Chapter 10 Parametric Equations and Polar Coordinates (參數方程式與極坐標)

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本章預定授課範圍

Section 10.1
Parametrizations of Plane Curves
(平面曲線的參數式)

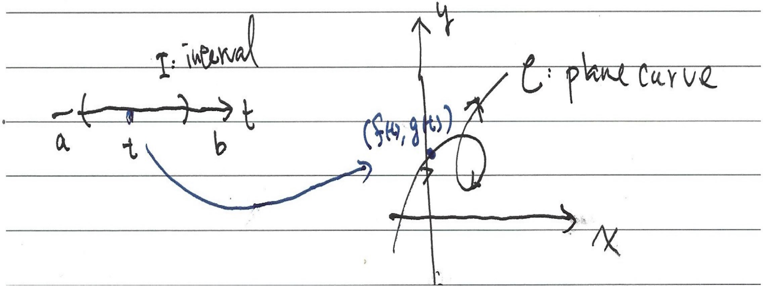
Def (參數曲線的定義)

Let I be an interval. A plane curve $\mathcal{C}$ is often defined by the graph of the parametric equations (參數方程式)

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad \forall\, t \in I,$$

where f and g are conti. functions of t, and t is a parameter (參數). In this case, $\mathcal{C}$ is called a parametric curve (參數曲線).

示意圖 (承上頁)



Example 1 (參數曲線的繪圖)

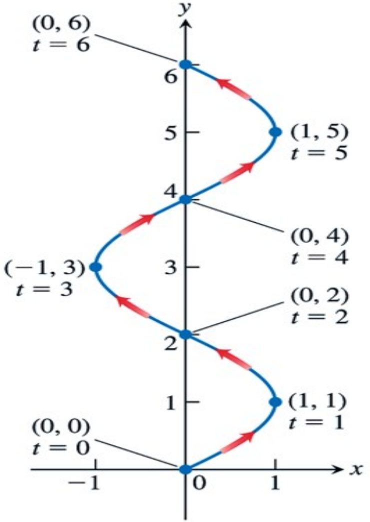
Sketch the curve defined by the parametric equations

$$x = f(t) = \sin\frac{\pi t}{2}, \quad y = g(t) = t$$

for $0 \le t \le 6$.

TABLE 10.1 Values of $x = \sin \pi t/2$ and $y = t$ for selected values of t.

t	x	y
0	0	0
1	1	1
2	0	2
3	−1	3
4	0	4
5	1	5
6	0	6



Graph the parametric curves

(a) $x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi.$

Solution of Example 3

(b) For $a \neq 0$, it is easily seen that

$$x^2 + y^2 = a^2 \cos^2 t + a^2 \sin^2 t = a^2,$$

and hence the graph of the parametric curve is a circle centered at $(0, 0)$ with radius $|a| > 0$. It starts from $(a, 0)$ at $t = 0$ and traces the circle once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$ finally.

(a) This is just a special case of Part (b) with $a = 1$!

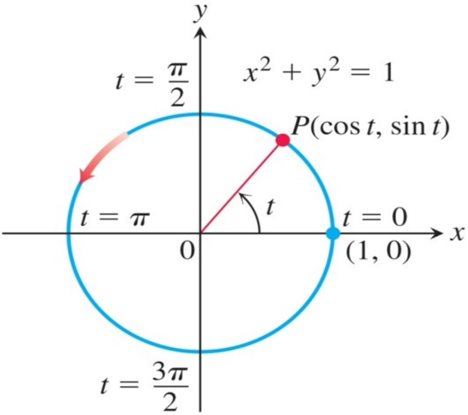


FIGURE 10.4 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 3).

Sketch and identify the graph of the parametric curve

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}$$

for $f > 0$. In this case, note that $x > 0$ for all $t > 0$.

For any $t > 0$, we first notice that

$$x + y = 2t, \quad x - y = \frac{2}{t},$$

and hence the poiint (x, y) lying on the given curve satisfies

$$(x + y)(x - y) = (2t)(2/t) = 4 \Longrightarrow x^2 - y^2 = 4 \text{ and } x > 0.$$

So, the graph of the parametric curve is the right-hand branch of he hyperbola (雙曲線) $x^2 - y^2 = 4$.

Example 7 的示意圖

TABLE 10.3 Values of $x = t + (1/t)$ and $y = t - (1/t)$ for selected values of t.

t	$1/t$	x	y
0.1	10.0	10.1	−9.9
0.2	5.0	5.2	−4.8
0.4	2.5	2.9	−2.1
1.0	1.0	2.0	0.0
2.0	0.5	2.5	1.5
5.0	0.2	5.2	4.8
10.0	0.1	10.1	9.9

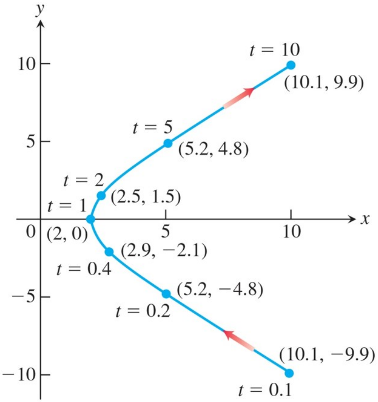


FIGURE 10.7 The curve for $x = t + (1/t)$, $y = t - (1/t)$, $t > 0$ in Example 7. (The part shown is for $0.1 \leq t \leq 10$.)

Section 10.2
Calculus with Parametric Curves
(參數曲線的微積分)

Thm (參數曲線的微分)

If $\mathcal{C}$ is a smooth curve defined by

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad \forall\, t \in I,$$

with $dx/dt = f'(t) \neq 0 \quad \forall\, t \in I$, then

(1) the slope of $\mathcal{C}$ at the point (x, y) is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \equiv m(t) \qquad \forall\, t \in I.$$

(2) the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{m'(t)}{f'(t)} \quad \forall\, t \in I.$$

Proof of above Thm

(1) From the Def. of dy/dx at the point $(x(t), y(r))$, we see that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{x(t+h) - x(t)}$$
$$= \lim_{h \to 0} \frac{[y(t+h) - y(t)]/h}{[x(t+h) - x(t)]/h} = \frac{y'(t)}{x'(t)}.$$

(2) Since $y' = dy/dx$ is a function of t, it follows from (1) that

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d(y')/dt}{dx/dt}.$$

Fincd the tangent line to the parametric curve

$$x = \sec t, \quad y = \tan t, \quad \frac{-\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$.

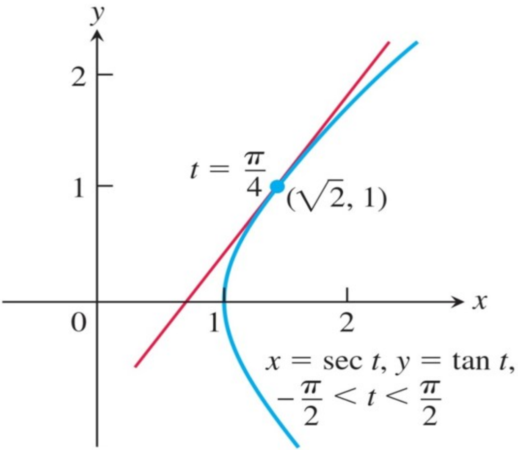


FIGURE 10.14 The curve in Example 1 is the right-hand branch of the hyperbola $x^2 - y^2 = 1$.

Solution of Example 1

Since the slope of the curve at $t = \pi/4$ is given by

$$m = \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\frac{\pi}{4}} = \left.\frac{\sec^2 t}{\sec t \tan t}\right|_{t=\frac{\pi}{4}}$$

$$= \left.\frac{\sec t}{\tan t}\right|_{t=\frac{\pi}{4}} = \frac{\sqrt{2}}{1} = \sqrt{2},$$

the equation of the tangent line to the given curve at $(\sqrt{2}, 1)$ is

$$y - 1 = \sqrt{2}(x - \sqrt{2}) \quad \text{or} \quad y = \sqrt{2}x - 1.$$

Example 2 (計算參數方程式的二階導數)

Find $\dfrac{d^2y}{dx^2}$ as a function of t if the parametric equations are

$$x = t - t^2 \quad \text{and} \quad y = t - t^3,$$

where $t \in \mathbb{R}$ is a parameter satisfying $1 - 2t \neq 0$ or $t \neq \frac{1}{2}$.

Notice that the first derivative is given by

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}.$$

Next, differentiating y' w.r.t. t gives

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{(-6t)(1-2t)-(1-3t^2)(-2)}{(1-2t)^2} = \frac{6t^2-6t+2}{(1-2t)^2},$$

and thus the second derivative is

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{dy'/dt}{1-2t} = \frac{6t^2-6t+2}{(1-2t)^3}.$$

- A plane curve $\mathcal{C}$ defined parametrically by

$$x = f(t) \quad \text{and} \quad y = g(t)$$

is smooth on $I = [a, b]$ if f', g' are conti. and not simultaneously zero on I.

- Let $\Delta = \{t_k \,|\, 0 \le k \le n\}$ be a partition of $I = [a, b]$ with $a = t_0 < t_1 < \cdots << t_n = b$. Then the length of $\mathcal{C}$ on $[t_{k-1}, t_k]$ is approximately evaluated by

$$\Delta L_k == \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2}$$

for each $k = 1, 2, \ldots, n$.

- For each k, from M.V.T. $\exists\, t_k^*, t_k^{**} \in [t_k, t_{k-1}]$ s.t.

$$\Delta L_k = \sqrt{[f'(t_k^*)\Delta t_k]^2 + [g'(t_k^{**})\Delta t_k]^2},$$

where $\Delta t_k = t_k - t_{k-1} \geq 0$ for $k = 1, 2, \ldots, n$.

- So, the length of $\mathcal{C}$ from $t = a$ to $t = b$ is approximate to

$$L \approx \sum_{k=1}^{n} \Delta L_k = \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\,(\Delta t_k).$$

Def (參數曲線的長度)

Let $\mathcal{C}$ be a smooth curve defined parametrically by

$$x = f(t) \quad \text{and} \quad y = g(t) \qquad \forall\, t \in I = [a, b].$$

If $\mathcal{C}$ does not intersect itself on I, then the (arc) length of $\mathcal{C}$ on I is

$$L = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\,(\Delta t_k)$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt \geq 0.$$

Find the length of the astroid (星形線) $\mathcal{C}$ defined by

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$$

Example 5 的示意圖

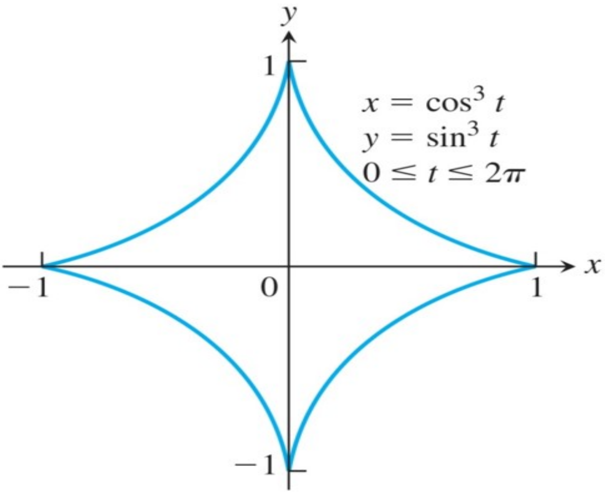


FIGURE 10.15 The astroid

Solution of Example 5 (1/2)

If we let $\mathcal{C}_1 = \{(x(t), y(t)) \mid 0 \le t \le \pi/2\}$, then

$$L(\mathcal{C}) = 4 \cdot L(\mathcal{C}_1) = 4 \int_0^{\pi/2} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

Since $x'(t) = -3\cos^2 t\ \sin t$ and $y'(t) = 3\sin^2 t \cos t$, we see that

$$\begin{aligned}\sqrt{[x'(t)]^2 + [y'(t)]^2} &= \sqrt{(-3\cos^2 t\ \sin t)^2 + (3\sin^2 t\cos t)^2} \\ &= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = 3\cos t \sin t\end{aligned}$$

for $0 \le t \le \dfrac{\pi}{2}$.

Solution of Example 5 (2/2)

Therefore, the length of the cureve $\mathcal{C}$ is

$$L(\mathcal{C}) = 4\int_0^{\pi/2} 3\cos t \sin t \, dt = 12\int_0^{\pi/2} \sin t \, d(\sin t)$$

$$= 6\sin^2 t\Big|_0^{\pi/2} = 6(1^2 - 0^2) = 6.$$

Useful Formulas

Recall the following identities for the sine and cosine functions:

1. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
2. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
3. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
4. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

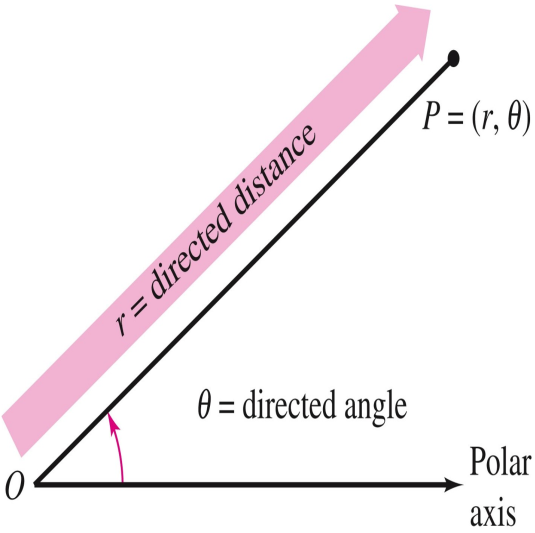
Section 10.3
Polar Coordinates
(極坐標)

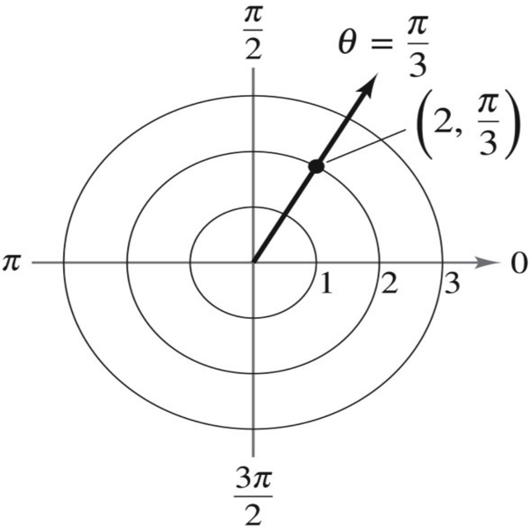
Def (極坐標的定義)

The polar coordinates (r, θ) of a point $P(x, y) \in \mathbb{R}^2$ is defined by

$r =$directed distance from the ple (極點) O to P.

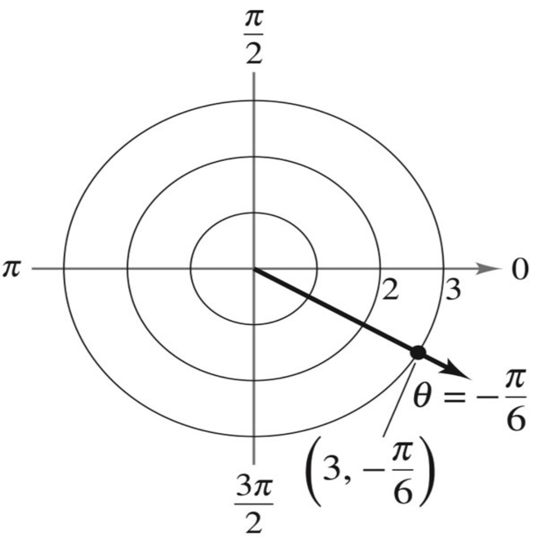
$\theta =$directed angle, conterclockwise from the polar axis (極軸) to the line $\overline{OP}$.

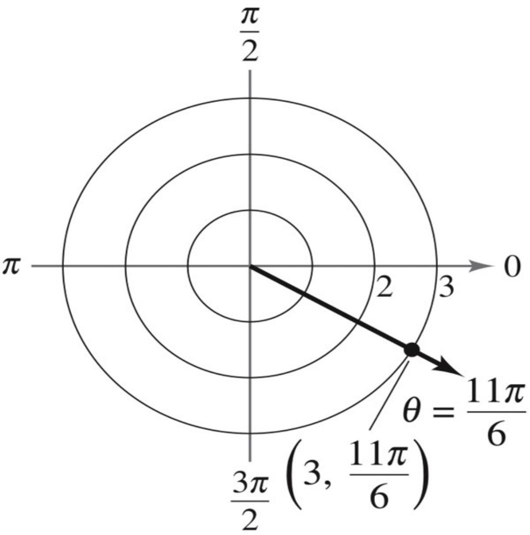

r = directed distance
P = (r, θ)
θ = directed angle
O
Polar
axis



$\frac{\pi}{2}$
$\theta = \frac{\pi}{3}$
$\left(2, \frac{\pi}{3}\right)$
π
0
1
2
3
$\frac{3\pi}{2}$

極坐標的示意圖 (3/4)




π/2
π
0
2
3
θ = 11π/6
(3, 11π/6)
3π/2

Notes

(1) The polar coordinates of the pole is $O = (0, \theta)$ for any $\theta \in \mathbb{R}$.

(2) The polar coordinates (r, θ) and $(r, \theta + 2n\pi)$ represent the same point in $\mathbb{R}^2$, i.e., $(r, \theta) = (r, \theta + 2n\pi) \quad \forall\, n \in \mathbb{Z}$.

(3) If $r > 0$, then $(-r, \theta) = (r, \theta + (2n+1)\pi) \quad \forall\, n \in \mathbb{Z}$.

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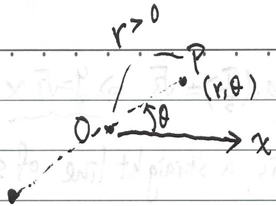


$(-r,\theta) = (r, \theta \pm \pi) = (r, \theta \pm 3\pi) = \cdots$

($(-r,\theta)$ 與 (r,θ) 互對稱於極點 O) ※

Polar Coordinates vs. Cartesian Coordinates

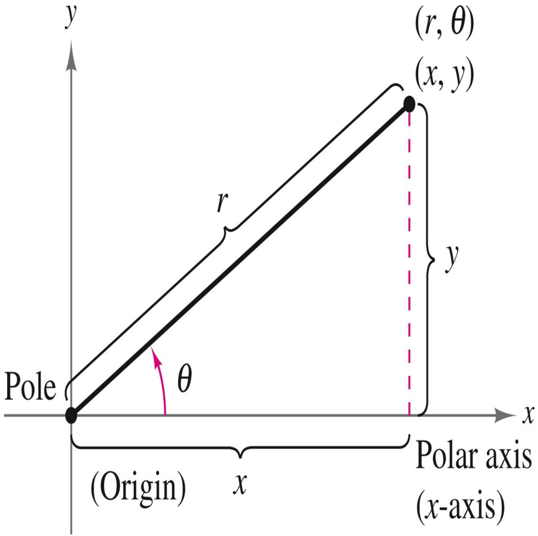
- Polar to Cartesian $(r, \theta) \longmapsto (x, y)$:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

- Cartesian to Polar $(x, y) \longmapsto (r, \theta)$:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

直角坐標和極坐標的關係



Notes (極坐標方程式的類型)

- Explicit Form (顯式): $r = f(\theta)$ or $\theta = g(r)$.
- Implicit Form (隱式): $F(r, \theta) = 0$.
- Only the graph of a polar equation in explicit form will be presented here.

(a) $r = 1$ and $r = -1$ are polar equations for the circle of radius 1 centered at O, since we know that

$$r = \pm 1 \Longrightarrow r^2 = 1 \Longrightarrow x^2 + y^2 = 1.$$

(b) $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$ and $\theta = \frac{-5\pi}{6}$ are polar equations for the line $y = (\tan\frac{\pi}{6})x = \frac{1}{\sqrt{3}}x$.

Example 2b 的示意圖

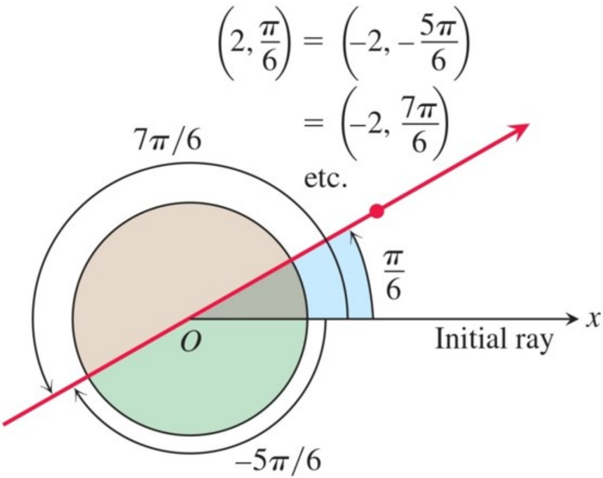


FIGURE 10.23 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs

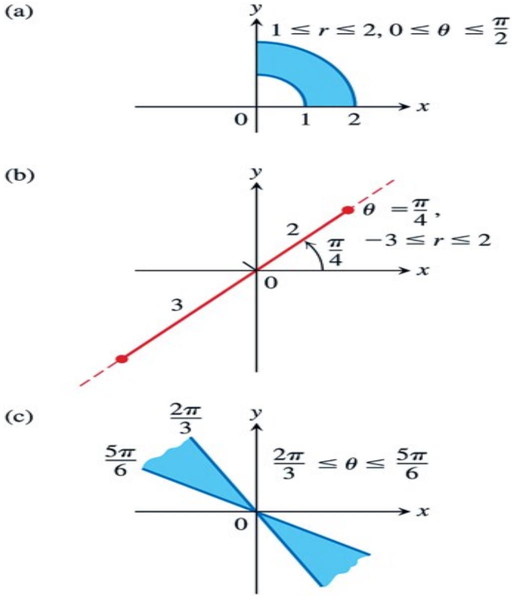
Example 3 (極坐標區域的例子)

Graph the following polar regions.

(a) $\mathcal{R} = \left\{(r,\theta)\,|\,1 \le r \le 2, 0 \le \theta \le \dfrac{\pi}{2}\right\}$.

(b) $\mathcal{R} = \left\{(r,\theta)\,|\,-3 \le r \le 2, \theta = \dfrac{\pi}{4}\right\}$.

(c) $\mathcal{R} = \left\{(r,\theta)\,|\,r \in \mathbb{R}, \dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}\right\}$.


(a)
y
1 ≤ r ≤ 2, 0 ≤ θ ≤ π/2
x
0
1
2
(b)
y
θ = π/4,
−3 ≤ r ≤ 2
2
π/4
x
0
3
(c)
2π/3
y
5π/6
2π/3 ≤ θ ≤ 5π/6
x
0

Example 5 (求出圓的極坐標方程式)

Find a polar equation for the circle

$$x^2 + (y - 3)^2 = 9$$

expressed as a Cartesian equation.

Example 5 的示意圖

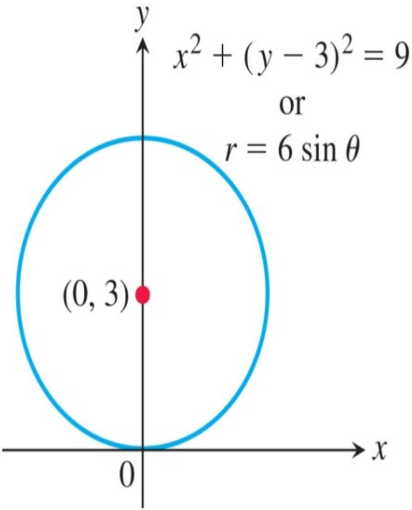


FIGURE 10.27 The circle in Example 5.

Solution of Example 5

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r^2 = x^2 + y^2 > 0$ and thus the given Cartesian equation becomes

$$x^2 + (y-3)^2 = 9 \Longrightarrow x^2 + y^2 - 6y = 0 \Longrightarrow r^2 - 6r\sin\theta = 0$$
$$\Longrightarrow r(r - 6\sin\theta) = 0 \Longrightarrow r = 6\sin\theta$$

for $0 \le \theta \le \pi$. (不是 $0 \le \theta \le 2\pi$ 喔!)

Find a Cartesian equation for each polar equation.

(a) $r\cos\theta = -4$.

(b) $r^2 = 4r\cos\theta$.

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4 \Longrightarrow x = -4$ is a vertical line intersecting the x-axis at the point $(-4, 0)$.

(b) $r^2 = 4r\cos\theta \Longrightarrow x^2 + y^2 = 4x \Longrightarrow x^2 - 4x + 4 + y^2 = 4 \Longrightarrow (x-2)^2 + y^2 = 4$ is a circle of radius 2 centered at $(2, 0)$.

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta} \Longrightarrow 2r\cos\theta - r\sin\theta = 4 \Longrightarrow 2x - y = 4$ $\Longrightarrow y = 2x - 4$ is a straight line of slope 2 passing through the points $(0, -4)$ and $(2, 0)$, respectively.

Section 10.4
Graphing Polar Coordinate Equations (描繪極坐標方程式的圖形)

Thm (Slope in Polar Form)

If f is a diff. function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

Proof of above Thm

Note that x and y can be expressed in terms of θ by

$$x = r\cos\theta = f(\theta)\cos\theta,$$
$$y = r\sin\theta = f(\theta)\sin\theta.$$

Thus, differentiating x and y w.r.t. the parameter θ, we obtain

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

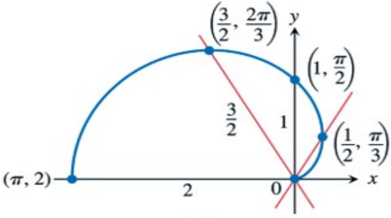
Sketch the graph of the cardioid (心臟線)

$$r = f(\theta) = 1 - \cos\theta$$

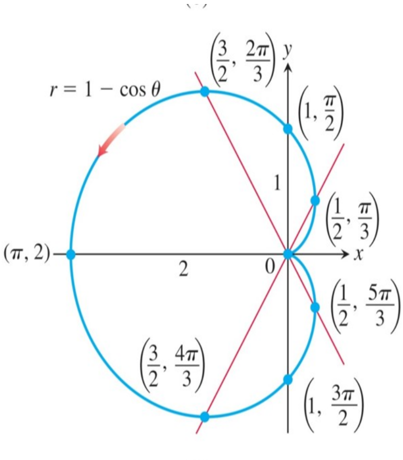
for $0 \leq \theta \leq 2\pi$.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)



(c)

FIGURE 10.29 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

A rose curve (玫瑰線) is described by the polar equation

$$r = f(\theta) = a\cos(n\theta) \quad \text{or} \quad r = f(\theta) = a\sin(n\theta),$$

where $a > 0$ and $n \in \mathbb{N}$.

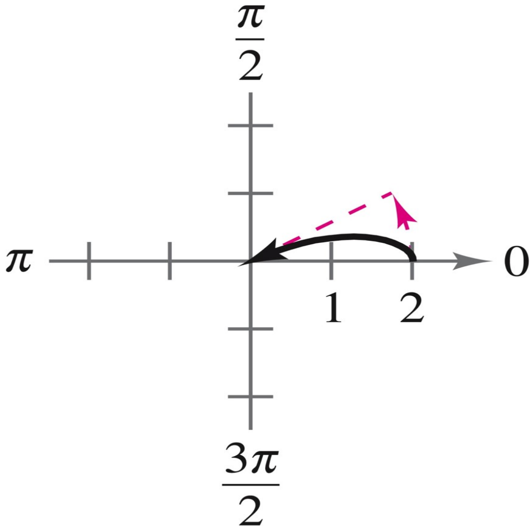
Example (三瓣玫瑰線; 補充題)

Graph a 3-leaf (or 3-petalled) rose curve of the polar equation

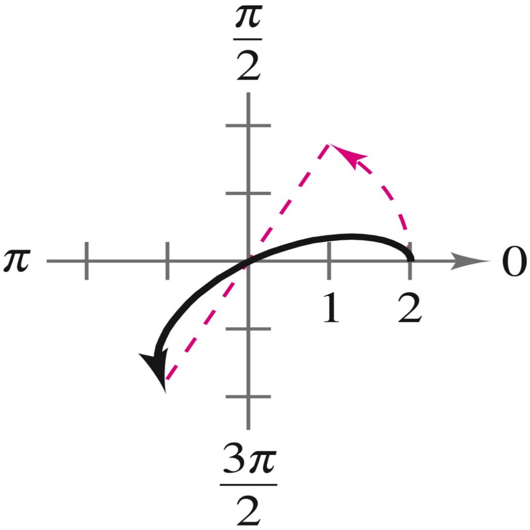
$$r = f(\theta) = 2\cos(3\theta), \quad 0 \le \theta \le \pi.$$

For some specific values of θ, we have the following table:

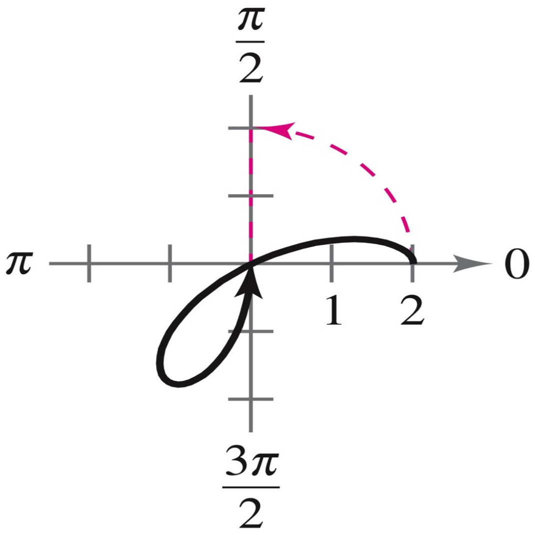
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	2	0	-2	0	2	0	-2


$\frac{\pi}{2}$
π
0
1
2
$\frac{3\pi}{2}$

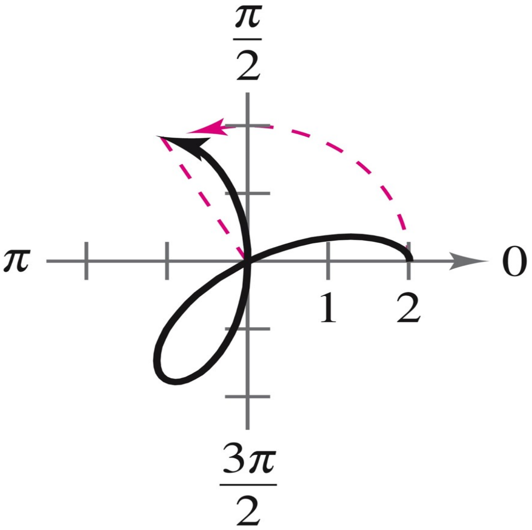
三瓣玫瑰線的動態示意圖



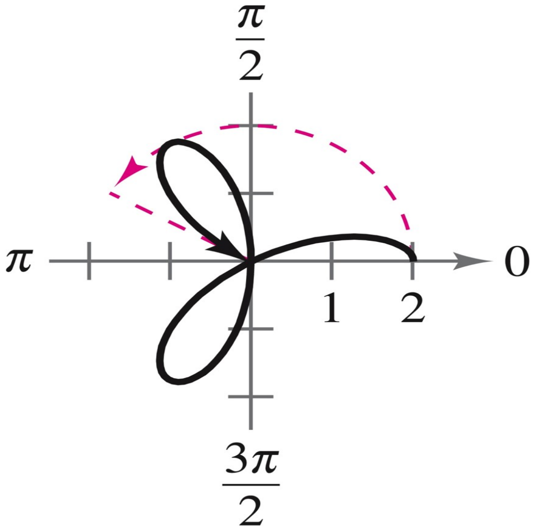
三瓣玫瑰線的動態示意圖



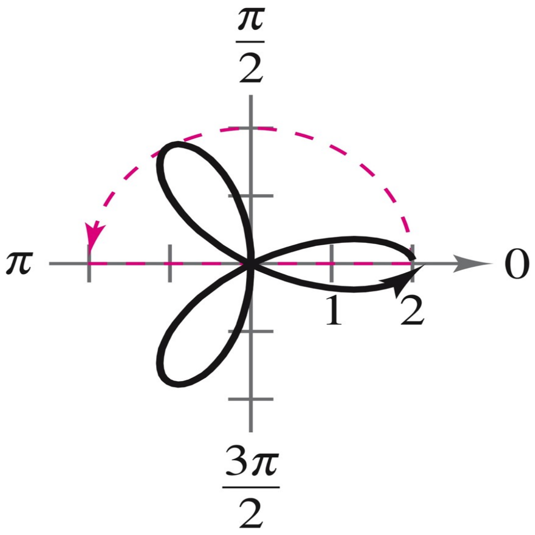
三瓣玫瑰線的動態示意圖



三瓣玫瑰線的動態示意圖



三瓣玫瑰線的動態示意圖



Section 10.5
Areas and Lengths in Polar Coordinates
(極坐標上的面積與弧長)

Partitions of a Polar Region

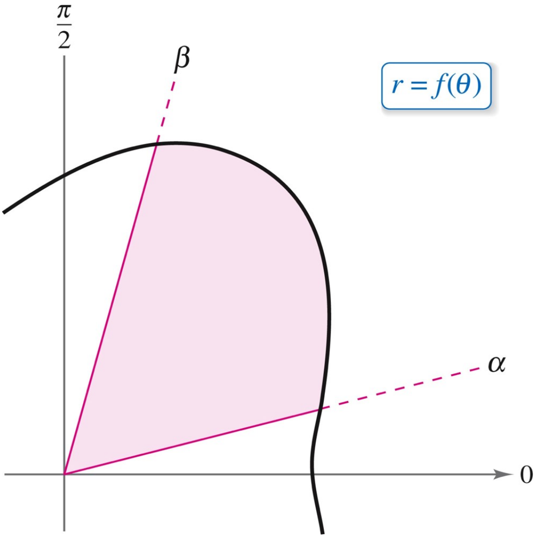
Consider a polar region given by

$$\mathcal{R} = \{(r, \theta) \,|\, \alpha \le \theta \le \beta,\ 0 \le r \le f(\theta)\}$$

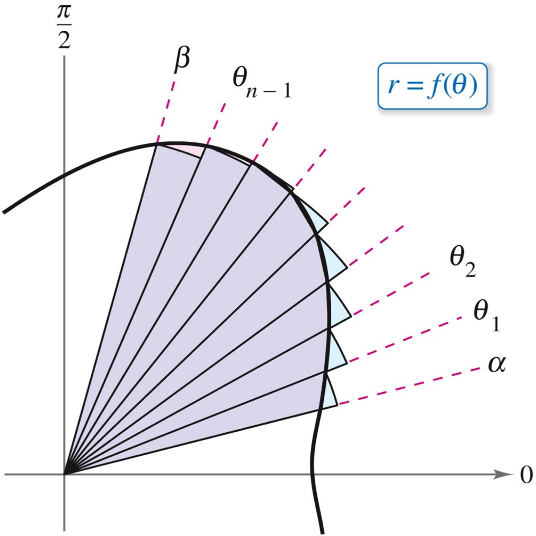
with $0 < \beta - \alpha \le 2\pi$. Let the region $\mathcal{R}$ be partitioned into n polar sectors (極坐標扇形) by the rays $r = \theta_i$ ($i = 0, 1, 2, \ldots, n$) with

$$\alpha \equiv \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{n-1} < \theta_n \equiv \beta.$$

極坐標區域分割的示意圖 (1/2)



極坐標區域分割的示意圖 (2/2)



Therefore, the area of the polar region $\mathcal{R}$ should be

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i)]^2 \Delta\theta_i = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i)]^2 \Delta\theta_i,$$

where $\Delta\theta_i \equiv \theta_i - \theta_{i-1}$ for $i = 1, 2, \ldots, n$.

Thm (極坐標區域的面積公式)

If $f(\theta)$ is conti. on $[\alpha, \beta]$ with $0 < \beta - \alpha \le 2\pi$, then the area of the polar region $\mathcal{R} = \{(r, \theta) \mid \alpha \le \theta \le \beta, 0 \le r \le f(\theta)\}$ is given by

$$A = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \frac{1}{2} [f(\theta_i)]^2 \Delta\theta_i = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta.$$

Example 1 (求極坐標區域的面積)

Find the area of the region enclosed by the cardioid

$$r = f(\theta) = 2(1 + \cos\theta), \quad 0 \leq \theta \leq 2\pi.$$

That is, find the area of the polar region defined by

$$\mathcal{R} = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq f(\theta)\}.$$

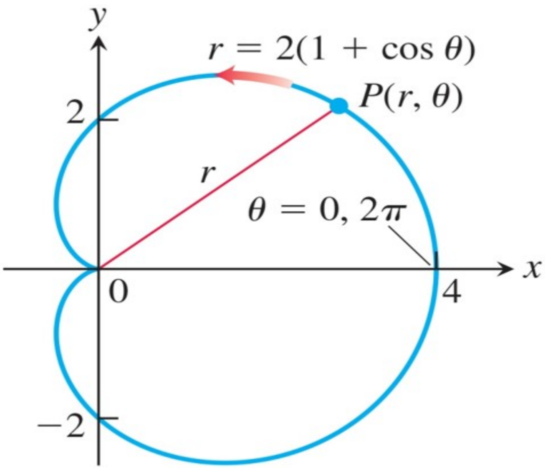


FIGURE 10.34 The cardioid in Example 1.

Solution of Example 1

The area of the region enclosed by the cardioid is

$$A = \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2\,d\theta = 2\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$
$$= 2\int_0^{2\pi}(1+2\cos\theta+\frac{1+\cos 2\theta}{2})\,d\theta = \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\,d\theta$$
$$= \left(3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi} = 6\pi - 0 = 6\pi.$$

Thm (Length of a Polar Curve)

If $f(\theta)$ and $f'(\theta)$ are conti. on $[\alpha, \beta]$ with $0 < \beta - \alpha \leq 2\pi$, then the (arc) length of a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Proof of above Thm

From the proof of the slope of a polar curve $r = f(\theta)$, we know that

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta \quad \text{and} \quad \frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta.$$

Then we immediately get

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \Big[-f(\theta)\sin\theta + f'(\theta)\cos\theta\Big]^2 \\ &\quad + \Big[f(\theta)\cos\theta + f'(\theta)\sin\theta\Big]^2 \\ &= [f(\theta)]^2 + [f'(\theta)]^2. \qquad \text{(Check!)}\end{aligned}$$

So, the arc length of the polar curve $r = f(\theta)$ is given by

$$L = \int_\alpha^\beta \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\, d\theta = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Example 4 (計算心臟線的長度)

Find the (arc) length of the cardioid

$$r = f(\theta) = 1 - \cos\theta$$

from $\theta = 0$ to $\theta = 2\pi$.

Example 4 的示意圖

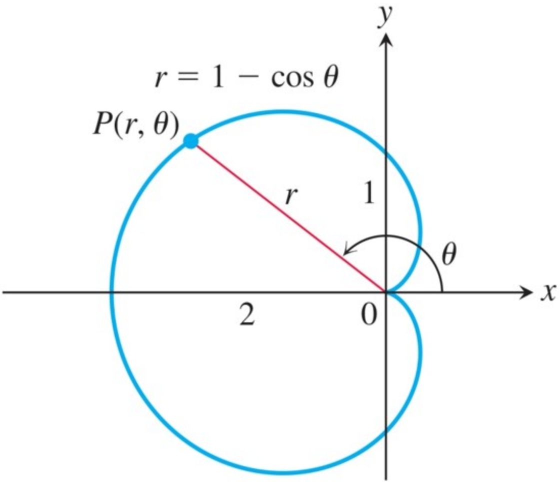


FIGURE 10.38 Calculating the length of a cardioid (Example 4).

The length of the given cardioid is

$$
\begin{aligned}
L &= \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \\
&= \left(-4\cos\frac{\theta}{2}\right)\Big|_0^{2\pi} = 4(1+1) = 8.
\end{aligned}
$$

Recall (半角公式)

- $\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}$ or $2\sin^2\frac{\theta}{2} = 1-\cos\theta$.
- $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$ or $2\cos^2\frac{\theta}{2} = 1+\cos\theta$.

Thank you for your attention!