2}{2} + \frac{(x-2)^2}{4} = 3 - \frac{3x}{2} + \frac{x^2}{4}. \end{aligned}$$

Example 2 的示意圖

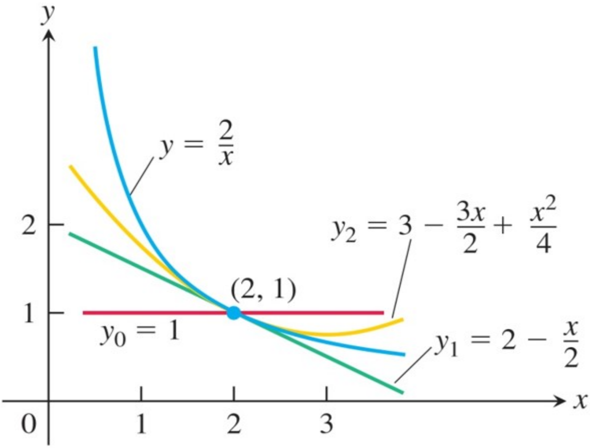


FIGURE 9.18 The graphs of $f(x) = 2/x$ and its first three polynomial approximations (Example 2).

Facts (冪級數的收斂行為)

For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, one of the following holds:

Type I: It converges only at $x = c$.

Type II: It converges only for $|x-c| < R$ with $R > 0$.

Type III: It converges for all $x \in \mathbb{R}$.

We denote by

$$R = \text{Radius of Convergence (收斂半徑)},$$
$$I = \text{Interval of Convergence (收斂區間)}.$$

Type I of Convergence

$$\sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges only at } x = c.$$

$$\Longrightarrow R = 0 \quad \text{and} \quad I = \{c\}.$$

For what values of x does the following power series

$$\sum_{n=0}^{\infty} n! x^n$$

converge? Give your reasons.

Let $a_n = n!x^n$ for $n \geq 0$. Then

(1) If $x = 0$, then $\sum_{n=0}^{\infty} a_n = a_0 = 1$ converges.

(2) For $x \neq 0$, we see that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!x^{n+1}}{n!x^n} \right| = \lim_{n\to\infty} (n+1)|x| = \infty.$$

So, $\sum a_n = \sum n!x^n$ diverges for $x \neq 0$ by the Ratio Test.

From (1) and (2), we know that $\sum_{n=0}^{\infty} n!x^n$ converges only at $x = 0$.

Type II of Convergence

$$\sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges absolutely for } |x-c|<R, \text{ and}$$

diverges for $|x-c|>R$.

$$\Longrightarrow R>0 \quad \text{and} \quad I=(c-R, c+R).$$

Endpoint Convergence of a Power Series

Note (冪級數的端點收斂)

Besides the interval of convergence $I = (c - R, c + R)$ in Type II, we may also have

$$I = [c - R, c + R), \quad (c - R, c + R] \quad \text{or} \quad [c - R, c + R]$$

for the endpoint convergence of a power series.

For what values of x does the following power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

converge? Give your reasons.

Solution of Example 3a (1/2)

Let $a_n = (-1)^{n-1}\dfrac{x^n}{n}$ for $n \geq 1$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n x^{n+1}}{n+1} \times \frac{n}{(-1)^{n-1}x^n}\right|$$
$$= \lim_{n\to\infty}\frac{n|x|}{n+1} = |x|.$$

So, $\sum a_n$ converges (absolutely) for $|x| < 1$ adn diverges for $|x| > 1$ by the Ratio Test.

(1) At $x = -1$, $\sum a_n = \sum \frac{-1}{n}$ diverges with $p = 1$.

(2) At $x = 1$, $\sum a_n = \sum \frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test.

From (1) and (2), the interval of convergence is $I = (-1, 1]$.

Remark

In fact, it is also true that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $-1 < x \leq 1$. See Example 6 shown below.

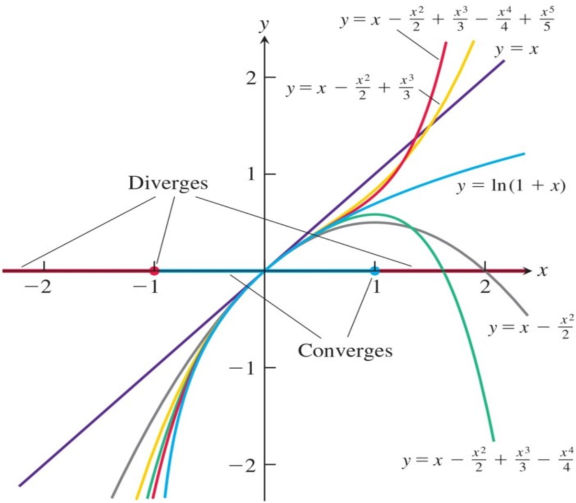


FIGURE 9.19 The power series $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ converges on the interval $(-1, 1]$.

The interval of convergence of the power series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

is $I = [-1, 1]$. In fact, it converges to $\tan^{-1} x$ there.

Type III of Convergence

$$\sum_{n=0}^{\infty} a_n(x-c)^n \text{ converges for all } x \in \mathbb{R}.$$

$$\Longrightarrow R = \infty \quad \text{and} \quad I = (-\infty, \infty).$$

Example 3c (Type III 的例子)

For what values of x does the following power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converge? Give your reasons.

Solution of Example 3c

Let $a_n = \frac{x^n}{n!}$ for $n \geq 0$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right|$$
$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \quad \forall x \in \mathbb{R}.$$

So, $\sum a_n = \sum \frac{x^n}{n!}$ converges (absolutely) for all $x \in \mathbb{R}$ by the Ratio Test.

Remarks

- From Example 3c and Thm 7, we immediately obtain the item (6) of Thm 5, i.e., $\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \forall\, x \in \mathbb{R}$.
- In Example 1 of Section 9.9, we will further show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \forall\, x \in \mathbb{R}.$$

For the cases of Type II or Type III, we consider a real-valued function f defined by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \forall\, x \in I,$$

where $I = (c-R, c+R)$ with $R > 0$ or $I = (-\infty, \infty)$.

Two Questions

- Is f differentiable and integrable on the open interval I?
- If yes, what are $f'(x)$ and $\displaystyle\int f(x)\,dx$?

Thm 21 (Term-by-Term Differentiation; 逐項微分)

If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is well-defied on $I = (c-R, c+R)$ with $R > 0$ being its radius of convergence, then f has derivatives of all orders with

$$f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}[a_n(x-c)^n] = \sum_{n=1}^{\infty} na_n(x-c)^{n-1} \quad \forall x \in I,$$

$$f''(x) == \sum_{n=2}^{\infty} n(n-1)a_n(x-c)^{n-2} \quad \forall x \in I,$$

$$\vdots$$

Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1.$$

Applying Thm 21 twice, we immediately obtain

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}, \quad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1 < x < 1.$$

Thm 22 (Term-by-Term Integration; 逐項積分)

If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is well-defied on $I = (c-R, c+R)$ with $R > 0$ being its radius of convergence, then

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \left[\int a_n(x-c)^n\,dx \right]$$
$$= C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1} \quad \forall x \in I,$$

where C is the constant of integration.

Remarks

(1) The radii of convergence of $f'(x)$ and $\int f(x)\,dx$ are the same as that of $f(x) = \sum a_n(x-c)^n$.

(2) But, their intervals of convergence may differ from f.

(微分和積分後的收斂半徑與 f 相同，但收斂區間略有不同!)

Since the interval of convergence of $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ is $I = [-1, 1)$ by the Ratio Test and the Alternating Series Test, we see that

- $f'(x) = \sum_{n=2}^{\infty} x^{n-1}$ converges for all $x \in (-1, 1)$.
- $\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$ converges for all $x \in [-1, 1]$.

Example 6 (Thm 22 的例子)

Since we have the geometric power series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad -1 < x < 1,$$

it follows from Thm 22 that

$$\ln(1+x) = \int \frac{dx}{1+x} = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, \quad -1 < x < 1.$$

Then $0 = \ln(1+0) = C + 0$ or $C = 0$. Thus, we see that

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x < 1,$$

but its interval of convergence is $I = (-1, 1]$!

Thm (收斂冪級數的運算)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be well-defined.

1. For any $k \in \mathbb{R}$, $f(kx) = \sum_{n=0}^{\infty} a_n (kx)^n = \sum_{n=0}^{\infty} (a_n k^n) x^n$.
2. For any $N \in \mathbb{N}$, $f(x^N) = \sum_{n=0}^{\infty} a_n (x^N)^n = \sum_{n=0}^{\infty} a_n x^{nN}$.
3. $f(x) \pm g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) \pm \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$.

Section 9.8
Taylor and Maclaurin Series
(泰勒級數與馬克勞林級數)

Thm (The Form of a Convergent Power Series)

If f is a real-valued function defined on $I = (c - R, c + R)$ by

$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n \quad \forall\, x \in I,$$

then f has derivatives of all orders on I and, moreover, we have

$$a_n = \frac{f^{(n)}(c)}{n!} \quad \forall\, n \in \mathbb{N},$$

with $0! = 1$ and $f^{(0)} = f$.

Proof of above Thm

- Clearly, $f^{(0)}(c) = f(c) = a_0$.
- For any $x \in I$, applying Thm 21, we see that

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \Longrightarrow f'(c) = a_1.$$

- Applying Thm 21 again, for any $x \in I$ we see that

$$f''(x) = 2a_2 + (3!)a_3(x-c) + (4\cdot 3)a_4(x-c)^2 + \cdots \Longrightarrow a_2 = \frac{f''(c)}{2!}.$$

- Again, applying Thm 21 for $f''(x)$, we obtain

$$f^{(3)}(x) = (3!)a_3 + (4!)a_4(x-c) + \cdots \Longrightarrow a_3 = \frac{f^{(3)}(c)}{3!}.$$

- Applying Thm 21 continuously, this completes the proof.

Def (泰勒級數的定義)

Suppose that f has derivatives of all orders at c.

(1) A power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

is called the Taylor series (泰勒級數) generated by f at $x = c$.

(2) If $c = 0$, a power series of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ is called the Maclaurin series (馬克勞林級數) of f.

Def (泰勒多項式的定義)

Suppose that f has n derivatives at $c \in \text{dom}(f)$.

(1) A polynomial of the form

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
$$= f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor poly. of order n generated by f at $x = c$.

(2) If $c = 0$, then $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$ is called the Maclaurin poly. of order n for f.

Example 2 (求 e^x 的泰勒級數與多項式)

Find the Taylor series and Taylor polynomials generated by

$$f(x) = e^x$$

at $x = 0$. That is, find the Maclaurin series and Maclaurin polynomials for f.

Since $f^{(k)}(x) = e^x \quad \forall\, k \in \mathbb{N}$, we see that

$$f^{(k)}(0) = e^0 = 1 \quad \forall\, k \in \mathbb{N}.$$

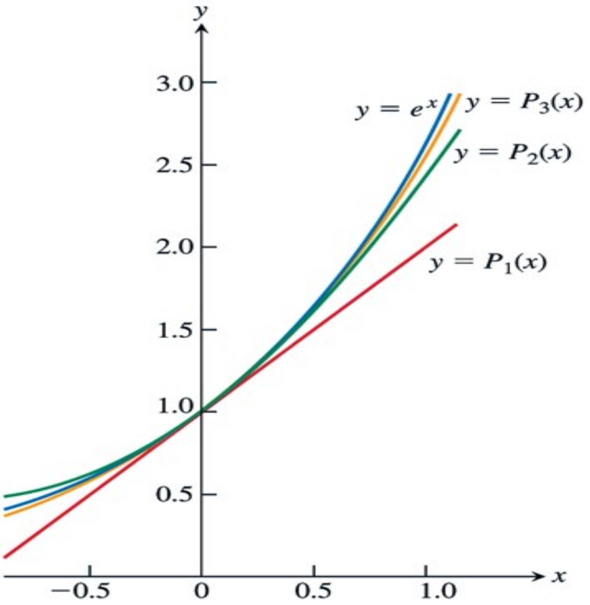
Thus the Taylor series generated by f at $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots,$$

and the Taylor poly. of order n generated by f at $x = 0$ is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}.$$

Example 2 的示意圖



Find the Taylor series and Taylor polynomials generated by

$$f(x) = \cos x$$

at $x = 0$. That is, find the Maclaurin series and Maclaurin polynomials for f.

For $f(x) = \cos x$, we first notice that

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x,$$

and hence $f^{(2k)}(0) = (-1)^k$ for $k \geq 0$ by mathematical induction. Thus the Maclurin series of f is

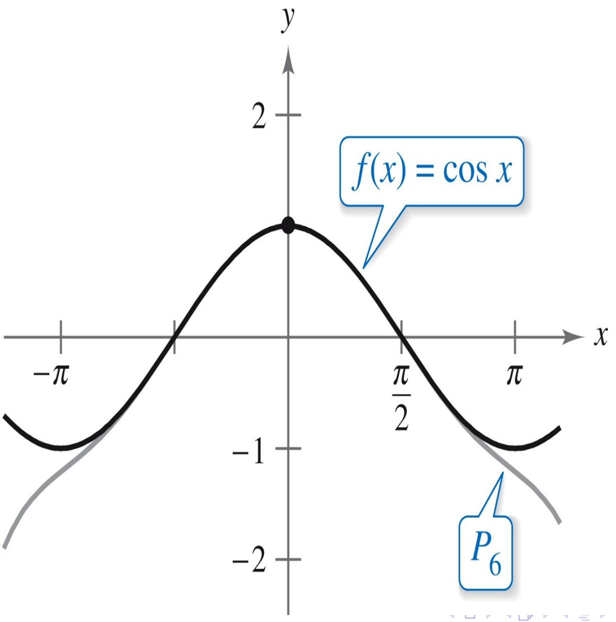
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots,$$

and the Maclaurin poly. of order $2n$ for f is given by

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n}{(2n)!} x^{2n},$$

where $n \in \mathbb{N}$.

馬克勞林多項式 $P_6(x)$ 的示意圖 (承上例)



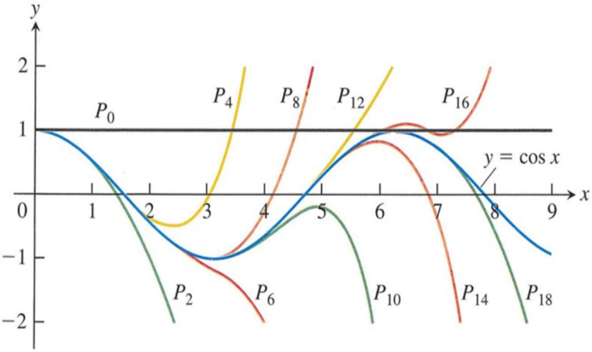


FIGURE 9.23 The polynomials

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$

converge to cos x as $n \to \infty$. We can deduce the behavior of cos x arbitrarily far away solely from knowing the values of the cosine and its derivatives at $x = 0$ (Example 3).

Remarks

- In Example 3, the Maclaurin polynomials of even orders are consistent with the fact that $f(x) = \cos x$ is an even function.
- It will be shown in Section 9.9 that the Maclaurin series derived in Examples 2 and 3 converge to e^x and $\cos x$, respectively, for all $x \in \mathbb{R}$.

Section 9.9
Convergence of Taylor Series
(泰勒級數的收斂性)

Def (泰勒多項式相關的剩餘項與誤差)

Let f have at least n derivatives on an interval I containing c.

(1) $R_n(x) \equiv f(x) - P_n(x)$ is called the remainder (剩餘項) associated with $P_n(x)$.

(2) $|R_n(x)| = |f(x) - P_n(x)|$ is the error associated with $P_n(x)$.

Thm 23 (Taylor's Theorem; 泰勒定理)

If f has $(n+1)$ derivatives on an interval I containing c, then $\forall\, x \in I$, $\exists\, z$ between x and c s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x) = P_n(x) + R_n(x),$$

where the Lagrange form of the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Suppose that f has derivatives of all orders on an open interval I containing c. It follows from Taylor's Thm (Thm 23) that $\forall\, x \in I$, $\exists\, z$ between x and c s.t.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x) = P_n(x) + R_n(x),$$

where the Lagrange form of the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

Main Question

When does the Taylor series generated by f at $x = c$ *always* converge to $f(x)$ on the open interval I?

Main Question

When does the Taylor series generated by f at $x = c$ *always* converge to $f(x)$ on the open interval I?

Thm (泰勒級數收斂性的等價條件)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \forall\, x \in I \Longleftrightarrow \lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in I.$$

Proof of above Thm

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \forall\, x \in I.$$

$$\iff f(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \lim_{n\to\infty} P_n(x) \quad \forall\, x \in I.$$

$$\iff \lim_{n\to\infty} \Big[f(x) - P_n(x) \Big] = 0 \quad \forall\, x \in I.$$

$$\iff \lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in I.$$

Remark

In order to prove that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = 0 \quad \forall\, x \in I,$$

we often use the Sandwich Theorem and the following fact

$$\lim_{n\to\infty} \frac{|x-c|^{n+1}}{(n+1)!} = 0 \quad \forall\, x \in \mathbb{R},$$

which has been shown in the item (6) of Thm 5.

Example 1 (e^x 的泰勒級數收斂性證明)

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for all $x \in \mathbb{R}$, and thus we can write

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \forall\, x \in \mathbb{R}.$$

Another Definition of the Euler's number e

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \approx 2.71828.$$

Proof of Example 1

For any $x \in \mathbb{R}$, from Taylor's Thm, $\exists$ z between 0 and x s.t.

$$f(x) = e^x = P_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + e^z \cdot \frac{x^{n+1}}{(n+1)!}.$$

Since the remainder $R_n(x)$ satisfies the following inequality

$$|R_n(x)| \le \max\{1, e^x\} \cdot \frac{|x|^{n+1}}{(n+1)!} \quad \forall\, x \in \mathbb{R},$$

it follows from the Sandwich Thm that

$$\lim_{n\to\infty} |R_n(x)| = 0 \Longrightarrow \lim_{n\to\infty} R_n(x) = 0 \quad \forall\, x \in \mathbb{R}$$

and thus we complete the proof.

Show that the Taylor series generated by $f(x) = \sin x$ at $x = 0$ converges to $f(x)$ for all $x \in \mathbb{R}$, and thus we can write

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \forall\, x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$, from Taylor's Thm, $\exists\, z$ between 0 and x s.t.

$$\sin x = P_{2n+1}(x) + R_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{f^{(2n+2)}(z) \cdot x^{2n+2}}{(2n+2)!}.$$

Note that the remainder $R_{2n+1}(x)$ satisfies the following inequality

$$|R_{2n+1}(x)| = \frac{|f^{(2n+2)}(z)| \cdot |x|^{2n+2}}{(2n+2)!} \le 1 \cdot \frac{|x|^{2n+2}}{(2n+2)!} \quad \forall\, x \in \mathbb{R},$$

since $f^{(2n+2)}(z) = \pm \sin z$ or $\pm \cos z$. Then, from the Sandwich Theorem, we see that

$$\lim_{n\to\infty} |R_{2n+1}(x)| = 0 \Longrightarrow \lim_{n\to\infty} R_{2n+1}(x) = 0 \quad \forall\, x \in \mathbb{R}$$

and hence this completes the proof.

(1) $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$.

(2) $\ln x = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ for $0 < x \leq 2$.

(3) $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $-\infty < x < \infty$.

(4) $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for $-\infty < x < \infty$.

(5) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for $-\infty < x < \infty$.

(6) $\sin^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} x^{2n+1}$ for $-1 \leq x \leq 1$.

(7) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $-1 \leq x \leq 1$.

(8) Binomial Series (二項式級數) with $k \in \mathbb{R}$:

$$(1+x)^k = 1 + \sum_{n=1}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n \quad \text{for } -1 < x < 1.$$

Thank you for your attention!