Chapter 12
Vector-Valued Functions and Motion in Space
(向量值函數與空間中的運動)

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本章預定授課範圍

Section 12.1
Curves in Space and Their Tangents
(空間中的曲線及其切線)

Def (向量值函數的定義)

(1) A function of the form

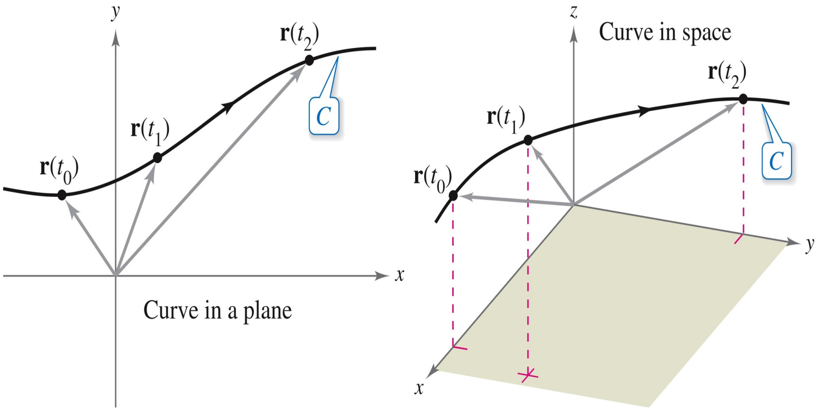
$$\mathbf{r}(t) = \vec{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} = \langle f(t), g(t) \rangle \in \mathbb{R}^2$$

or

$$\mathbf{r}(t) = \vec{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} = \langle f(t), g(t), h(t) \rangle \in \mathbb{R}^3$$

is a vector(-valued) function, where f, g, h are real-valued functions of parameter t.

(2) In this case, we say that f, g, h are the component functions (分量函數) of $\mathbf{r}(t)$.



y
$\mathbf{r}(t_2)$
C
$\mathbf{r}(t_1)$
$\mathbf{r}(t_0)$
x
Curve in a plane
z
Curve in space
$\mathbf{r}(t_2)$
$\mathbf{r}(t_1)$
$\mathbf{r}(t_0)$
C
y
x

The **helix** defined by the vector function

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}, \quad -\infty < t < \infty.$$

winds around the cylinder $x^2 + y^2 = 1$ because

$$x^2 + y^2 = (\cos t)^2 + (\sin t)^2 = 1 \quad \forall\, t \in \mathbb{R}.$$

Each time t increases by 2π, the curve completes one turn around the cylinder.

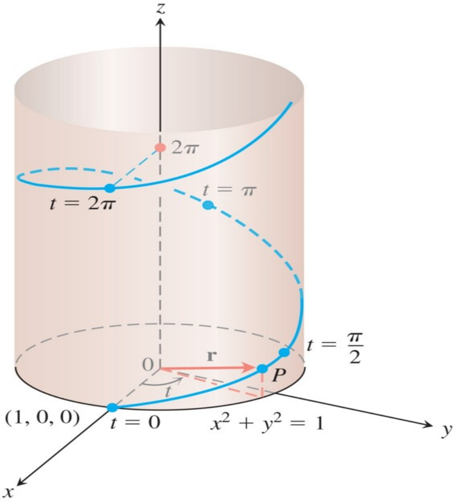


FIGURE 12.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

Example 1 的示意圖 (2/3)

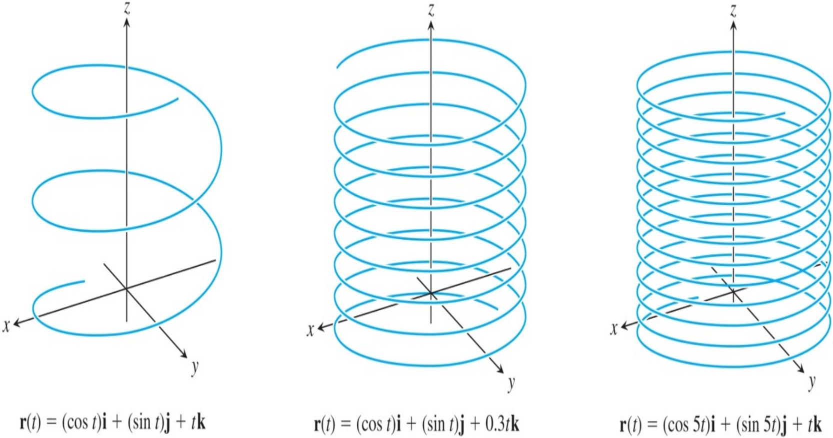
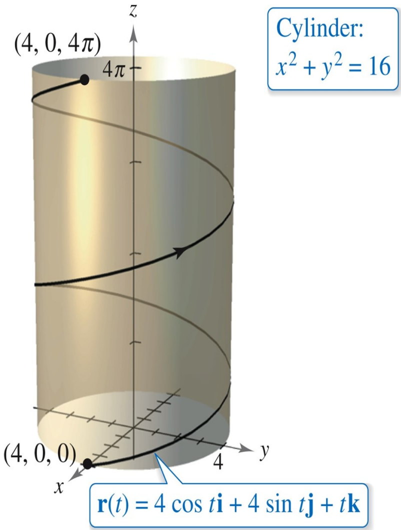
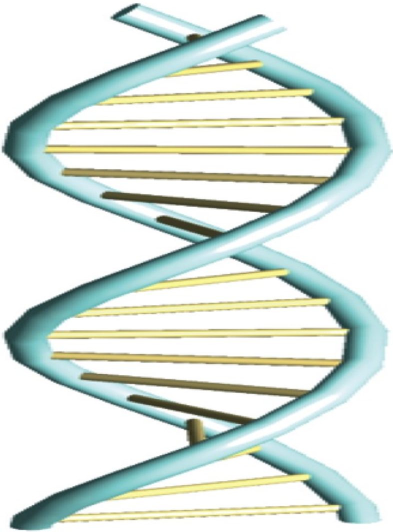


FIGURE 12.4 Helices spiral upward around a cylinder, like coiled springs.


z
(4, 0, 4π)
4π
Cylinder:
$x^2 + y^2 = 16$
(4, 0, 0)
4
y
x
$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}$



Def (向量值函數的極限)

Let $\mathbf{r}(t)$ be a vector function defined on an interval $I \setminus \{t_0\}$. We say that $\mathbf{r}(t)$ has a limit (vector) $\mathbf{L}$ as t approaches t_0, denoted by

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L},$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $0 < |t - t_0| < \delta$ and $t \in I$, then $\|\mathbf{r}(t) - \mathbf{L}\| = |\mathbf{r}(t) - \mathbf{L}| < \varepsilon$.

Thm (向量值函數的極限公式)

Suppose that $\lim_{t\to t_0} f(t)$, $\lim_{t\to t_0} g(t)$ and $\lim_{t\to t_0} h(t)$ both exist.

(1) For $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, we have

$$\lim_{t\to t_0} \mathbf{r}(t) = \left[\lim_{t\to t_0} f(t)\right]\mathbf{i} + \left[\lim_{t\to t_0} g(t)\right]\mathbf{j}.$$

(2) For $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, we have

$$\lim_{t\to t_0} \mathbf{r}(t) = \left[\lim_{t\to t_0} f(t)\right]\mathbf{i} + \left[\lim_{t\to t_0} g(t)\right]\mathbf{j} + \left[\lim_{t\to t_0} h(t)\right]\mathbf{k}.$$

For $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, we see that

$$\begin{aligned}\lim_{t\to\pi/4} \mathbf{r}(t) &= \left[\lim_{t\to\pi/4} \cos t\right]\mathbf{i} + \left[\lim_{t\to\pi/4} \sin t\right]\mathbf{j} + \left[\lim_{t\to\pi/4} t\right]\mathbf{k} \\ &= \frac{\sqrt{2}}{2}\,\mathbf{i} + \frac{\sqrt{2}}{2}\,\mathbf{j} + \frac{\pi}{4}\,\mathbf{k}.\end{aligned}$$

Def (向量值函數的連續性)

Let $\mathbf{r}(t)$ be a vector-valued function defined on the interval I containing t_0.

(1) $\mathbf{r}(t)$ is conti. at $t = t_0$ if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.

(2) $\mathbf{r}(t)$ is conti. on I if it is conti. at every point $t_0 \in I$.

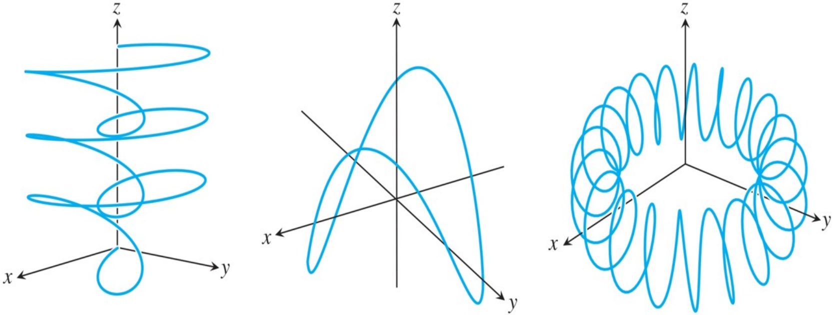
The Equivalence of Continuity

Thm (連續性的等價條件)

Let $f(t)$, $g(t)$ and $h(t)$ be real-valued functions defined on I.

$$\begin{aligned} & \mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \text{ is conti. on } I. \\ \Longleftrightarrow\ & f(t), g(t) \text{ and } h(t) \text{ are conti. on } I. \end{aligned}$$

All the vector functions shown in Figure 12.2 are conti. on $\mathbb{R}$.



$\mathbf{r}(t) = (\sin 3t)(\cos t)\mathbf{i} + (\sin 3t)(\sin t)\mathbf{j} + t\mathbf{k}$

(a)

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}$

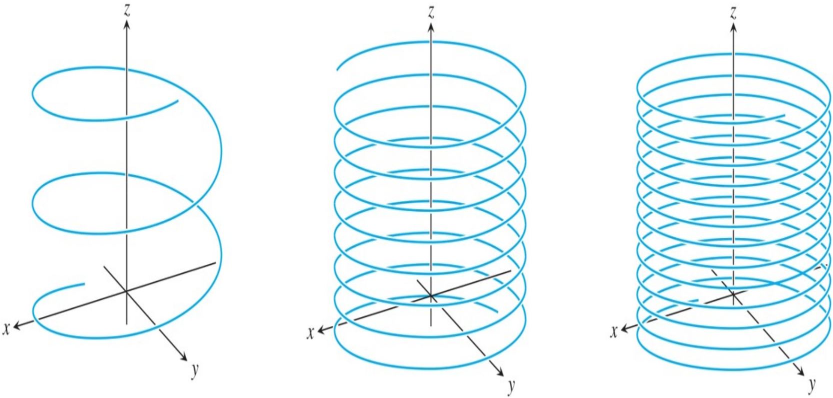
(b)

$\mathbf{r}(t) = (4 + \sin 20t)(\cos t)\mathbf{i} + (4 + \sin 20t)(\sin t)\mathbf{j} + (\cos 20t)\mathbf{k}$

(c)

FIGURE 12.2 Space curves are defined by the position vectors $\mathbf{r}(t)$.

All the vector functions shown in Figure 12.4 are conti. on $\mathbb{R}$.



$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0.3t\mathbf{k}$ $\qquad$ $\mathbf{r}(t) = (\cos 5t)\mathbf{i} + (\sin 5t)\mathbf{j} + t\mathbf{k}$

FIGURE 12.4 Helices spiral upward around a cylinder, like coiled springs.

Example 3b (不連續的向量值函數)

The vector-valued function defined by

$$\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + \lfloor t \rfloor\,\mathbf{k}$$

is disconti. at each $t \in \mathbb{Z}$, since the greatest integer function $\lfloor t \rfloor$ has a jump discontinuity at each $t \in \mathbb{Z}$.

Def (向量值函數的微分)

Let $\mathbf{r}(t)$ be a vector-valued function defined on the open interval I.

(1) $\mathbf{r}(t)$ is diff. at $t \in I$ if its derivative at t

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} \quad \exists.$$

(2) $\mathbf{r}(t)$ is diff. on I if it is diff. at every parameter $t \in I$.

Thm (向量值函數的微分公式)

Let f, g and h be diff. on an open interval I.

1. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)j\,\mathbf{j} \in \mathbb{R}^2$, then
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} \quad \forall t \in I.$$
2. If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \in \mathbb{R}^3$, then
$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k} \quad \forall t \in I.$$

Def (速度向量與加速度向量)

If a particle (質點) moves along a smooth curve $\mathcal{C}$ represented by the vector-valued function $\mathbf{r}(t)$ $\forall\, t \in I$, where I is some interval of time instants, we define

(1) the velocity (vector) at time t: $\mathbf{v}(t) := \mathbf{r}'(t)$.

(2) the acceleration (vector) at time t: $\mathbf{a}(t) := \mathbf{v}'(t) = \mathbf{r}''(t)$.

(3) the speed (速率) at time t: speed $:= |\mathbf{v}(t)| = |\mathbf{r}'(t)|$.

(4) the direction of motion at time t: $\dfrac{\mathbf{v}(\mathbf{t})}{|\mathbf{v}(\mathbf{t})|}$ with $\mathbf{v}(t) \neq \mathbf{0}$.

Find the velocity, acceleration and speed of a particle whose motion in space is given by the position vector

$$\mathbf{r}(t) = (2\cos t)\,\mathbf{i} + (2\sin t)\,\mathbf{j} + (5\cos^2 t)\,\mathbf{k}.$$

Sketch the velocity vector $\mathbf{v}\left(\dfrac{7\pi}{4}\right)$.

Solution of Example 4

The velocity vector, acceleration vector and speed at time t are

$$\begin{aligned}
\mathbf{v}(t) = \mathbf{r}'(t) &= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 10\cos t\sin t\,\mathbf{k} \\
&= -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k}, \\
\mathbf{a}(t) = \mathbf{v}'(t) &= -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k}, \\
\text{Speed} = |\mathbf{v}(t)| &= \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} \\
&= \sqrt{4 + 25\sin^2 2t}.
\end{aligned}$$

When $t = 7\pi/4$, we see that

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + 5\,\mathbf{k}, \quad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j}.$$

Example 4 的示意圖

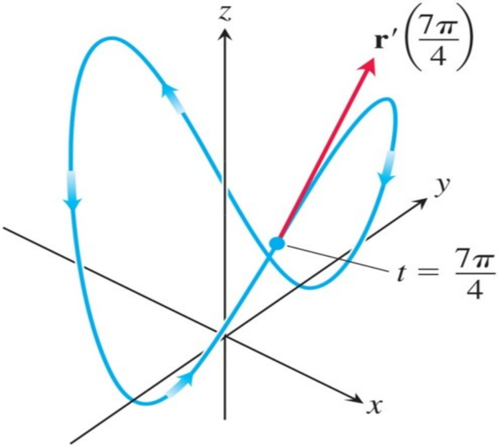


FIGURE 12.7 The curve and the velocity vector when $t = 7\pi/4$ for the motion given in Example 4.

Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be diff. vector-valued functions of t, and let $w(t)$ be a diff. real-valued function of t.

(1) $\frac{d}{dt}\Big[\mathbf{r}(t) \pm \mathbf{u}(t)\Big] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$.

(2) $\frac{d}{dt}\Big[c\mathbf{r}(t)\Big] = c\mathbf{r}'(t) \quad \forall\, c \in \mathbb{R}$.

(3) $\frac{d}{dt}\Big[w(t)\mathbf{r}(t)\Big] = w'(t)\mathbf{r}(t) + w(t)\mathbf{r}'(t)$.

(4) $\frac{d}{dt}\Big[\mathbf{r}(t) \bullet \mathbf{u}(t)\Big] = \mathbf{r}'(t) \bullet \mathbf{u}(t) + \mathbf{r}(t) \bullet \mathbf{u}'(t)$.

(5) $\frac{d}{dt}\Big[\mathbf{r}(t) \times \mathbf{u}(t)\Big] = \mathbf{r}'(t) \times \mathbf{u}(t) + \mathbf{r}(t) \times \mathbf{u}'(t)$.

(6) Chain Rule: $\frac{d}{dt}\Big[\mathbf{r}(w(t))\Big] = w'(t)\mathbf{r}'(w(t))$.

Section 12.2
Integrals of Vector Functions
(向量值函數的積分)

Def (向量值函數的不定積分)

(1) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} \in \mathbb{R}^2$, define

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j}.$$

(2) If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} \in \mathbb{R}^3$, define

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k}.$$

Integrating each component function of $\mathbf{r}$ w.r.t. t, we have

$$\begin{aligned}\int (\cos t\,\mathbf{i} + \mathbf{j} - 2t\,\mathbf{k})\,dt &= \Big(\int \cos t\,dt\Big)\,\mathbf{i} + \Big(\int dt\Big)\,\mathbf{j} - \Big(\int 2t\,dt\Big)\,\mathbf{k} \\ &= (\sin t + C_1)\,\mathbf{i} + (t + C_2)\,\mathbf{j} - (t^2 + C_3)\,\mathbf{k} \\ &= \sin t\,\mathbf{i} + t\,\mathbf{j} - t^2\,\mathbf{k} + \mathbf{C},\end{aligned}$$

where $\mathbf{C} = C_1\,\mathbf{i} + C_2\,\mathbf{j} - C_3\,\mathbf{k}$ is a constant vector of integration.

Def (向量值函數的定積分)

If f, g and h are integrable on $[a, b]$, then so is the vector-valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ and the definite integral of $\mathbf{r}$ from a to b is defined by

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k},$$

which is a constant vector in $\mathbb{R}^3$.

Example 2 (計算向量值函數的定積分)

Integrating each component function of $\mathbf{r}$ from 0 to π, we have

$$\begin{aligned}\int_0^{\pi}(\cos t\,\mathbf{i}+\mathbf{j}-2t\,\mathbf{k})\,dt &= \left(\int_0^{\pi}\cos t\,dt\right)\mathbf{i}+\left(\int_0^{\pi}dt\right)\mathbf{j}-\left(\int_0^{\pi}2t\,dt\right)\mathbf{k}\\ &= \left(\sin t\Big|_0^{\pi}\right)\mathbf{i}+\left(t\Big|_0^{\pi}\right)\mathbf{j}-\left(t^2\Big|_0^{\pi}\right)\mathbf{k}\\ &= (0-0)\,\mathbf{i}+(\pi-0)\,\mathbf{j}-(\pi^2-0^2)\,\mathbf{k}\\ &= 0\,\mathbf{i}+\pi\,\mathbf{j}-\pi^2\,\mathbf{k}=\pi\,\mathbf{j}-\pi^2\,\mathbf{k}\in\mathbb{R}^3.\end{aligned}$$

Find the glider's position function $\mathbf{r}(t)$ if its acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = -3\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j} + 2\,\mathbf{k},$$

$\mathbf{v}(0) = 3\,\mathbf{j}$ and $\mathbf{r}(0) = 4\,\mathbf{i} = \langle 4, 0, 0\rangle$.

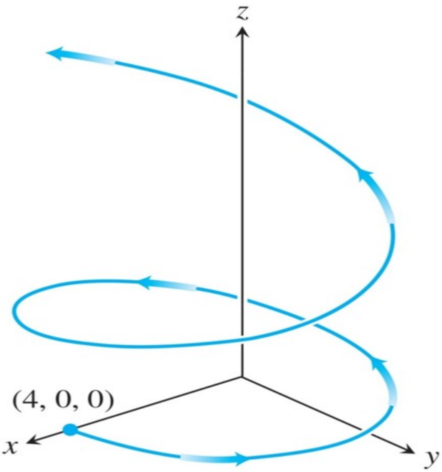


FIGURE 12.9 The path of the hang glider in Example 3. Although the path spirals around the z-axis, it is not a helix.

After integrating $\mathbf{a}(t)$ w.r.t. t, we immediately obtain

$$\mathbf{v}(t) = \int \mathbf{v}'(t)\,dt = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2t\,\mathbf{k} + \mathbf{D},$$

where $\mathbf{D} \in \mathbb{R}^3$ is a constant vector. Then $\mathbf{D} = \mathbf{0}$ follows from $\mathbf{v}(0) = 3\,\mathbf{j}$. Thus, the velocity (vector) of the glider is given by

$$\mathbf{r}'(t) = \mathbf{v}(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2t\,\mathbf{k}.$$

Next, integrating $\mathbf{v}(t) = \mathbf{r}'(t)$ w.r.t. t, we see that

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C},$$

where $\mathbf{C} \in \mathbb{R}^3$ is a constant vector. Then $\mathbf{C} = \mathbf{i}$ follows from $\mathbf{r}(0) = 4\,\mathbf{i}$. So, the glider's position at time t is determined by

$$\mathbf{r}(t) = (1 + 3\cos t)\,\mathbf{i} + 3\sin t\,\mathbf{j} + t^2\,\mathbf{k}.$$

Thank you for your attention!