Chapter 11
Vectors and the Geometry of Space (向量與空間幾何)

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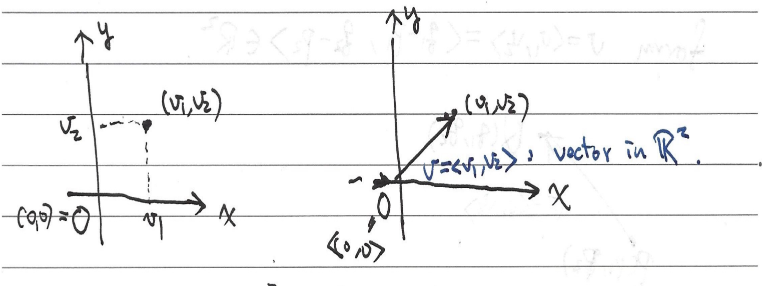
本章預定授課範圍

Section 11.0
Definitions and Preliminaries
(定義和預備知識)

Two-Dimensional Euclidean (Vector) Space

$$\begin{aligned}\mathbb{R}^2 &= \{(v_1, v_2) \mid v_1, v_2 \in \mathbb{R}\} \\ &= \{\mathbf{v} = \langle v_1, v_2 \rangle \mid \mathbf{v} \text{ is a vector (向量)}\}\end{aligned}$$



- Vector Addition (向量加法):

$$\mathbf{v} + \mathbf{u} = \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle = \langle v_1 + u_1, v_2 + u_2 \rangle \quad \forall\, \mathbf{v}, \mathbf{u} \in \mathbb{R}^2.$$

- Scalar Multiplication (純量乘法):

$$c\mathbf{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle \quad \forall\, c \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^2.$$

- Length or Norm (範數) of a Vector:

$$|\mathbf{v}| = \|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| := \sqrt{v_1^2 + v_2^2} \geq 0 \quad \forall\, \mathbf{v} \in \mathbb{R}^2.$$

- $\mathbf{v} \in \mathbb{R}^2$ is a unit vector (單位向量) if $|\mathbf{v}| = \|\mathbf{v}\| = 1$.

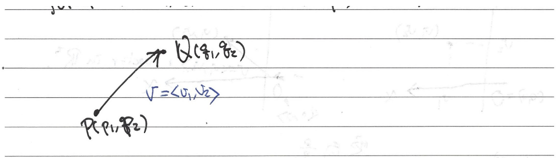
Definitions in $\mathbb{R}^2$ (2/2)

- If $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^2$, then

$$\mathbf{v} = \langle v_1, v_2 \rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} \quad \forall\, \mathbf{v} \in \mathbb{R}^2.$$

- If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2)$ to $Q(q_1, q_2)$, then it has the component form (分量形式)

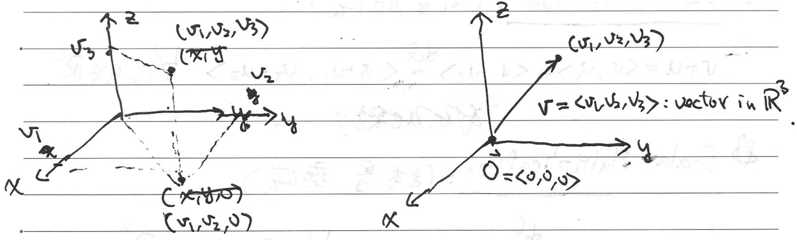
$$\mathbf{v} = \langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle \in \mathbb{R}^2.$$



Vectors in Space (空間向量)

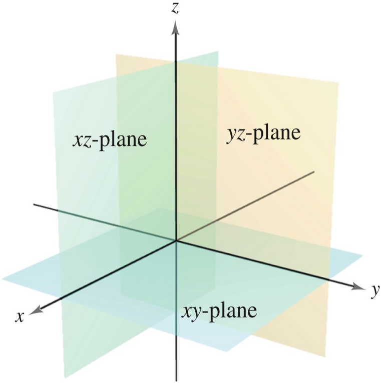
Three-Dimensional Euclidean (Vector) Space

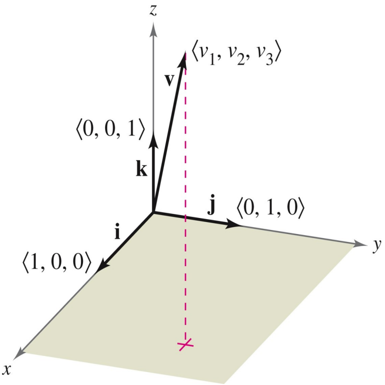
$$\begin{aligned}\mathbb{R}^3 &= \{(v_1, v_2, v_3) \mid v_1, v_2, v_3 \in \mathbb{R}\} \\ &= \{\mathbf{v} = \langle v_1, v_2, v_3 \rangle \mid \mathbf{v} \text{ is a vector in space}\}\end{aligned}$$



☆ $\vec{0} = \langle 0,0,0 \rangle$ is the zero vector in $\mathbb{R}^3$.

Vectors in Space 的示意圖





Definitions in $\mathbb{R}^3$ (1/2)

- Vector Addition (向量加法):

$$\begin{aligned}\mathbf{v}+\mathbf{u} &= \langle v_1, v_2, v_3\rangle + \langle u_1, u_2, u_3\rangle \\ &= \langle v_1+u_1, v_2+u_2, v_3+u_3\rangle \quad \forall\, \mathbf{v}, \mathbf{u} \in \mathbb{R}^3.\end{aligned}$$

- Scalar Multiplication (純量乘法):

$$c\mathbf{v} = c\langle v_1, v_2, v_3\rangle = \langle cv_1, cv_2, cv_3\rangle \quad \forall\, c \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^3.$$

- Length or Norm of a Vector:

$$|\mathbf{v}| = \|\mathbf{v}\| = \|\langle v_1, v_2, v_3\rangle\| := \sqrt{v_1^2+v_2^2+v_3^2} \geq 0 \quad \forall\, \mathbf{v} \in \mathbb{R}^3.$$

- $\mathbf{v} \in \mathbb{R}^3$ is a unit vector if $|\mathbf{v}| = \|\mathbf{v}\| = 1$.

Definitions in $\mathbb{R}^3$ (2/2)

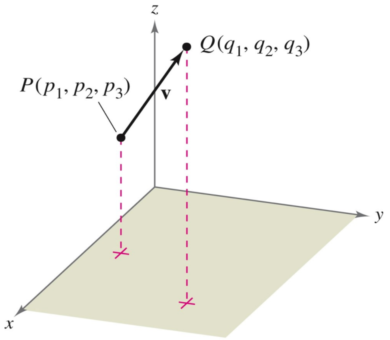
- If $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$ and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are standard unit vectors in $\mathbb{R}^3$, then

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k} \quad \forall\, \mathbf{v} \in \mathbb{R}^3.$$

- If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then it has the component form

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle \in \mathbb{R}^3.$$

向量 $\overrightarrow{PQ}$ 的示意圖 (承上頁)



$$\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ ($n = 2$ or 3) and $c, d \in \mathbb{R}$. Then

(1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(2) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(3) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(4) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(5) $c(d\mathbf{u}) = (cd)\mathbf{u} = d(c\mathbf{u})$

(6) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(8) $1\mathbf{u} = \mathbf{u}$, $0\mathbf{u} = \mathbf{0}$

Def (向量點積的定義)

(1) The dot product of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 \in \mathbb{R}.$$

(2) The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \in \mathbb{R}.$$

(3) In some textbooks, the dot product of two vectors is also called the inner product (內積) of the vectors.

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ ($n = 2$ or 3) and $c \in \mathbb{R}$. Then

(1) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ (2) $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$

(3) $c(\mathbf{u} \bullet \mathbf{v}) = (c\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (c\mathbf{v})$ (4) $\mathbf{u} \bullet \mathbf{0} = \mathbf{0} \bullet \mathbf{u} = 0$

(5) $\cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(6) $\mathbf{v} \bullet \mathbf{v} = \|\mathbf{v}\|^2$ and thus $\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$.

Some Special Vectors

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$.

- $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the direction of $\mathbf{v} \neq \mathbf{0}$.
  (沿著 v 方向的單位向量)
- $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors (平行向量) if $\exists\, 0 \neq c \in \mathbb{R}$ s.t. $\mathbf{u} = c\mathbf{v}$.
- $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors (垂直向量) if $\mathbf{u} \bullet \mathbf{v} = 0$.

Example (平行與垂直向量的夾角)

- If $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors, then $\mathbf{u} = c\mathbf{v}$ for some $0 \neq c \in \mathbb{R}$ and hence we see that

$$\cos\theta = \frac{(c\mathbf{v}) \bullet \mathbf{v}}{\|c\mathbf{v}\| \, \|\mathbf{v}\|} = \frac{c(\mathbf{v} \bullet \mathbf{v})}{|c| \, \|\mathbf{v}\|^2} = \frac{c}{|c|} = \pm 1.$$

  So, $\theta = 0$ if $c > 0$ and $\theta = \pi$ if $c < 0$.

- If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors, then $\mathbf{u} \bullet \mathbf{v} = 0$ and thus

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = 0 \Longrightarrow \theta = \frac{\pi}{2}.$$

Def (空間向量的外積)

The cross product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

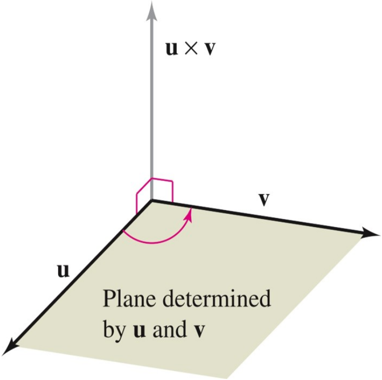
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \text{(對第一列作行列式降階!)}$$

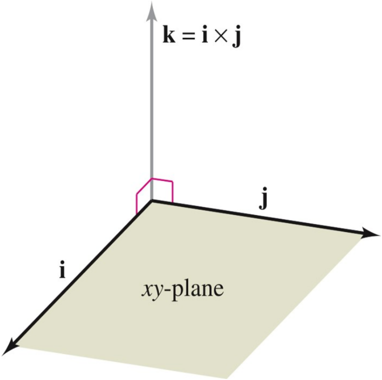
$$:= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Notes

- $\mathbf{u} \times \mathbf{v}$ is a vector in $\mathbb{R}^3$, but $\mathbf{u} \bullet \mathbf{v}$ is a scalar.
- $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0 = (\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v}$, i.e., the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, respectively.
- $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$, i.e., they are parallel vectors, but in the opposite directions.

外積的示意圖 (承上頁)





Properties of Cross Products

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$. Then

(1) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (2) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(3) $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$ (4) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(5) $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}$ (6) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Example (計算三維向量的外積)

The cross product of $\mathbf{u} = \langle 1, -2, 1 \rangle$ and $\mathbf{v} = \langle 3, 1, -2 \rangle$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} \quad \text{(對第一列作行列式降階!)}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{k}$$

$$= 3\,\mathbf{i} + 5\,\mathbf{j} + 7\,\mathbf{k} = \langle 3, 5, 7 \rangle \in \mathbb{R}^3.$$

Section 11.6
Cylinders and Quadratic Surfaces
(柱狀曲面與二次曲面)

Type I: Cylindrical Surfaces (柱狀曲面或是柱面)

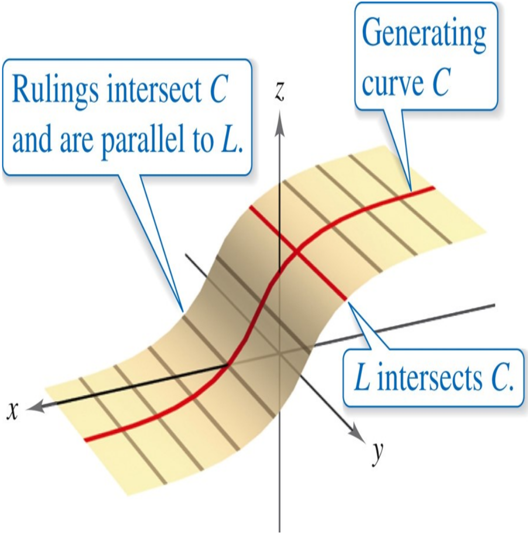
If the line L is not parallel to the plane containing a curve C, then

$$\mathcal{S} = \{\ell \,|\, \ell \text{ is a line parallel to } L \text{ and intersecting } C\}$$

is a cylindrical surface, or simply a cylinder.

- C is the generating curve (生成曲線) of the cylinder $\mathcal{S}$.

柱面的示意圖 (承上頁)


Rulings intersect C and are parallel to L.
Generating curve C
L intersects C.
z
x
y

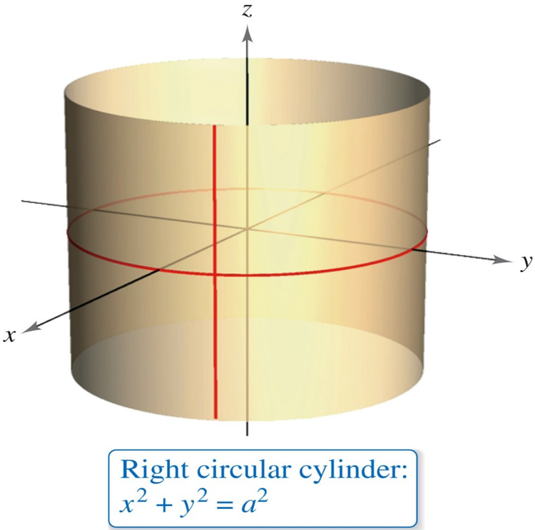
Example (圓形柱面的例子; 補充題)

The right circular cylinder (圓柱面) with radius $a > 0$ is defined by

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = a^2\}.$$

Then the generating curve C of the cylinder $\mathcal{S}$ lying in the xy-plane is $x^2 + y^2 = a^2$.

圓柱曲面的示意圖 (承上頁)



Example 1 (抛物柱面的例子)

The generating curve of the parabolic cylinder

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid y = x^2\}$$

is the parabola $y = x^2$ lying in the xy-plane.

Example 1 的示意圖

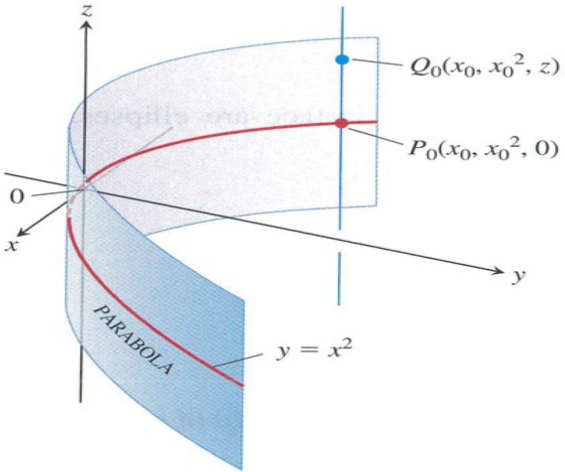


FIGURE 11.45 Every point of the cylinder in Example 1 has coordinates of the form (x_0, x_0^2, z).

Type II: Quadratic Surfaces (二次曲面)

The general equation of a quadratic surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where the coefficients $A, B, \cdots, J$ are real numbers.

Type II: Quadratic Surfaces

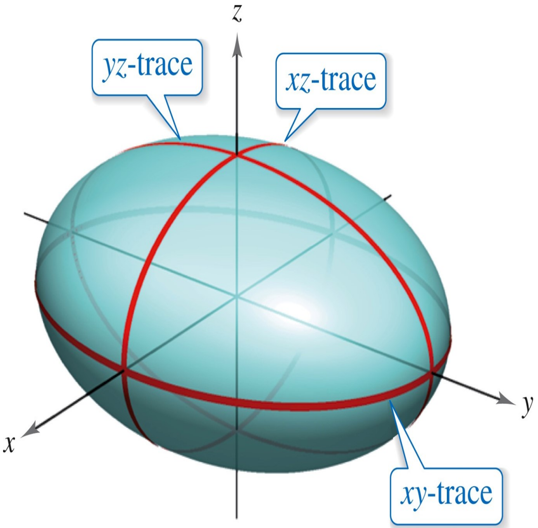
(1) Ellipsoid: (橢圓曲面)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{with } a, b, c > 0.$$

Them the xy-trace, xz-trace and yz-trace of the surface are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2}+\frac{z^2}{c^2}=1 \quad \text{and} \quad \frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Ellipsoid 的示意圖 (承上頁)



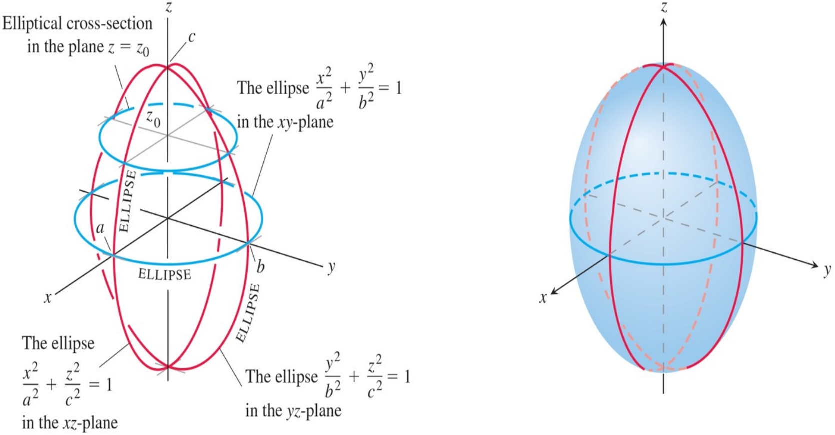


FIGURE 11.46 The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in Example 2 has elliptical cross-sections in each of the three coordinate planes.

Example 4 (橢圓曲面的例子)

Identify the surface given by the equation

$$x^2 + y^2 + 4z^2 - 2x + 4y + 1 = 0.$$

After completing the squares for the given equation, we have

$$(x-1)^2 + (y+2)^2 + 4z^2 + 1 = 1 + 4$$
$$\Longrightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} + \frac{z^2}{1} = 1,$$

which is an ellipsoid centered at $(1, -2, 0)$ whose three semiaxes have lengths 2, 2 and 1, respectively.

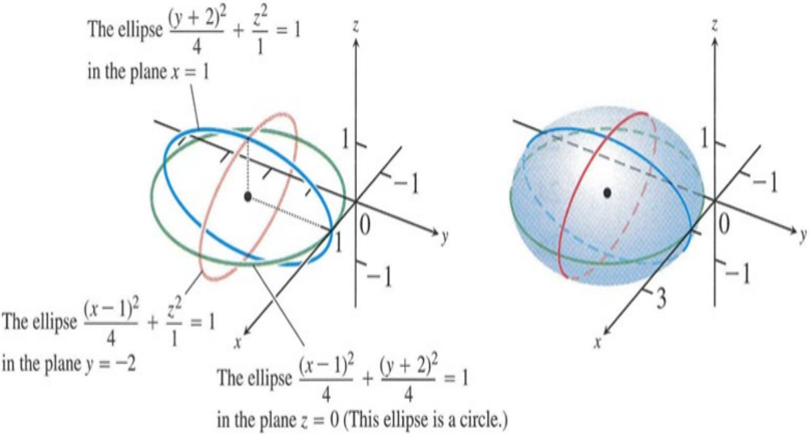


FIGURE 11.48 An ellipsoid centered at the point $(1, -2, 0)$.

Type II: Quadratic Surfaces

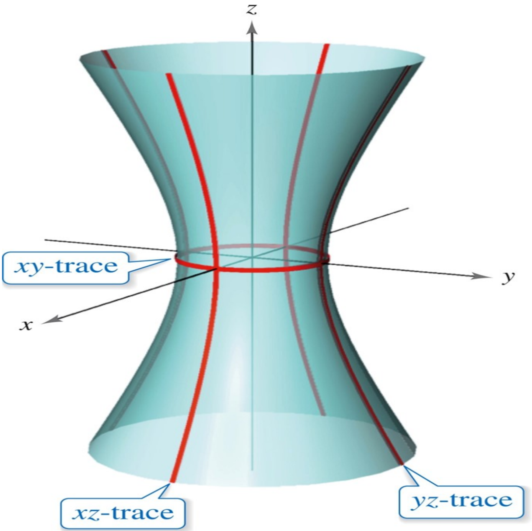
(2) Hyperboloid of One Sheet: (單片雙曲面)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{with } a, b, c > 0.$$

Them the xy-trace, xz-trace and yz-trace of the surface are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hyperboloid of One Sheet 的示意圖



Type II: Quadratic Surfaces

(3) Hyperboloid of Two Sheets: (雙片雙曲面)

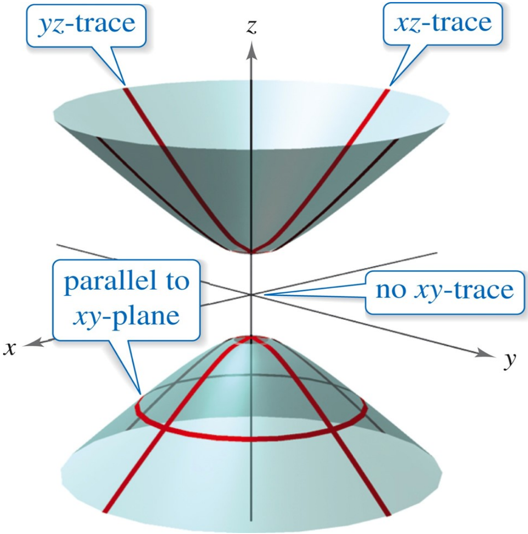
$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{with } a, b, c > 0.$$

Them the xz-trace and yz-trace of the surface are

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1 \quad \text{and} \quad \frac{z^2}{c^2} - \frac{y^2}{b^2} = 1,$$

but no xy-trace!

Hyperboloid of Two Sheets 的示意圖



Type II: Quadratic Surfaces

(4) Elliptic Cone: (橢圓錐面)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \text{with } a, b, c > 0.$$

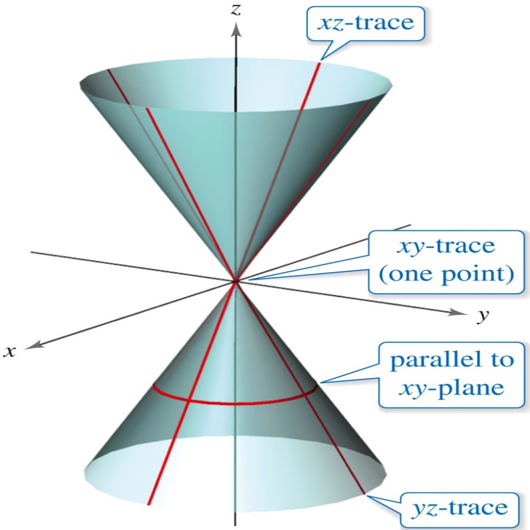
Them the xy-trace, xz-trace and yz-trace of the surface are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k \quad \text{for some } k \geq 0,$$

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0 \Longrightarrow z = \pm\frac{c}{a}x \text{ and}$$

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \Longrightarrow z = \pm\frac{c}{b}y.$$

Elliptic Cone 的示意圖



Type II: Quadratic Surfaces

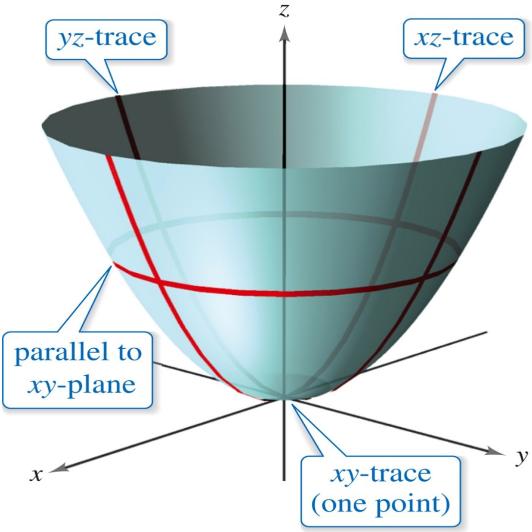
(5) Elliptic Paraboloid: (橢圓拋物面)

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{with } a, b > 0.$$

Them the *xy*-trace, *xz*-trace and *yz*-trace of the surface are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k \quad \text{for some } k \geq 0, \quad z = \frac{x^2}{a^2} \quad \text{and} \quad z = \frac{y^2}{b^2}.$$

Elliptic Paraboloid 的示意圖



Type II: Quadratic Surfaces

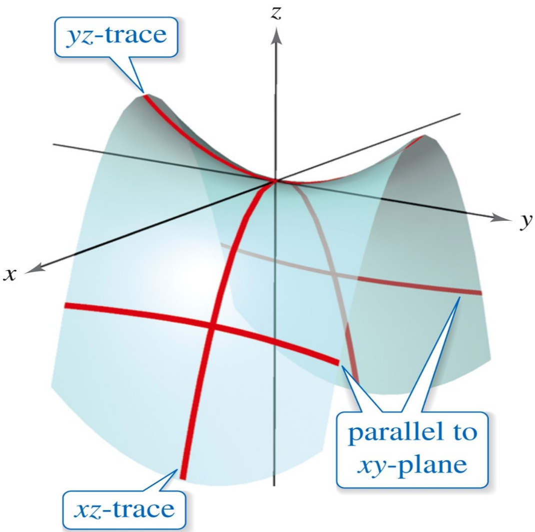
(6) Hyperbolic Paraboloid: (雙曲拋物面)

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2} \quad \text{with } a, b > 0.$$

Them the *xy*-trace, *xz*-trace and *yz*-trace of the surface are

$$y = \pm\frac{b}{a}x, \quad z = -\frac{x^2}{a^2} \quad \text{and} \quad z = \frac{y^2}{b^2}.$$

Hyperbolic Paraboloid 的示意圖



Example 3 (雙曲拋物面的例子)

For the hyperbolic paraboloid of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c} \quad \text{with} \quad a, b, c > 0,$$

we see that $z = \frac{c}{b^2} y^2$ if $x = 0$ (yz-plane), $z = \frac{-c}{a^2} x^2$ if $y = 0$ (xz-plane), and in the planes parallel to the xy-plane

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z_0}{c}, \quad z = z_0 > 0,$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{-z_0}{c}, \quad z = z_0 < 0,$$

are hyperbolas whose focal axes are parallel to the y-axis and the x-axis, respectively.

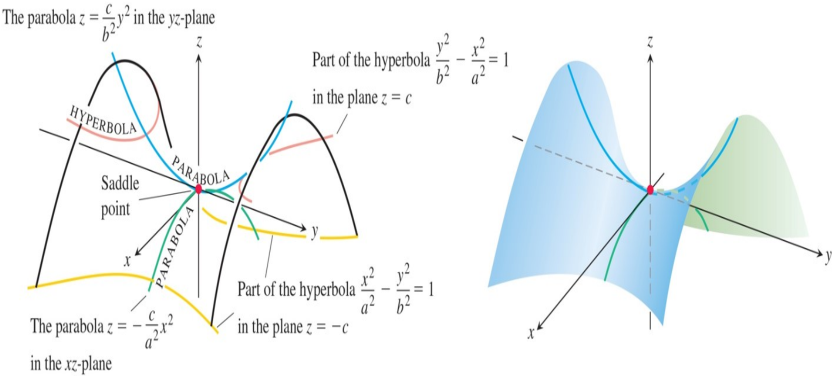


FIGURE 11.47 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

Thank you for your attention!