Chapter 13
Partial Derivatives
(偏導數、偏導函數)

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本章預定授課範圍

Section 13.1
Functions of Several Variables
(多變量函數)

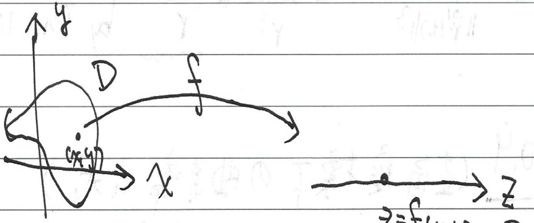
Type I: Functions of Two Variables (雙自變量函數)

Let $D \subseteq \mathbb{R}^2$. A function $z = f(x, y)$ of variables x and y is a rule that assigns to each $(x, y) \in D$ a unique value $f(x, y) \in \mathbb{R}$.

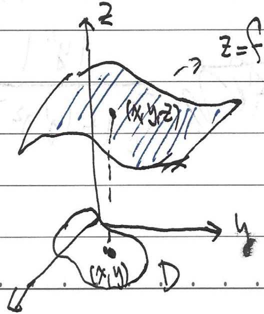
(1) $D = \text{dom}(f)$ is the domain of f.

(2) $\text{range}(f) = \{z = f(x, y) \in \mathbb{R} \mid (x, y) \in D\}$ is the range of f.

(3) x and y are the independent variables (自變數), and z is the dependent variable (應變數) of f.

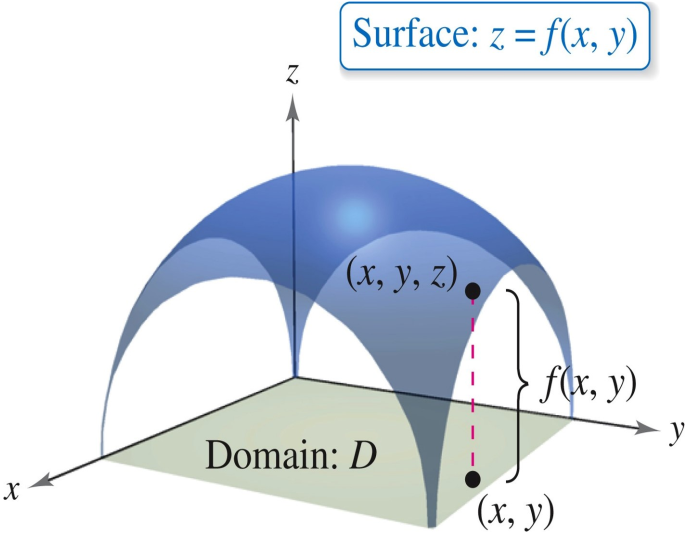


Note: the graph of $z=f(x,y)$ is a surface in $\mathbb{R}^3$.



$z=f(x,y)$, 的圖形是 $\mathbb{R}^3$ 中的曲面!

函數 $z = f(x, y)$ 的示意圖



Example 2 (描述 $z = f(x, y)$ 的定義域和值域)

The domain and range of $z = f(x, y) = \sqrt{y - x^2}$ are given by

$$\begin{aligned} D = \mathrm{dom}(f) &= \{(x, y) \in \mathbb{R}^2 \mid y - x^2 \geq 0\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}, \\ \mathrm{range}(f) &= \{z = f(x, y) = \sqrt{y - x^2} \mid (x, y) \in D\} \\ &= \{z = \sqrt{y - x^2} \geq 0 \mid y \geq x^2\} = [0, \infty). \end{aligned}$$

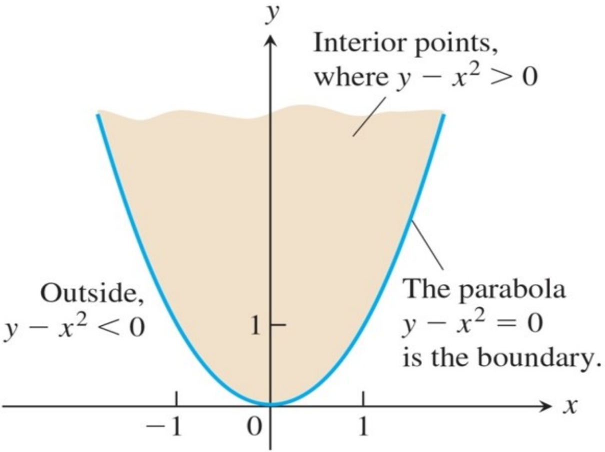


FIGURE 13.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

Def (函數 $f(x, y)$ 的等位線或等高線及其圖形)

Let $f(x, y)$ be a real-valued function defined on $D \subseteq \mathbb{R}^2$.

(1) For any $c \in \mathbb{R}$, the set $\mathcal{C}$ in the xy-plane defined by

$$\mathcal{C} = \{(x, y) \in D \,|\, f(x, y) = c\}$$

is called a level curve or contour curve of f.

(2) The graph of f is the set in space defined by

$$G(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3 \,|\, (x, y) \in D\},$$

which is often called the surface $z = f(x, y)$.

Graph the following real-valued function

$$f(x, y) = 100 - x^2 - y^2$$

and find the level curves of f for $c = 0,\ 51,\ 75$ and 100, respectively.

The level curves of f for the given values of c are

$$\begin{aligned}
\mathcal{C}_1 &= \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 100\},\\
\mathcal{C}_2 &= \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 51\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 49\},\\
\mathcal{C}_3 &= \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 75\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 25\},\\
\mathcal{C}_4 &= \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 100\} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 0\}\\
&= \{(x, y) \in \mathbb{R}^2 \mid x = y = 0\} = \{(0, 0)\}.
\end{aligned}$$

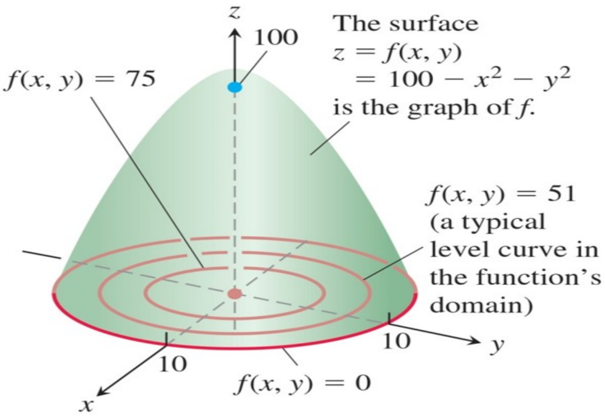
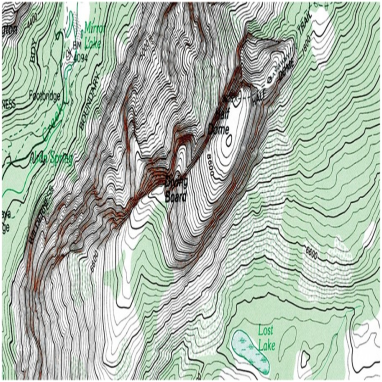


FIGURE 13.5 The graph and selected level curves of the function $f(x, y)$ in Example 3. The level curves lie in the xy-plane, which is the domain of the function $f(x, y)$.

等高線的應用

Alfred B. Thomas/Earth Scenes/Animals Animals; USGS



Type II: Functions of Three Variables (三自變量函數)

Let $D \subseteq \mathbb{R}^3$. A function $w = f(x, y, z)$ of x, y and z is a rule that assigns to each $(x, y, z) \in D$ a unique value $f(x, y, z) \in \mathbb{R}$.

(1) $D = \text{dom}(f)$ is the domain of f.

(2) $\text{range}(f) = \{w = f(x, y, z) \in \mathbb{R} \mid (x, y, z) \in D\}$ is the range of f.

(3) x, y, z are the independent variables, and w is the dependent variable of f.

Def (函數 $f(x, y, z)$ 的等位面及其圖形)

Let $f(x, y, z)$ be a real-valued function defined on $D \subseteq \mathbb{R}^3$.

(1) For any $c \in \mathbb{R}$, the set $\mathcal{S}$ in space defined by

$$\mathcal{S} = \{(x, y, z) \in D \,|\, f(x, y, z) = c\}$$

is called a level surface (等位面) of f.

(2) The graph of f is defined by the set

$$G(f) = \{(x, y, z, f(x, y, z)) \in \mathbb{R}^4 \,|\, (x, y, z) \in D\},$$

which can not be visualized in $\mathbb{R}^3$!

Example 4 (等位面的例子)

The level surface of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ given by

$$\begin{aligned}\mathcal{S} &= \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = \sqrt{x^2 + y^2 + z^2} = c\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = c^2\}\end{aligned}$$

is a sphere of radius $c \geq 0$ centered at the origin.

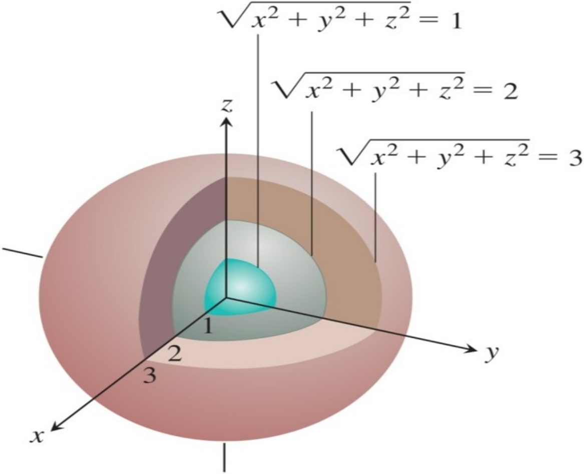


FIGURE 13.8 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 4).

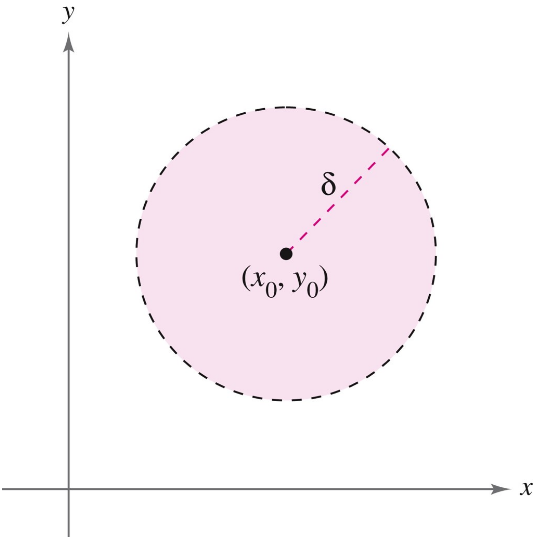
Section 13.2
Limits and Continuity in Higher Dimensions
(在高維度上的極限與連續性)

Open Disks and Interior Points

- The open disk of radius $\delta > 0$ centered at $(x_0, y_0) \in \mathbb{R}^2$ is

$$N_\delta(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}.$$

- Let $\mathcal{R}$ be a plane region. If $\exists\, \delta > 0$ s.t. $N_\delta(x_0, y_0) \subseteq \mathcal{R}$, then we say that (x_0, y_0) is an interior point (內點) of $\mathcal{R}$.

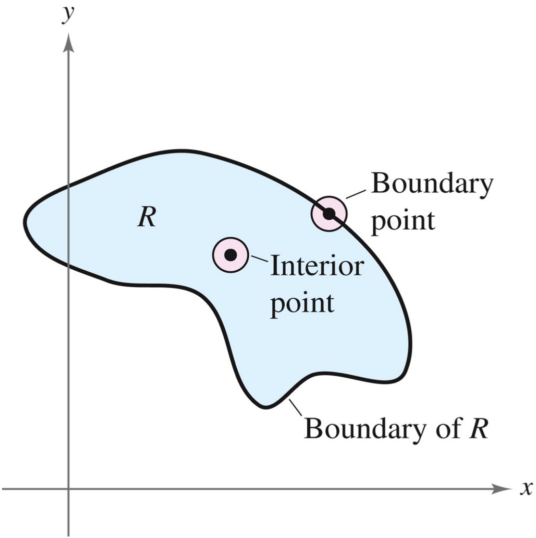

y
δ
(x_0, y_0)
x

Open and Closed Regions

We say that the plane region $\mathcal{R}$ is

- an open region if every point of $\mathcal{R}$ is an interior point.
- a closed region if $\mathcal{R}$ contains all its boundary points (邊界點).

Closed Region 的示意圖 (承上頁)



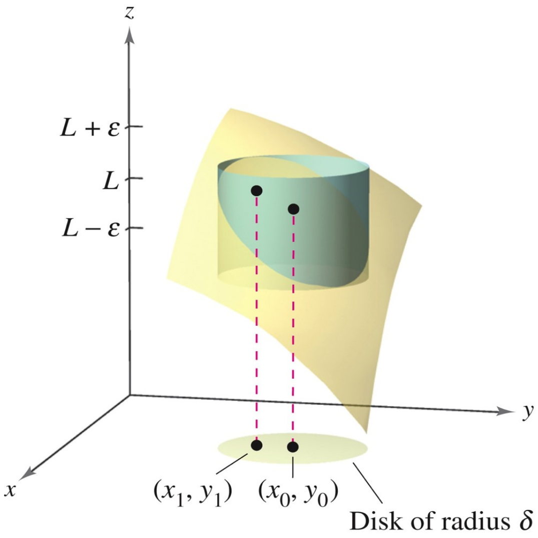
Def (雙自變量函數的極限)

Let $f(x, y)$ be a real-valued function defined on $D \setminus \{(x_0, y_0)\}$, where D is an open region in $\mathbb{R}^2$. We say that f has a limit L as (x, y) approaches (x_0, y_0), denoted by

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $(x, y) \in D \cap \Big(N_\delta(x_0, y_0) \setminus \{(x_0, y_0)\}\Big) \Longrightarrow |f(x, y) - L| < \varepsilon$.

雙自變量函數的極限 (承上頁)



Example (證明簡單的極限公式; 補充題)

For $f(x, y) = x$ and $f(x, y) = y$, the following limits

$$\lim_{(x,y)\to(x_0,y_0)} x = x_0, \qquad \lim_{(x,y)\to(x_0,y_0)} y = y_0$$

are true by considering $\delta = \varepsilon > 0$ in the Def. and the inequality

$$|x - x_0| \; \Big(\text{or } |y - y_0|\Big) \leq \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Thm 1 (雙自變量函數的極限法則; 1/2)

Assume $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, $\lim_{(x,y)\to(x_0,y_0)} g(x, y) = M$ and $k \in \mathbb{R}$.

(1) $\lim_{(x,y)\to(x_0,y_0)} k = k$, $\lim_{(x,y)\to(x_0,y_0)} |x| = |x_0|$, $\lim_{(x,y)\to(x_0,y_0)} |y| = |y_0|$.

(2) $\lim_{(x,y)\to(x_0,y_0)} [f(x, y) \pm g(x, y)] = L \pm M$.

(3) $\lim_{(x,y)\to(x_0,y_0)} [kf(x, y)] = kL$ and $\lim_{(x,y)\to(x_0,y_0)} [f(x, y)g(x, y)] = LM$.

(4) $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{L}{M}$ with $M \neq 0$.

Thm 1 (雙自變量函數的極限法則; 2/2)

(5) $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n \quad \forall\, n \in \mathbb{N}.$

(6) $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} \quad \forall\, n \in \mathbb{N}$, provided that $L \geq 0$ when n is even.

(7) $\lim_{(x,y)\to(x_0,y_0)} |f(x,y)| = 0 \iff \lim_{(x,y)\to(x_0,y_0)} f(x,y) = 0.$

(8) If $h(z)$ is conti. at $z = L$, then

$$\lim_{(x,y)\to(x_0,y_0)} h(f(x,y)) = h\left(\lim_{(x,y)\to(x_0,y_0)} f(x,y)\right) = h(L).$$

Applying the rationalizing technique and Thm 1, we see that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x-y} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) \\
&= \Big(\lim_{(x,y)\to(0,0)} x\Big)\Big[\Big(\lim_{(x,y)\to(0,0)} \sqrt{x}\Big)+\Big(\lim_{(x,y)\to(0,0)} \sqrt{y}\Big)\Big] \\
&= 0(0+0) = 0.
\end{aligned}$$

Example 3 (使用三明治定理求極限)

Find the following limit for a function of x and y

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$$

if it exists. Give your reasons.

Example 3 的示意圖

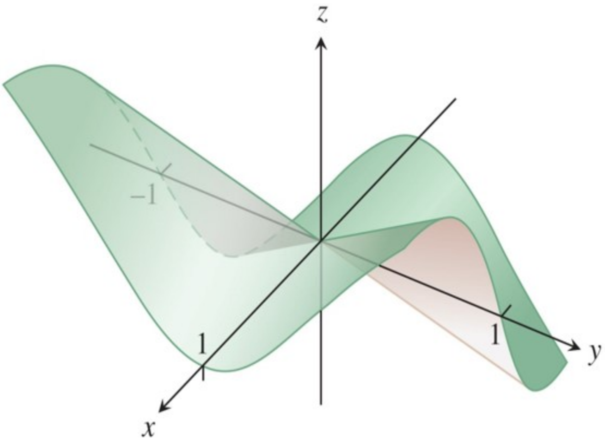


FIGURE 13.13 The surface graph suggests that the limit of the function in Example 3 must be 0, if it exists.

Let $f(x, y) = \dfrac{4xy^2}{x^2 + y^2}$. We first notice that

$$0 \le |f(x, y)| = \frac{4|x|y^2}{x^2 + y^2} = 4|x| \cdot \left(\frac{y^2}{x^2 + y^2}\right) \le 4|x|$$

for all $(x, y) \ne (0, 0)$. Since $\lim\limits_{(x,y)\to(0,0)} 4|x| = 0 = \lim\limits_{(x,y)\to(0,0)} 0$, it follows from the Sandwich Thm that $\lim\limits_{(x,y)\to(0,0)} |f(x, y)| = 0 \Longrightarrow$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

Fact (極限存在的等價條件)

The following statements are equivalent.

(1) $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

(2) $f(x,y) \to L$ as $(x,y) \to (x_0,y_0)$ along all paths passing (x_0,y_0).

Thm (Two-Path Test; 雙路徑測試)

If the following limits satisfy

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } \mathcal{C}_1}} f(x,y) = L_1 \neq L_2 = \lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{along } \mathcal{C}_2}} f(x,y)$$

for two different paths $\mathcal{C}_1$ and $\mathcal{C}_2$ passing (x_0, y_0), then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \quad \nexists.$$

Recall (單變數函數的雙路徑測試法則)

If the one-sided limits of $y = f(x)$ at x_0 satisfy

$$\lim_{x \to x_0^-} f(x) = L_1 \neq L_2 = \lim_{x \to x_0^+} f(x),$$

then $\lim_{x \to x_0} f(x) \nexists$ and thus f has a jump discontinuity at x_0.

Example 4 (極限不存在的例子)

If $f(x, y) = \frac{y}{x}$ for all $x \neq 0$, does the following limit

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{y}{x}$$

exist? Give your reasons.

Firstly, as $(x, y) \to (0, 0)$ along $\mathcal{C}_1 : \; y = 0$, we immediately obtain

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } \mathcal{C}_1}} f(x, y) = \lim_{x\to 0} f(x, 0) = \lim_{x\to 0} \frac{0}{x} = 0. \qquad (*)$$

On the other hand, as $(x, y) \to (0, 0)$ along $\mathcal{C}_2 : \; y = x$, we see that

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } \mathcal{C}_2}} f(x, y) = \lim_{x\to 0} f(x, x) = \lim_{x\to 0} \frac{x}{x} = 1. \qquad (**)$$

From (*) and (**), $\lim_{(x,y)\to(0,0)} f(x, y) \nexists$ by the Two-Path Test.

Def (雙自變量函數的連續性)

Let $f(x, y)$ be a real-valued function defined on the open region $D \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in D$.

(1) f is conti. at (x_0, y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

(2) f is conti. on D if it is conti. at every point $(x_0, y_0) \in D$.

Thm (雙自變量函數的連續性質)

(1) If $k \in \mathbb{R}$ and f, g are conti. at (x_0, y_0), then kf, $f \pm g$, fg and f/g with $g(x_0, y_0) \neq 0$ are conti. at (x_0, y_0), respectively.

(2) Continuity of Composite Functions:
If f is conti. at (x_0, y_0) and g is conti. at $f(x_0, y_0)$, then $(g \circ f)(x, y) := g(f(x, y))$ is conti. at (x_0, y_0).

Example 5 (判斷函數 $f(x, y)$ 的連續性)

Show that the function of x and y defined by

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is conti. on the open region $D = \mathbb{R}^2 \setminus \{(0, 0)\}$.

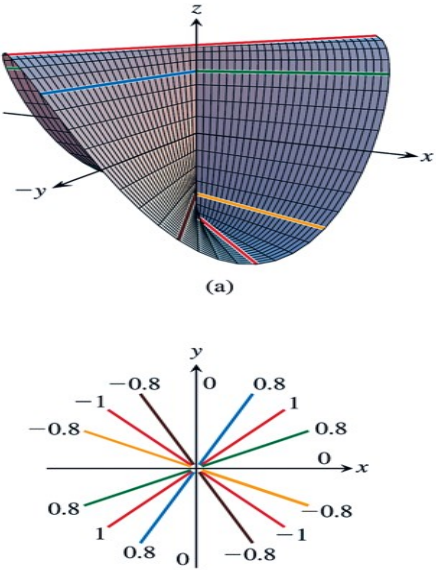
Solution of Example 5

As $(x, y) \to (0, 0)$ along $\mathcal{C} : y = mx$, we immediately obtain

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}}} f(x, y) = \lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

It follows from the Two-Path Test that $\lim\limits_{(x,y)\to(0,0)} f(x, y) \nexists$, and thus f is NOT conti. at $(0, 0)$. In addition, f is conti. at each point $(x, y) \neq (0, 0)$ by above Thm and we thus complete the proof.

Example 5 的示意圖



(a)

(b)

Example 6 (沿著特殊曲線證明極限不存在; 補充題)

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} \nexists.$

Note (沿著通過原點的直線)

For the path $\mathcal{C}: \ y = mx$, it can be easily seen that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}}} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{2mx^3}{x^4+m^2x^2} = 0.$$

As $(x, y) \to (0, 0)$ along $\mathcal{C} : y = kx^2$ with $k \neq 0$, we have

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } \mathcal{C}}} f(x, y) = \lim_{x\to 0} \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

From the Two-Path Test, we conclude that $\lim\limits_{(x,y)\to(0,0)} f(x, y) \nexists$.

Section 13.3
Partial Derivatives
(偏導數、偏導函數)

Def (雙自變量函數的偏微分; 1/2)

Let $f(x, y)$ bee defined on an open region $D \subseteq \mathbb{R}^2$.

(1) The first partial derivative of f w.r.t. x is

$$f_x(x, y) := \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$=$ 將 y 視為常數, 僅對 x 微分

Partial Derivatives of $f(x, y)$

Def (雙自變量函數的偏微分; 2/2)

(2) The first partial derivative of f w.r.t. y is

$$f_y(x, y) := \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$= $ 將 x 視為常數, 僅對 y 微分

Notations for Partial Derivatives

Equivalent Notations of f_x and f_y

Let $z = f(x, y)$ be a real-valued function of x and y.

1. $$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = D_x f = D_1 f.$$

2. $$z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = D_y f = D_2 f.$$

Example 3 (計算一階偏導函數)

Find f_x and f_y as functions of x and y if the function is defined by

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Note that $D = \text{dom}(f) = \{(x, y) \in \mathbb{R}^2 \mid y + \cos x \neq 0\}$.

The first partial derivatives of f are given by

$$f_x = \frac{0 - (2y)(0 - \sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2} \quad \text{and}$$
$$f_y = \frac{2(y + \cos x) - (2y)(1 + 0)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2},$$

which are well-defined on $D = \text{dom}(f)$.

Find $\frac{\partial z}{\partial x}$ assuming that the following nonlinear equation

$$yz - \ln z = x + y$$

defines z implicitly as a function of x and y, and the partial derivative $\frac{\partial z}{\partial x}$ exists.

Solution of Example 4

Applying the technique of implicit differentiation, we see that

$$0 \cdot z + y\frac{\partial z}{\partial x} - \frac{1}{z} \cdot \frac{\partial z}{\partial x} = 1 + 0 = 1$$
$$\Longrightarrow \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1 \Longrightarrow \frac{yz-1}{z} \cdot \frac{\partial z}{\partial x} = 1$$
$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{z}{yz-1} \text{ for } yz - 1 \neq 0 \text{ or } yz \neq 1.$$

Let $f(x, y)$ bee defined on an open region $D \subseteq \mathbb{R}^2$ and let $(x_0, y_0) \in D$ be given arbitrarily.

- $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\dfrac{df(x, y_0)}{dx}\right|_{x=x_0}$ is the slope of the tangent line to the curve $z = f(x, y_0)$ at the point $(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$.
- $\left.\dfrac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\dfrac{df(x_0, y)}{dy}\right|_{y=y_0}$ is the slope of the tangent line to the curve $z = f(x_0, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ in the vertical plane $y = y_0$.

Example 5 (求出空間曲線的切線斜率)

The plane $x = 1$ intersects the paraboloid given by

$$z = f(x, y) = x^2 + y^2$$

in a parabola (拋物線). Find the slope of the tangent line to the parabola at the point $(1, 2, 5) \in \mathbb{R}^3$.

Example 5 的示意圖

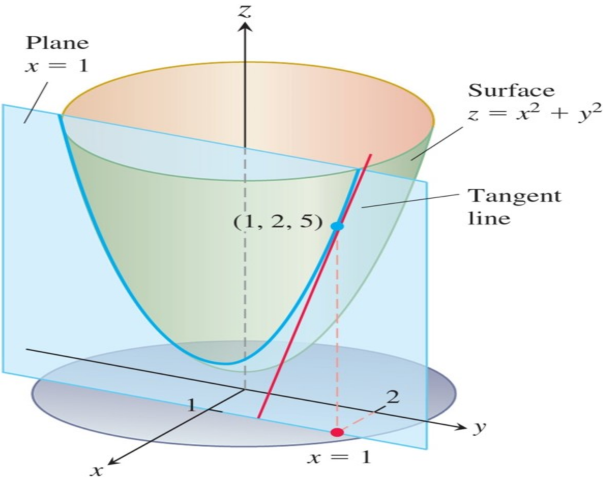


FIGURE 13.19 The tangent line to the curve of intersection of the plane $x = 1$ and the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$ (Example 5).

Let $z = f(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$.

Method 1 : The slope of the tangent line to the parabola $z = f(1, y) = 1 + y^2$ at the point $(1, 2, 5)$ is

$$\frac{\partial f}{\partial y}\Big|_{(1,2)} = \frac{\partial}{\partial y}(x^2 + y^2)\Big|_{(1,2)} = 2y\Big|_{y=2} = 4.$$

Method 2 : The slope of the tangent line to the parabola $z = f(1, y) = 1 + y^2$ at the point $(1, 2, 5)$ is

$$\frac{df(1, y)}{dy}\Big|_{y=2} = \frac{d}{dy}(1 + y^2)\Big|_{y=2} = 2y\Big|_{y=2} = 4.$$

Def (三自變量函數的偏微分; 1/3)

Let $f(x, y, z)$ bee defined on an open region $D \subseteq \mathbb{R}^3$.

(1) The first partial derivative of f w.r.t. x is

$$f_x(x, y, z) := \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x, y, z)}{\Delta x}$$

$=$ 將 y 和 z 視為常數, 僅對 x 微分

Def (三自變量函數的偏微分; 2/3)

(2) The first partial derivative of f w.r.t. y is

$$f_y(x, y, z) := \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$

$$= \text{將 } x \text{ 和 } z \text{ 視為常數, 僅對 } y \text{ 微分}$$

Def (三自變量函數的偏微分; 3/3)

(3) The first partial derivative of f w.r.t. z is

$$f_z(x, y, z) := \lim_{\Delta z \to 0} \frac{f(x, y, z + \Delta z) - f(x, y, z)}{\Delta z}$$

$=$ 將 x 和 y 視為常數, 僅對 z 微分

Example 6 (計算偏導函數 f_z)

Find $f_z = \dfrac{\partial f}{\partial z}$ if the real-valued function is defined by

$$f(x, y, z) = x\sin(y + 3z)$$

for all $(x, y, z) \in \mathbb{R}^3$.

Solution of Example 6

The partial derivative of f w.r.t. z is given by

$$\begin{aligned} f_z = \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\Big[x\sin(y+3z)\Big] \\ &= 0\cdot\sin(y+3z) + x\cdot\frac{\partial}{\partial z}\Big[\sin(y+3z)\Big] \\ &= x\cos(y+3z)\cdot\frac{\partial}{\partial z}(y+3z) = 3x\cos(y+3z). \end{aligned}$$

Consider a real-valued function of x and y defined by

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0. \end{cases}$$

(a) Find the limit of f as (x,y) approaches $(0,0)$ along $y = x$.

(b) Find the limit of f as (x,y) approaches $(0,0)$ along $y = 0$.

(c) Prove that f is not conti. at $(0,0)$.

(d) Show that $f_x(0,0)$ and $f_y(0,0)$ both exist.

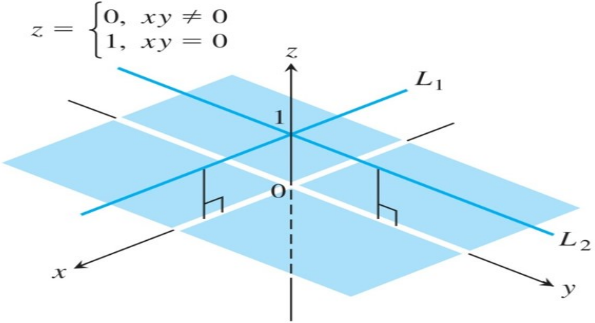


FIGURE 13.21 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 (lying 1 unit above the xy-plane) and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

(a) As $(x, y) \to (0, 0)$ along the line $\mathcal{C}_1 : \ y = x$, we obtain

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}_1}} f(x, y) = \lim_{x\to 0} f(x, x) = 0.$$

(b) As $(x, y) \to (0, 0)$ along the line $\mathcal{C}_2 : \ y = 0$, we see that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \mathcal{C}_2}} f(x, y) = \lim_{x\to 0} f(x, 0) = 1.$$

Solution of Example 8 (2/2)

(c) From the parts (a) and (b), we know that $\lim_{(x,y)\to(0,0)} f(x,y) \nexists$ by the Two-Path Test and hence f is NOT conti. at the origin.

(d) By Def., the first partial derivatives of f at $(0,0)$ are given by

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1-1}{\Delta x} = 0,$$
$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{1-1}{\Delta y} = 0.$$

Remarks (函數滿足連續性的充分條件)

Let f be a function defined on the open region D with $(x_0, y_0) \in D$.

(1) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist $\not\Longrightarrow$ f is conti. at (x_0, y_0). See Example 8 for more details.

(2) f is diff. at $(x_0, y_0) \Longrightarrow$ it is conti. at $((x_0, y_0)$. See Thm 4 below for more details.

Let $z = f(x, y)$ be a real-valued function of x and y.

- First Partial Derivatives: (一階偏導數)

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}.$$

- Second Partial Derivatives: (二階偏導數)

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$
$$f_{yy} = \frac{\partial^2 f}{\partial y^2} := \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

- Third Partial Derivatives: (8 個三階偏導數)

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} := \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad f_{xxy} = \frac{\partial^3 f}{\partial y \partial x^2} := \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right),$$

$$f_{xyx} = \frac{\partial^3 f}{\partial x \partial y \partial x} := \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right), \quad f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} := \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right),$$

$$f_{yyx} = \frac{\partial^3 f}{\partial x \partial y^2} := \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right), \quad f_{yyy} = \frac{\partial^3 f}{\partial y^3} := \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right),$$

$$f_{yxx} = \frac{\partial^3 f}{\partial x^2 \partial y} := \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right), \quad f_{yxy} = \frac{\partial^3 f}{\partial y \partial x \partial y} := \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}\right).$$

Thm 2 (The Mixed Derivative Theorem)

If $f(x, y)$ is a real-valued function s.t. f_x, f_y, f_{xy} and f_{yx} are conti. on an open region $\mathcal{R}$, then

$$f_{xy}(x, y) = f_{yx}(x, y) \quad \forall\, (x, y) \in \mathcal{R}.$$

Example 9 (驗證 Thm 2 的例子)

Find the second partial derivatives of the function

$$f(x, y) = x \cos y + y e^x,$$

and thus verify the correctness of Thm 2.

Solution of Example 9

Since the first partial derivatives of f are given by

$$f_x = \cos y + ye^x \quad \text{and} \quad f_y = -x\sin y + e^x,$$

it is easily seen that the second partial derivatives of f are

$$f_{xx} = ye^x, \quad f_{yy} = -x\cos y$$
$$f_{xy} = -\sin y + e^x, \quad f_{yx} = -\sin y + e^x.$$

Note that the first and second partial derivatives of f are conti. on $\mathrm{dom}(f) = \mathbb{R}^2$, and this example shows the correctness of Thm 2.

Differentiability of $y = f(x)$, Revisited

Recall (單變數函數的可微分等價條件)

$$
\begin{aligned}
& y = f(x) \text{ is diff. at } x = x_0. \\
\iff & f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \quad \exists. \\
\iff & \lim_{\Delta x \to 0} \left[\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right] = 0. \\
\iff & \lim_{\Delta x \to 0} \frac{\Delta y - f'(x_0)\Delta x}{\Delta x} = 0, \text{ with } \Delta y := f(x_0+\Delta x) - f(x_0). \\
\iff & \Delta y = f'(x_0)\Delta x + \varepsilon_1 \Delta x, \text{ with } \varepsilon_1 = \varepsilon_1(x_0, \Delta x) \to 0 \text{ as } \Delta x \to 0.
\end{aligned}
$$

Def (雙自變量函數的可微分性)

Let $z = f(x, y)$ be defined on an open region D with $(x_0, y_0) \in D$. Suppose that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist.

(1) f is diff. at (x_0, y_0) if $\exists$ two functions ε_1 and ε_2 s.t.

$$\begin{aligned}\Delta z &:= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,\end{aligned}$$

with $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

(2) f is diff. on D if it is diff. at every point $(x_0, y_0) \in D$.

Example (使用定義證明可微分性; 補充題)

Let $f(x, y) = x^2 + 3y \quad \forall\,(x, y) \in \mathbb{R}^2 = D = \text{dom}(f)$. For any $*(x, y) \in D$, if we let $\varepsilon_1 := \Delta x$ and $\varepsilon_2 := 0$, then

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= (x + \Delta x)^2 + 3(y + \Delta y) - x^2 - 3y\\ &= (2x)(\Delta x) + 3(\Delta y) + (\Delta x)^2 + 0(\Delta y)\\ &= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y,\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Thus, it follows immediately from the Defi. that f is diff. on $D = \mathbb{R}^2$.

Cor of Thm 3 (可微分的充分條件)

Let $f(x, y)$ be a real-valued function s.t. its first partial derivatives both exist on an open region $D \subseteq \mathbb{R}^2$. If f_x and f_y are conti. on D, then f is diff. on D.

Example (使用 Cor of Thm 3 證明可微分性; 補充題)

For $f(x, y) = x^2 + 3y$, we see that $f_x = 2x$ and $f_y = 3$ are both conti. on $D = \text{dom}(f) = \mathbb{R}^2$. Thus, it follows from Cor of Thm 3 that f is diff. on D.

Thm 4 (可微分 $\Longrightarrow$ 連續)

Let $f(x, y)$ be a real-valued function defined on an open region $D \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in D$.

(1) If f is diff. at (x_0, y_0), then f is conti. at (x_0, y_0).

(2) If f is NOT conti. at (x_0, y_0), then f is NOT diff. at (x_0, y_0).

Proof of Thm 4

If f is diff. at (x_0, y_0), then $\exists$ two functions ε_1 and ε_2 s.t. for any $x := x_0 + \Delta x$ and $y := y_0 + \Delta y$, we have

$$\begin{aligned} f(x,y) - f(x_0,y_0) &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \\ &= f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ &= \Big[f_x(x_0,y_0)+\varepsilon_1\Big](x - x - x_0) + \Big[f_y(x_0,y_0)+\varepsilon_2\Big](y - y - y_0), \end{aligned}$$

with $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus, we see that

$$\lim_{(x,y)\to(x_0,y_0)} \Big[f(x,y) - f(x_0,y_0)\Big] = 0 \Longrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0),$$

i.e., f is continuous at (x_0, y_0).

Def (可微分的等價定義)

Let $z = f(x, y)$ be defined on an open region D with $(x_0, y_0) \in D$. Suppose that $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist. We say that f is diff. at (x_0, y_0) if

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta z - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0,$$

where $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$.

Example (補充題; 連續 $\not\Rightarrow$ 可微分)

The function $f(x, y) = \sqrt{|xy|}$ is conti. at $(0, 0)$, but it is NOT diff. at $(0, 0)$. Why?

Example (Cor of Thm 3 的反例; 補充題)

A diff. function with disconti. first partial derivatives is given by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Verify that $f_x(0,0) = f_y(0,0) = 0$ and f is diff. at the origin by above Def., but f_x and f_y are NOT conti. at $(0,0)$! Why?

Section 13.4
The Chain Rule
(連鎖律)

Type I: One Parameter (Thm 5)

If $w = f(x, y)$ is a diff. function of x and y, $x = g(t)$ and $y = h(t)$ are diff. functions of parameter t, then $w = w(t)$ is diff. and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example 1 (Thm 5 的例子)

Find $\dfrac{dw}{dt}$ if $w = w(t)$ is a function of t defined by

$$w = xy, \quad x = \cos t, \quad y = \sin t,$$

and evaluate $\left.\dfrac{dw}{dt}\right|_{t=\pi/2}$.

From Thm 5, the derivative of w w.r.t. t is given by

$$\begin{aligned}\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = y(-\sin t) + x(\cos t)\\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) = \cos^2 t - \sin^2 t\\ &= \frac{1+\cos 2t}{2} - \frac{1-\cos 2t}{2} = \cos 2t.\end{aligned}$$

Moreover, $\left.\dfrac{dw}{dt}\right|_{t=\pi/2} = \cos 2t\Big|_{t=\pi/2} = \cos\pi = -1.$

Type II: Two Parameters (Cor of Thm 7)

If $w = f(x, y)$ is a diff. function of x and y, $x = g(r, s)$ and $y = h(r, s)$ are diff. functions of parameters r and s, then $w = w(r, s)$ is a diff. function of r and s, and

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

Example 4 (雙參數連鎖律的例子)

Express $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s,$$

where r and s are the parameters.

Applying the Chain Rule (Cor of Thm 7), we see that

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = (2x)(1) + (2y)(1)\\ &= 2(x+y) = 2(2r) = 4r,\\ \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = (2x)(-1) + (2y)(1)\\ &= 2(y-x) = 2(2s) = 4s.\end{aligned}$$

Type III: Multiple Parameters (多重參數)

If $w = f(x_1, x_2, \cdots, x_n)$ is a diff. function of $x_1, x_2, \cdots, x_n$, and $x_j = x_j(t_1, t_2, \cdots, t_m)$ is a diff. function of $t_1, t_2, \cdots, t_m$ for each $j = 1, 2, \ldots, n$, then $w = w(t_1, t_2, \cdots, t_m)$ is diff. and

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1},\\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2},\\
&\vdots\\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}.
\end{aligned}$$

(若參數個數為 m 時, 總共有 m 條連鎖律公式!)

Thm 8 (隱微分的公式)

Suppose that all first partial derivatives of F exist.

(1) If the equation $F(x, y) = 0$ defines y implicitly as a diff. function of x, then

$$\frac{dy}{dx} = \frac{-F_x}{F_y} \quad \text{with } F_y \neq 0.$$

(2) If the equation $F(x, y, z) = 0$ defines z implicitly as a diff. function of x and y, then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \quad \text{with } F_z \neq 0.$$

Proof of Thm 8 (1/2)

Suppose that $y = y(x)$ is a diff. function of x satisfying $F(x, y) = F(x, y(x)) = 0$. Then

$$
\begin{aligned}
& F_x \cdot \frac{dx}{dx} + F_y \cdot \frac{dy}{dx} = 0 \\
\Longrightarrow\ & F_x \cdot 1 + F_y \cdot \frac{dy}{dx} = 0 \\
\Longrightarrow\ & \frac{dy}{dx} = \frac{-F_x}{F_y} \text{ with } F_y \neq 0.
\end{aligned}
$$

Suppose that $z = z(x, y)$ is a diff. function of x and y satisfying $F(x, y, z) = F(x, y, z(x, y)) = 0$. Then

$$F_x \cdot \frac{dx}{dx} + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x} = F_x + F_z \cdot \frac{\partial z}{\partial x} = 0,$$
$$F_x \cdot 0 + F_y \cdot \frac{dy}{dy} + F_z \cdot \frac{\partial z}{\partial y} = F_y + F_z \cdot \frac{\partial z}{\partial y} = 0.$$

Therefore, we immediately obtain the following formulas

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} \quad \text{with } F_z \neq 0.$$

Use Thm 8 to find $\dfrac{dy}{dx}$ if the nonlinear equation

$$F(x, y) = y^2 - x^2 - \sin xy = 0$$

defines y implicitly as a diff. function of x.

Solution of Example 5

Since the first partial derivatives of F are given by

$$F_x = -2x - y\cos xy \quad \text{and} \quad F_y = 2y - x\cos xy,$$

it follows immediately from Thm 8 that

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(-2x - y\cos xy)}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}.$$

Example 6 (Thm 8 的例子)

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(0, 0, 0)$ if

$$F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y = 0,$$

where $z = z(x, y)$ is assumed to be a diff. function of x and y.

Since the first partial derivatives of F are given by

$$F_x = 3x^2 + yze^{xz}, \quad F_y = e^{xz} - z\sin y, \quad F_z = 2z + xye^{xz} + \cos y,$$

it follows immediately from Thm 8 that

$$\frac{\partial z}{\partial x}\Big|_{(0,0,0)} = \frac{-F_x}{F_z}\Big|_{(0,0,0)} = \frac{-3x^2 - yze^{xz}}{2z + xye^{xz} + \cos y}\Big|_{(0,0,0)} = 0$$

and

$$\frac{\partial z}{\partial y}\Big|_{(0,0,0)} = \frac{-F_y}{F_z}\Big|_{(0,0,0)} = \frac{-e^{xz} + z\sin y}{2z + xye^{xz} + \cos y}\Big|_{(0,0,0)} = -1.$$

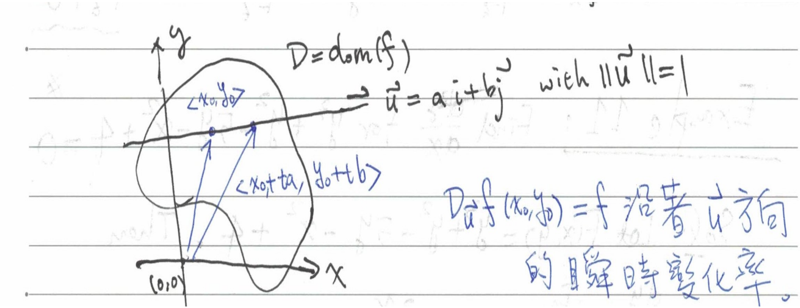
Section 13.5
Directional Derivatives and Gradient Vectors
(方向導數與梯度向量)

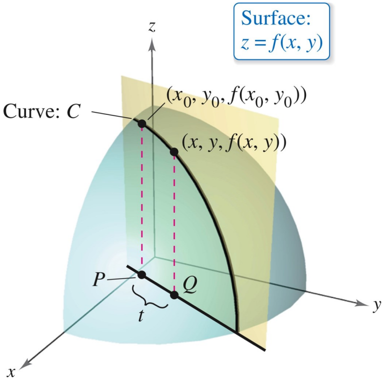
Def (雙自變量函數的方向導數)

Let $f(x, y)$ be a real-valued function defined on D with $(x_0, y_0) \in D$. The directional derivative of f at $P_0(x_0, y_0)$ in the direction of a unit vector $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is defined by

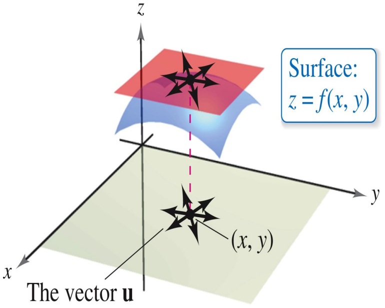
$$\left(\frac{df}{dt}\right)_{\mathbf{u},P_0} = D_{\mathbf{u}}f(P_0) := \lim_{t\to 0}\frac{f(x_0 + ta, y_0 + tb) - f(x_0, y_0)}{t}.$$

(函數 f 在 P_0 處沿著單位向量 u 的方向導數)


y
x
(0,0)
D=dom(f)
<x0,y0>
<x0+ta, y0+tb>
u = a i + b j with ||u|| = 1
Du f (x0,y0) = f 沿著 u 方向
的瞬時變化率。



Surface:
$z = f(x, y)$
z
Curve: C
$(x_0, y_0, f(x_0, y_0))$
$(x, y, f(x, y))$
P
Q
t
y
x



z
Surface:
$z = f(x, y)$
y
(x, y)
x
The vector **u**

Use the definition to find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of $\mathbf{u} = \dfrac{1}{\sqrt{2}}\,\mathbf{i} + \dfrac{1}{\sqrt{2}}\,\mathbf{j}$.

The directional derivative of f at P_0 in the direction of $\mathbf{u}$ is

$$\begin{aligned} D_{\mathbf{u}}f(P_0) &= \lim_{t\to 0} \frac{f(1+t/\sqrt{2}, 2+t/\sqrt{2}) - f(1,2)}{t} \\ &= \lim_{t\to 0} \frac{(1+t/\sqrt{2})^2 + (1+t/\sqrt{2})(2+t/\sqrt{2}) - (1^2+1\cdot 2)}{t} \\ &= \lim_{t\to 0} \frac{(2/\sqrt{2})t + (t^2/2) + (3/\sqrt{2})t + (t^2/2)}{t} \\ &= \lim_{t\to 0} \frac{(5/\sqrt{2})t + t^2}{t} = \lim_{t\to 0}\left(\frac{5}{\sqrt{2}} + t\right) = \frac{5}{\sqrt{2}}. \end{aligned}$$

How to evaluate $D_{\mathbf{u}}f(x_0, y_0)$?

Thm (方向導數的計算公式)

Let $f(x, y)$ be diff. on an open region D with $(x_0, y_0) \in D$. If $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is a unit vector in $\mathbb{R}^2$, then

$$D_{\mathbf{u}}f(x_0, y_0) = a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0).$$

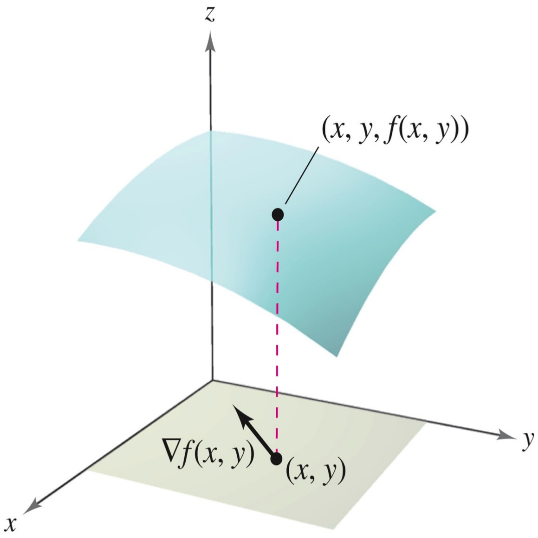
Since f is diff. on D, $g(t) := f(x_0 + ta, y_0 + tb)$ is also diff. for sufficiently small $t \neq 0$. It follows from Chain Rule (Thm 5) that

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= \lim_{t \to 0} \frac{f(x_0 + at, y_0 + bf) - f(x_0, y_0)}{t} \\ &= \lim_{t \to 0} \frac{g(t) - g(0)}{t - 0} = g\,'(0) \\ &= a \cdot f_x(x_0 + ta, y_0 + tb)\Big|_{t=0} + b \cdot f_y(x_0 + ta, y_0 + tb)\Big|_{t=0} \\ &= a \cdot f_(x_0, y_0) + b \cdot f_y(x_0, y_0). \end{aligned}$$

Def (梯度向量的定義)

Let $f(x, y)$ be a function of x and y s.t. f_x and f_y both exist. The gradient (vector) of f at (x, y) is

$$\nabla f(x, y) := f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} \in \mathbb{R}^2.$$


z
(x, y, f(x, y))
y
$\nabla f(x, y)$
(x, y)
x

Equivalent Formula for Evaluating $D_{\mathbf{u}} f(x_0, y_0)$

Thm 9 (方向導數的等價公式)

If $f(x, y)$ is diff. on an open region $D \subseteq \mathbb{R}^2$ and $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ is a unit vector in $\mathbb{R}^2$, then

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \bullet \mathbf{u} \quad \forall\,(x_0, y_0) \in D.$$

Proof of Thm 9

The proof follows immediately from above Thm, since we have

$$D_{\mathbf{u}}f(x_0, y_0) = a \cdot f_x(x_0, y_0) + b \cdot f_y(x_0, y_0) = \nabla f(x_0, y_0) \bullet \mathbf{u} \quad \forall\, (x_0, y_0) \in D,$$

which can be evaluated by the inner product of vectors ∇f and $\mathbf{u}$.

Remark (一階偏導數的另類解釋)

From Thm 9, the first partial derivatives of f can be interpreted as

$$f_x(x_0, y_0) = \nabla f(x_0, y_0) \bullet \mathbf{i} = D_{\mathbf{i}} f(x_0, y_0),$$
$$f_y(x_0, y_0) = \nabla f(x_0, y_0) \bullet \mathbf{j} = D_{\mathbf{j}} f(x_0, y_0).$$

Example 2 (Thm 9 的例子)

Find the derivative of the real-valued function

$$f(x, y) = xe^y + \cos(xy)$$

at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\,\mathbf{i} - 4\,\mathbf{j}$.

Solution of Example 2

Note that the unit vector in the direction of $\mathbf{v}$ is given by

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\,\mathbf{i} - 4\,\mathbf{j}}{\sqrt{(3)^2 + (4)^2}} = \frac{3}{5}\,\mathbf{i} - \frac{4}{5}\,\mathbf{j}.$$

Since the gradient vector of f at the point $(2, 0)$ is

$$\nabla f(2,0) = (e^y - y\sin xy)\Big|_{(2,0)}\mathbf{i} + (xe^y - x\sin xy)\Big|_{(2,0)}\mathbf{j} = \mathbf{i} + 2\,\mathbf{j},$$

it follows from Thm 9 that the directional derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is given by

$$D_{\mathbf{u}}f(2,0) = \nabla f(2,0) \bullet \mathbf{u} = (1)(\frac{3}{5}) + (2)(\frac{-4}{5}) = -1.$$

Example 2 的示意圖

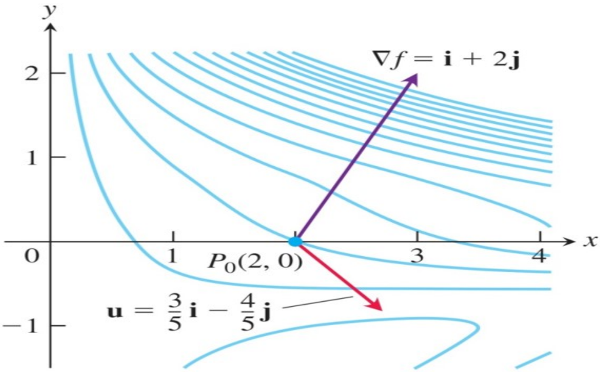


FIGURE 13.29 Picture ∇f as a vector in the domain of f. The figure shows a number of level curves of f. The rate at which f changes at (2, 0) in the direction $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = -1$, which is the component of ∇f in the direction of unit vector $\mathbf{u}$ (Example 2).

Remark (方向導數的另一種公式)

If $f(x, y)$ is a real-valued function s.t. its first partial derivatives exist on an open region D, and $\mathbf{u}$ is a unit vector in $\mathbb{R}^2$, then for any $(x, y) \in D$, we see that

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \bullet \mathbf{u} \\ &= \|\nabla f(x, y)\| \, \|\mathbf{u}\| \, \cos\theta \\ &= \|\nabla f(x, y)\| \, \cos\theta, \qquad (0 \le \theta \le \pi) \end{aligned}$$

where θ is the angle between vectors $\nabla f(x, y)$ and $\mathbf{u}$.

Properties of $D_{\mathbf{u}}f(x, y)$

(1) The function f increases most rapidly in the direction of ∇f, and the derivative in this direction is

$$D_{\mathbf{u}}f(x, y) = \|\nabla f(x, y)\| \cos(0) = \|\nabla f(x, y)\|$$

if $\mathbf{u} = \nabla f / \|\nabla f\|$ is a unit vector in the direction of ∇f.

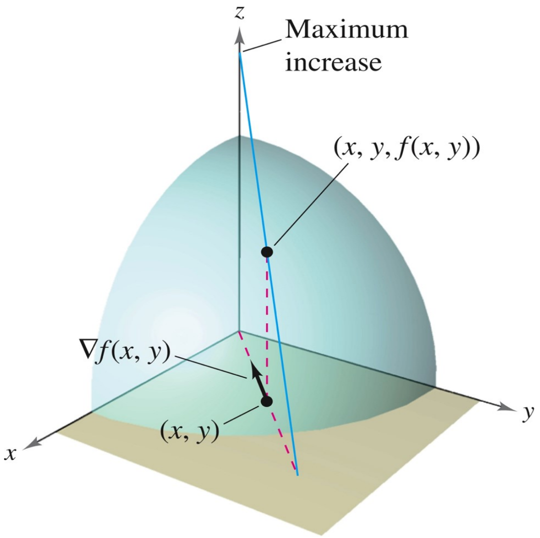
(2) The function f decreases most rapidly in the direction of $-\nabla f$, and the derivative in this direction is

$$D_{\mathbf{u}}f(x, y) = \|\nabla f(x, y)\| \cos(\pi) = -\|\nabla f(x, y)\|$$

if $\mathbf{u} = -\nabla f / \|\nabla f\|$ is a unit vector in the direction of $-\nabla f$.

(3) Any direction $\mathbf{u}$ orthogonal to $\nabla f \neq \mathbf{0}$ is a direction of zero change in f because $D_{\mathbf{u}}f(x, y) = \|\nabla f(x, y)\| \cos(\frac{\pi}{2}) = 0$.

函數值增加最快的方向



Example 3 (計算函數值增減最快的方向)

Find the directions in which the function of x and y defined by

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

(a) increases most rapidly at $(1,1)$, and

(b) decreases most rapidly at $(1,1)$.

(c) What are the directions of zero change in f at $(1,1)$?

Solution of Example 3 (1/2)

We first notice that the gradient of f at any $(x, y) \in \mathbb{R}^2 = \text{dom}(f)$ is

$$\nabla f(x, y) = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j} = x\,\mathbf{i} + y\,\mathbf{j}.$$

(a) The direction where f increases most rapidly at $(1, 1)$ is

$$\mathbf{u} = \frac{\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{j}.$$

Solution of Example 3 (2/2)

(b) The direction where f decreases most rapidly at $(1,1)$ is

$$-\mathbf{u} = \frac{-\nabla f(1,1)}{\|\nabla f(1,1)\|} = \frac{-\mathbf{i}-\mathbf{j}}{\sqrt{2}} = \frac{-1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change in f are

$$\mathbf{n} = \frac{-1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j},$$

since $D_{\mathbf{u}}f(1,1) = \nabla f(1,1) \bullet \mathbf{u} = 0$ if $\mathbf{u} = \pm\mathbf{n}$.

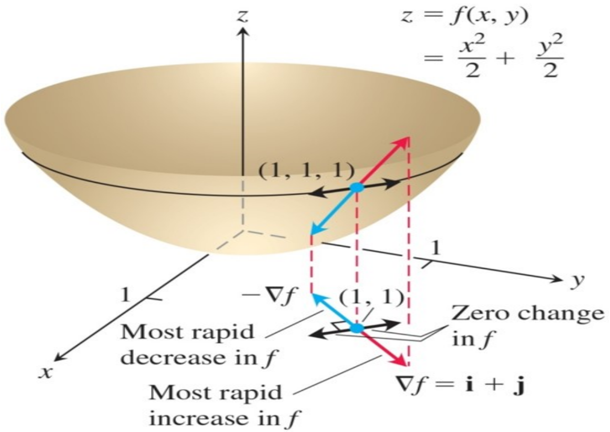


FIGURE 13.30 The direction in which $f(x, y)$ increases most rapidly at $(1, 1)$ is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$ (Example 3).

Thm (二維梯度向量垂直於等位線)

If $f(x, y)$ is diff. at a point $P(x_0, y_0)$ with the gradient vector

$$\nabla f(P) = \nabla f(x_0, y_0) \neq \mathbf{0},$$

then $\nabla f(P)$ is orthogonal (or normal) to the level curve $\mathcal{C}$ passing through the point $P(x_0, y_0)$.

Remark (等位線的切線方程式)

The equation of the tangent line to a level curve through (x_0, y_0) is

$$\nabla f(x_0, y_0) \bullet \langle x - x_0, y - y_0 \rangle = 0$$
$$\iff f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0,$$

where (x, y) is any point lying on the tangent line.

Example 4 (計算橢圓的切線方程式)

Find an equation for the tangent to the ellipse (橢圓)

$$f(x, y) = \frac{x^2}{4} + y^2 = 2$$

at the point $(-2, 1)$.

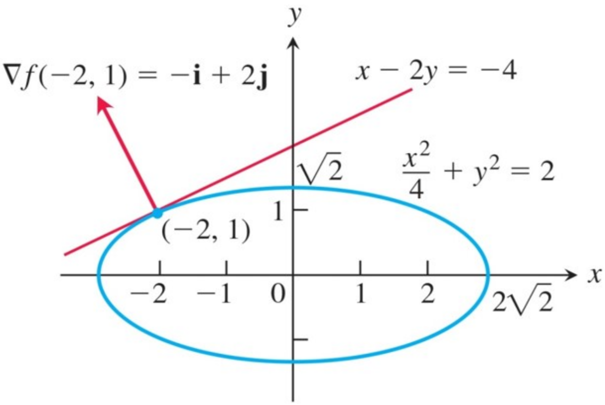


FIGURE 13.32 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

Solution of Example 4

Since the gradient vector of f at $(-2, 1)$ is

$$\nabla f(-2,1) = \left(\frac{x}{2}\,\mathbf{i} + 2y\,\mathbf{j}\right)\Big|_{(-2,1)} = -\mathbf{i} + 2\,\mathbf{j},$$

the equation of the tangent to the curve $f(x, y) = 2$ at $(-2, 1)$ is

$$(-1)(x+2) + (2)(y-1) = 0 \quad \text{or} \quad x - 2y = -4.$$

Let $f(x, y, z)$ be diff. on an open region $D \subseteq \mathbb{R}^3$.

(1) The gradient (vector) of f at $(x, y, z) \in D$ is

$$\nabla f(x, y, z) = f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k} \in \mathbb{R}^3.$$

(2) If $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is a unit vector in $\mathbb{R}^3$, then

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= a \cdot f_x(x, y, z) + b \cdot f_y(x, y, z) + c \cdot f_z(x, y, z) \\ &= \nabla f(x, y, z) \bullet \mathbf{u} \qquad \forall\, (x, y, z) \in D. \end{aligned}$$

(3) The direction of maximum increase of f is $\nabla f(x, y, z)$, and $D_{\mathbf{u}} f(x, y, z) = \|\nabla f(x, y, z)\|$ if $\mathbf{u} = \nabla f(x, y, z)/\|\nabla f(x, y, z)\|$.

(4) The direction of minimum increase of f is $-\nabla f(x, y, z)$, and $D_{\mathbf{u}} f(x, y, z) = -\|\nabla f(x, y, z)\|$ if $\mathbf{u} = -\nabla f(x, y, z)/\|\nabla f(x, y, z)\|$.

(5) For any $(x_0, y_0, z_0) \in D$, $\nabla f(x_0, y_0, z_0)$ is normal to the level surface $\mathcal{S}$ through (x_0, y_0, z_0).

Example 6 (計算 $f(x, y, z)$ 的方向導數)

Consider a function of x, y and z defined by

$$f(x, y, z) = x^3 - xy^2 - z.$$

(a) Find the derivative of f at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\,\mathbf{i} - 3\,\mathbf{j} + 6\,\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Note that the direction of $\mathbf{v}$ and the gradient vector of f are

$$\mathbf{u} = \frac{2}{7}\,\mathbf{i} - \frac{3}{7}\,\mathbf{j} + \frac{6}{7}\,\mathbf{k}, \quad \nabla f = (3x^2 - y^2)\,\mathbf{i} - 2xy\,\mathbf{j} - \mathbf{k}.$$

(a) The derivative of f at P_0 in the direction of $\mathbf{v}$ is

$$\begin{aligned} D_{\mathbf{u}}f(P_0) &= \nabla f(P_0) \bullet \mathbf{u} = (2\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}) \bullet \left(\frac{2}{7}\,\mathbf{i} - \frac{3}{7}\,\mathbf{j} + \frac{6}{7}\,\mathbf{k}\right) \\ &= (2)(2/7) + (-2)(-3/7) + (-1)(6/7) = \frac{4}{7}. \end{aligned}$$

(b) f increases most rapidly at the point P_0 in the direction of $\nabla f((P_0) = 2\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$, and decreases most rapidly in the direction of $-\nabla f(P_0) = -2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. Moreover, the rates of change in these directions are, respectively,

$$\|\nabla f(P_0)\| = \sqrt{4+4+1} = 3 \quad \text{and} \quad -\|\nabla f(P_0)\| = -3.$$

Thm (三維梯度向量垂直於等位面)

If $F(x, y, z)$ is diff. at the pooint $P(x_0, y_0, z_0)$ with the gradient

$$\nabla F(P) = \nabla F(x_0, y_0, z_0) \neq \mathbf{0},$$

then $\nabla F(P)$ is orthogonal to the level surface $\mathcal{S}$ passing through the point P.

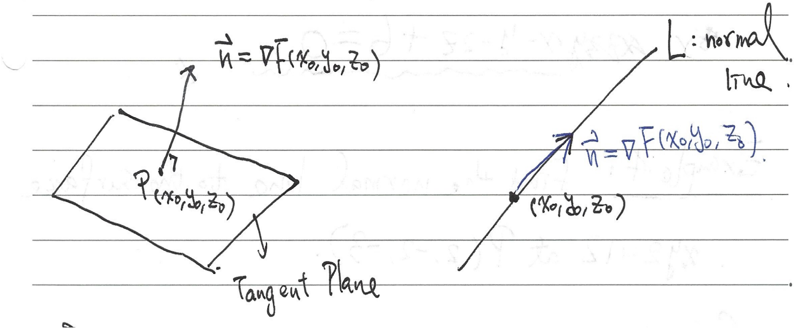
Section 13.6
Tangent Planes and Differentials
(切平面與微分)

Def (切平面與法線的定義)

Let $F(x, y, z)$ be diff. at the point $P(x_0, y_0, z_0)$ on the surface $\mathcal{S}$ given by $F(x, y, , z) = 0$ with $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$.

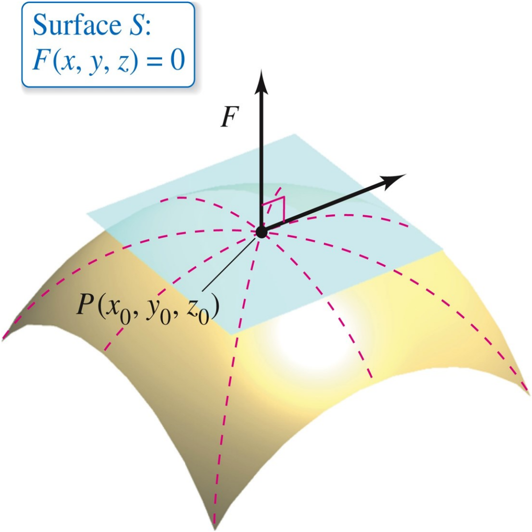
(1) The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the tangent plane (切平面) to $\mathcal{S}$ at P.

(2) The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the normal line (法線) to $\mathcal{S}$ at P.



$\vec{n} = \nabla F(x_0, y_0, z_0)$
$P(x_0, y_0, z_0)$
Tangent Plane
L : normal line.
$\vec{n} = \nabla F(x_0, y_0, z_0)$
(x_0, y_0, z_0)

切平面的示意圖



Thm (切平面的方程式)

The equation of the tangent plane to a surface $\mathcal{S}$ given by $F(x, y, z) = 0$ at $P(x_0, y_0, z_0)$ is

$$F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0.$$

Proof of above Thm

For any point (x, y, z) on the tangent plane, we see that

$$\langle x - x_0, y - y_0, z - z_0 \rangle \bullet \nabla F(P) = 0.$$

Since $\nabla F(P) = \langle F_x(P), F_y(P), F_z(P) \rangle$, the equation of the tangent plane is obtained consequently.

Equation of a Line in Space

Recall (空間直線的方程式)

The equation of a straight line through $(x_0, y_0, z_0) \in \mathbb{R}^3$ having the direction of $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ is given by

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$$

for $-\infty < t < \infty$.

Note: The equation of a normal line can be obtained from above prarmetric euations of x, y and z.

Example 1 (計算切平面與法線)

Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1, 2, 4)$.

Solution of Example 1

Since the gradient vector of f at $P_0(1, 2, 4)$ is given by

$$\nabla f(P_0) = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\Big|_{P_0} = 2\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k},$$

an equation of th tangent plane to the given surfce at P_0 shall be

$$2(x-1) + 4(y-2) + (1)(z-4) = 0 \quad \text{or} \quad 2x + 4y + z = 14.$$

Moreover, the line normal to the given surface at P_0 is

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t$$

for $-\infty < t < \infty$.

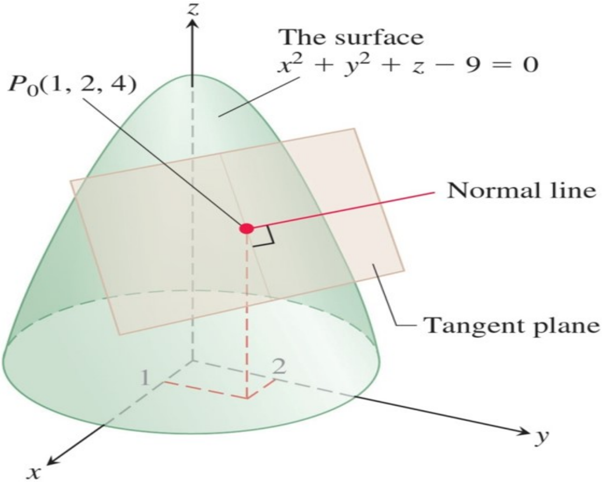


FIGURE 13.34 The tangent plane and normal line to this level surface at P_0 (Example 1).

Thm (曲面 $z = f(x, y)$ 的切平面方程式)

The plane tangent to the surface $z = f(x, y)$ of a diff. function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Proof of above Thm

This result holds by considering $F(x, y, z) := f(x, y) - z = 0$ directly, and its partial derivatives are given by

$$F_x(P_0) = f_x(x_0, y_0), \quad F_y(P_0) = f_y(x_0, y_0), \quad F_z(P_0) = -1.$$

Find the plane tangent to the following surface

$$z = f(x, y) = x \cos y - ye^x$$

at the point $P_0(0, 0, 0)$.

Since the first partial derivatives of f at the point $(0,0)$ are

$$f_x(0,0) = (\cos y - ye^x)\Big|_{(0,0)} = 1 - 0 = 1 \quad \text{and}$$
$$f_y(0,0) = (-x\sin y - e^x)\Big|_{(0,0)} = 0 - 1 = -1,$$

the tangent plane of the surface $z = f(x,y)$ at $P_0(0,0,0)$ is

$$1 \cdot (x-0) + (-1)\cdot(y-0) - (z-0) = 0 \quad \text{or} \quad x - y - z = 0.$$

Example 3 (兩曲面交線的切線參數式)

The two surfaces given by

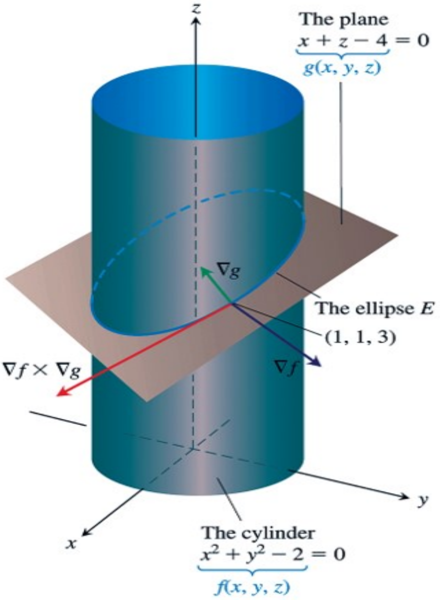
$$f(x, y, z) = x^2 + y^2 - 2 = 0$$

and

$$g(x, y, z) = x + z - 4 = 0$$

meet in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Example 3 的示意圖



Solution of Example 3

Note that the gradient vectors of f and g at $P_0(1,1,3)$ are given by

$$\nabla f(P_0) = (2x\,\mathbf{i} + 2y\mathbf{j} + 0\,\mathbf{k})\Big|_{P_0} = 2\,\mathbf{i} + 2\,\mathbf{j}, \quad \nabla g(P_0) = \mathbf{i} + \mathbf{k}.$$

The tangent line is orthogonal to both $\nabla f(P_0)$ and $\nabla g(P_0)$, and thus parallel to the following vector

$$\mathbf{v} = \nabla f(P_0) \times \nabla g(P_0) = (2\,\mathbf{i} + 2\,\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}.$$

So, the tangent line to rhe ellipse E of intersection at P_0 is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t, \quad -\infty < t < \infty.$$

Thm (估計 f 沿著 $\mathbf{u}$ 方向的函數值變化量)

The change in the value of a diff. function f when moving a small distance dt from the point P_0 in a specific direction of the unit vector $\mathbf{u}$ is approximately estimated by

$$\left(\frac{df}{dt}\right)_{\mathbf{u},P_0} = D_{\mathbf{u}}f(P_0) = \nabla f(P_0) \bullet \mathbf{u}$$

$$\Longrightarrow \Delta f \approx df = [\nabla f(P_0) \bullet \mathbf{u}]\, dt.$$

Example 4 (估計函數值變化量的例子)

Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change when moving $dt = 0.1$ unit from the point $P_0(0, 1, 0)$ straight forward $P_1(2, 2, -2)$.

Solution of Example 4

Firstly, the unit vector along the direction of $\overrightarrow{P_0P_1} = \langle 2, 1, -2 \rangle$ is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{\|\overrightarrow{P_0P_1}\|} = \frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}.$$

Since the gradient vector of f at $P_0(0, 1, 0)$ is given by

$$\nabla f(P_0) = \left[y\cos x\,\mathbf{i} + (\sin x + 2z)\,\mathbf{j} + 2y\,\mathbf{k}\right]\Big|_{P_0} = \mathbf{i} + 2\,\mathbf{k},$$

the approximate value of change in f when moving $dt = 0.1$ unit from the point P_0 in the direction of $\mathbf{u}$ is

$$df = \left[(\mathbf{i} + 2\,\mathbf{k}) \bullet \left(\frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}\right)\right](0.1) = \left(\frac{-2}{3}\right)(0.1) = \frac{-1}{15} \text{ unit.}$$

Example 4 的示意圖

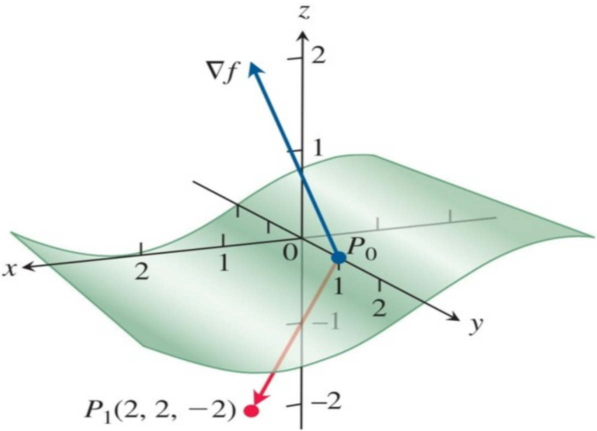


FIGURE 13.36 As $P(x, y, z)$ moves off the level surface at P_0 by 0.1 unit directly toward P_1, the function f changes value by approximately -0.067 unit (Example 4).

Def (Total Differential 的定義)

Let $z = f(x, y)$ be a real-valued function of x and y.

(1) The differentials of x and y are $dx = \Delta x$ and $dy = \Delta y$, where Δx and Δy are (small) increments of x and y.

(2) The total differential (全微分) of z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Note (函數 $f(x, y)$ 的線性近似公式)

If $dx = \Delta x$ and $dy = \Delta y$ are sufficiently small, then

$$\begin{aligned}\Delta z &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &\approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.\end{aligned}$$

So, the approximate vale of $f(x_0 + \Delta x, y_0 + \Delta y)$ is obtained by

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Section 13.7
Extreme Values and Saddle Points
(極值與鞍點)

Def (雙自變量函數的極值定義)

Let $f(x, y)$ be a function defined on $D \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in D$.

(1) $f(x_0, y_0)$ is a local minimum value of f if $f(x, y) \geq f(x_0, y_0)$ for all $(x, y) \in D$ near (x_0, y_0).

(2) $f(x_0, y_0)$ is a local maximum value of f if $f(x, y) \leq f(x_0, y_0)$ for all $(x, y) \in D$ near (x_0, y_0).

(3) $f(x_0, y_0)$ is an absolute minimum value of f if $f(x, y) \geq f(x_0, y_0) \quad \forall\, (x, y) \in D$.

(4) $f(x_0, y_0)$ is an absolute maximum value of f if $f(x, y) \leq f(x_0, y_0) \quad \forall\, (x, y) \in D$.

Def (臨界點的定義)

Let $f(x, y)$ be a real-valued function defined on an open region D. The point $(x_0, y_0) \in D$ is called a critical point (臨界點) of f if either

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0,$$

or

$$\text{one of } f_x(x_0, y_0) \text{ and } f_y(x_0, y_0) \nexists.$$

Thm 10 (局部極值只發生在臨界點上)

Let $f(x, y)$ be a real-valued function defined on an open region D. If f has a local maximum or minimum value at $(x_0, y_0) \in D$, then (x_0, y_0) is a critical point of f.

Suppose that f has a relative maximum at (x_0, y_0) and that f_x, f_y both exist on D. Then it is easily seen that

$$0 \le \lim_{\Delta x \to 0-} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} = f_x(x_0, y_0) = \lim_{\Delta x \to 0+} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \le 0,$$

and

$$0 \le \lim_{\Delta y \to 0-} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} = f_y(x_0, y_0) = \lim_{\Delta y \to 0+} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \le 0.$$

So, $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, i.e., (x_0, y_0) is a critical point of f.

Example 1 (Thm 10 的例子)

Find the local extreme values of the real-valued function

$$f(x, y) = x^2 + y^2 - 4y + 9$$

on its domain $D = \mathrm{dom}(f) = \mathbb{R}^2$.

Example 1 的示意圖

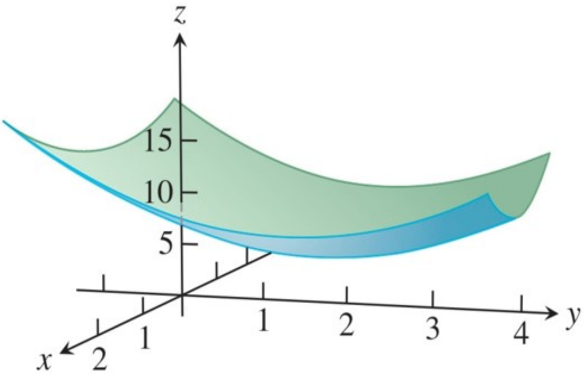


FIGURE 13.46 The graph of the function $f(x, y) = x^2 + y^2 - 4y + 9$ is a paraboloid which has a local minimum value of 5 at the point $(0, 2)$ (Example 1).

Note that if the first partial derivatives of f satisfy

$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y - 4 = 0,$$

then $x = 0$ and $y = 2$, and thus $(0, 2)$ is the only critical point of f. Furthermore, since it is easily seen that

$$f(x, y) = x^2 + y^2 - 4y + 9 = x^2 + (y-2)^2 + 5 \geq 5 = f(0, 2) \quad \forall\, (x, y) \in \mathbb{R}^2,$$

$f(0, 2) = 5$ is an absolute min. value of f and henec it is also a local min. value of f.

Def (曲面上鞍點的定義)

Let $f(x, y)$ be a diff. function defined on an open region $D \subseteq \mathbb{R}^2$.

(1) We say that f has a saddle point at a critical point $(a, b) \in D$ if $\exists\ (x_1, y_1), (x_2, y_2) \in D$ near (a, b) s.t. $f(x_1, y_1) > f(a, b)$ and $f(x_2, y_2) < f(a, b)$.

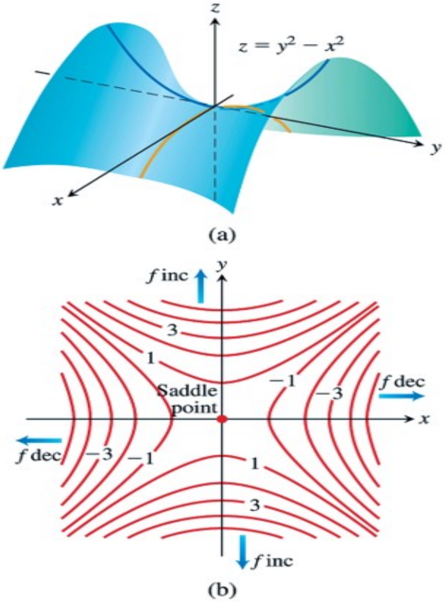
(2) In this case, the point $(a, b, f(a, b))$ is called a saddle point of the surface $z = f(x, y)$.

Example 2 (鞍點的例子)

For the surface $z = f(x, y) = y^2 - x^2$, its first partial derivatives are given by

$$f_x = -2x \quad \text{and} \quad f_y = 2y,$$

and thus $(0, 0)$ is the only critical point of f with $f(0, 0) = 0$. Since $f(0, y) = y^2 > 0 = f(0, 0)$ for $y \neq 0$ and $f(x, 0) = -x^2 < 0 = f(0, 0)$ for $x \neq 0$, it follows from Def. that f has a saddle point $(0, 0, 0)$ at the critical point $(a, b) = (0, 0) \in \mathbb{R}^2 = \text{dom}(f)$.


z
$z = y^2 - x^2$
y
x
(a)
y
f inc
3
1
Saddle
point
−1
−3
f dec
x
f dec
−3
−1
1
3
f inc
(b)

Thm 11 (二階偏導數測試)

Suppose $f(x, y)$ has conti. second partial derivatives on an open region D containing (a, b). If $f_x(a, b) = f_y(a, b) = 0$ and

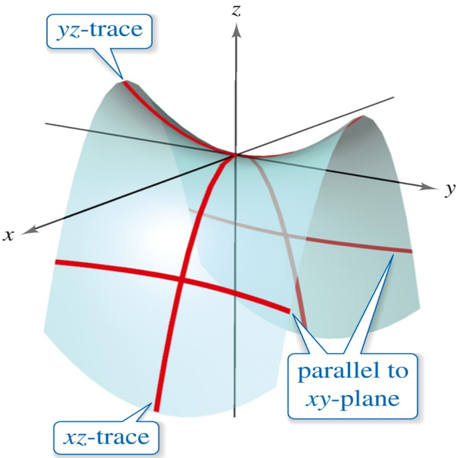
$$d := f_{xx}(a, b) f_{yy}(a, b) - \left[f_{xy}(a, b) \right]^2,$$

then

1. $d > 0$ and $f_{xx}(a, b) > 0 \implies f$ has a local min. value at (a, b).
2. $d > 0$ and $f_{xx}(a, b) < 0 \implies f$ has a local max. value at (a, b).
3. $d < 0 \implies f$ has a saddle point at (a, b).
4. $d = 0 \implies$ the test is inconclusive.

鞍點 (Saddle Point) 的示意圖

The origin $(0, 0, 0)$ is a saddle point of the surface $z = y^2 - x^2$.



Example 3 (Thm 11 的例子)

Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

on its domain $D = \text{dom}(f) = \mathbb{R}^2$.

Solving the following system of equations

$$f_x = y - 2x - 2 = 0 \quad \text{and} \quad f_y = x - 2y - 2 = 0,$$

we then have $x = y = -2$ and hence $(-2, -2)$ is the only critical point of f.

Solution of Example 3 (2/2)

Since the second partial derivatives of f are

$$f_{xx} = -2 < 0, \quad f_{yy} = -2, \quad f_{xy} = 1,$$

we see that $d = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - (1)^2 = 3 > 0$. Therefore, it follows from Thm 11 that $f(-2,-2) = 8$ is a local maximum value of f.

Example 4 (Thm 11 的例子)

Find the local extreme values of the function

$$f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$$

on its domain $D = \mathrm{dom}(f) = \mathbb{R}^2$.

Solution of Example 4 (1/2)

Firstly, we need to solve the system of nonlinear equations

$$f_x = -6x + 6y = 0 \quad \text{and} \quad f_y = 6y - 6y^2 + 6x = 0,$$

and then $x = y$ from the first equation. Substituting this into the second equation, we immediately obtain

$$6x - 6x^2 + 6x = 12x - 6x^2 = 6x(2 - x) = 0 \Longrightarrow x = 0 \text{ or } 2.$$

So, $(0, 0)$ and $(2, 2)$ are critical points of f, respectively.

Note that second partial derivatives of f satisfy $f_{xx} = -6 < 0$ and

$$d = f_{xx}f_{yy} - (f_{xy})^2 = (-6)(6 - 12y) - (6)^2 = 72(y - 1).$$

For these two critical points of f, from Thm 11 we see that

(1) f has a saddle point at $(0,0)$ because $4d = -72 < 0$, and

(2) $f(2,2) = 8$ is a local maximum value because $f_{xx} = -6 < 0$ and $d = 72 > 0$.

Example 4 的示意圖

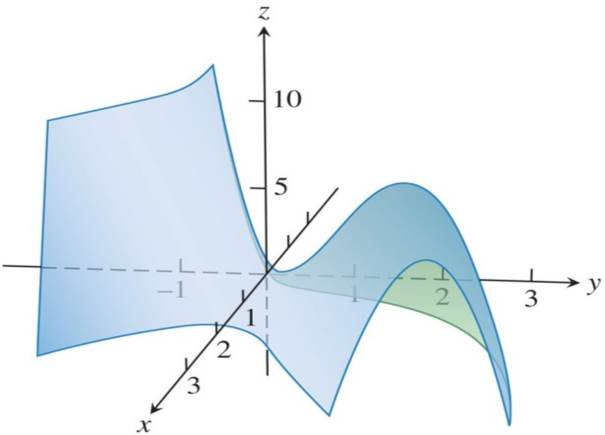


FIGURE 13.48 The surface $z = 3y^2 - 2y^3 - 3x^2 + 6xy$ has a saddle point at the origin and a local maximum at the point (2, 2) (Example 4).

Section 13.8
Lagrange Multipliers
(拉格朗日乘子)

Consider the optimization problem (最佳化問題或優化問題)

$$\begin{aligned} &\text{maximize/minimize } f(x,y) \\ &\text{subject to } g(x,y)=c \ \Big(\text{and } (x,y)\in D=\text{dom}(f)\Big). \end{aligned} \qquad \text{(P1)}$$

- $f(x,y)$ is called the objective function (目標函數).
- The level curve $g(x,y)=c$ is called a constraint (限制式或約束條件) of the optimization problem (P1).

Thm (Lagrange's Theorem)

Suppose f and g have conti. first partial derivatives. If the optimization problem (P1) has an extremum at $P_0(x_0, y_0) \in D$ with $\nabla g(P_0) \neq \mathbf{0}$, then ∇f and ∇g are parallel at P_0, i.e., $\exists\, \lambda \in \mathbb{R}$ s.t.

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

(f 和 g 在 (x_0, y_0) 處的梯度向量是平行的!)

Proof of above Thm (1/2)

Assume that the smooth curve $\mathcal{C}: \; g(x, y) = c$ is represented by a vector-valued function of the form

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \quad \forall\, t \in I = [a, b]$$

with $\mathbf{r}(t_0) = x(t_0)\,\mathbf{i} + y(t_0)\,\mathbf{j} = x_0\,\mathbf{i} + y_0\,\mathbf{j}$ for some $t_0 \in I$. Thus, $\mathbf{r}'(t) \neq \mathbf{0} \;\; \forall\, t \in I$ and $h(t) := f(x(t), y(t))$ is diff. on I, where the value $h(t_0) = f(x_0, y_0)$ is a local extremum of the problem (P1).

Proof of above Thm (2/2)

It follows from Chain Rule (Thm 5) at $t = t_0$ or $P_0(x_0, y_0)$ that

$$0 = h'(t_0) = f_x(P_0)x'(t_0) + f_y(P_0)y'(t_0) = \nabla f(P_0) \bullet \mathbf{r}'(t_0).$$

That is, $\nabla f(P_0)$ is orthogonal to the level cureve $\mathcal{C}:\ g(x, y) = c$.. Consequently, $\nabla f(P_0)$ and $\nabla g(P_0)$ are parallel vectors, since ∇g is also orthogonal to $\mathcal{C}$ at P_0.

How to find $P_0(x_0, y_0)$?

Remarks (拉格朗日乘子法的求解過程)

(1) λ is called the Lagrange multiplier (拉格朗日乘子).

(2) From above Thm, we need to solve a system of nonlinear equations (非線性方程組)

$$\begin{cases} \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0), \\ g(x_0, y_0) = c \end{cases} \quad \text{or} \quad \begin{cases} f_x(x_0, y_0) = \lambda g_x(x_0, y_0), \\ f_y(x_0, y_0) = \lambda g_y(x_0, y_0), \\ g(x_0, y_0) = c \end{cases}$$

for finding the optimizer $P_0(x_0, y_0) \in D = \mathrm{dom}(f)$.

Find the largest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} = 1.$

等位線 $g(x, y) = 1$ 的示意圖

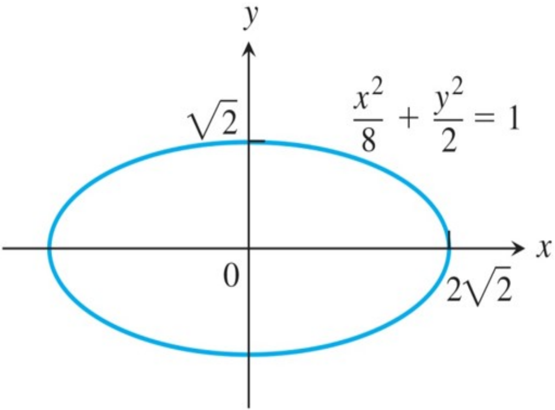


FIGURE 13.55 Example 3 shows how to find the largest and smallest values of the product xy on this ellipse.

From the Lagrange's Thm, we need to solve

$$y = f_x = \lambda g_x = \frac{\lambda x}{4}, \quad x = f_y = \lambda g_y = \lambda y, \quad \frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Then $\frac{4y}{x} = \lambda = \frac{x}{y} \Longrightarrow x^2 = 4y^2$ and thus we see that

$$\frac{x^2}{8} + \frac{y^2}{2} = 1 \Longrightarrow y^2 = \frac{4y^2}{8} + \frac{y^2}{2} = 1 \Longrightarrow y = \pm 1.$$

Moreover, we further have $x^2 = 4y^2 = 4(1) = 4 \Longrightarrow x = \pm 2$.

Since the function $f(x, y) = xy$ takes the extreme values at the four points $(\pm 2, 1)$ and $(\pm 2, -1)$, we thus conclude that the largest and smallest values of f are given by

$$f(2, 1) = f(-2, -1) = 2 \quad \text{and} \quad f(-2, 1) = f(2, -1) = -2,$$

respectively, in this case.

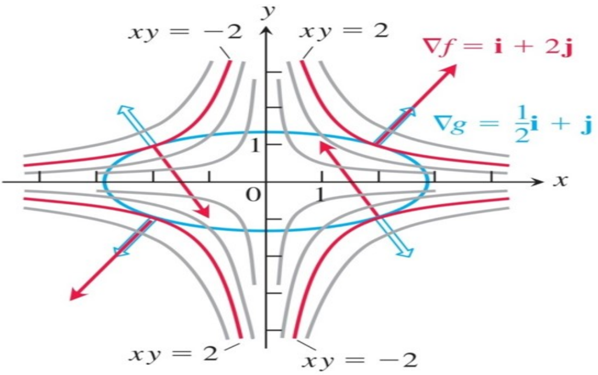


FIGURE 13.56 When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse where ∇f (red) is a scalar multiple of ∇g (blue) (Example 3).

Example 4 (雙自變量的約束優化問題)

Find the maximum and minimum values of the function

$$f(x, y) = 3x + 4y$$

on the unit circle $g(x, y) = x^2 + y^2 = 1$.

Solution of Example 4 (1/2)

From the Lagrange's Thm, we need to solve

$$3 = f_x = \lambda g_x = 2\lambda x, \quad 4 = f_y = \lambda g_y = 2\lambda y, \quad x^2 + y^2 = 1.$$

Then $\dfrac{3}{2x} = \lambda = \dfrac{4}{2y} \Longrightarrow 8x = 6y$ or $4x = 3y$. Thus we see that

$$x^2 + y^2 = 1 \Longrightarrow \frac{25y^2}{16} = \left(\frac{3y}{4}\right)^2 + y^2 = 1 \Longrightarrow y = \pm\frac{4}{5}.$$

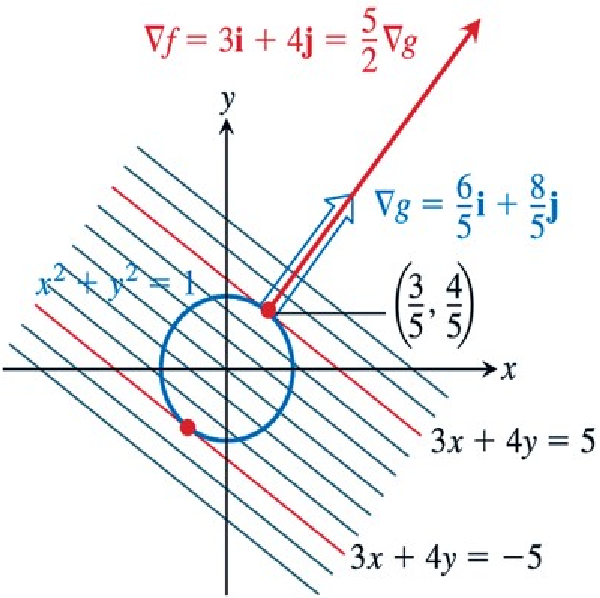
Moreover, we further have $x = \dfrac{3y}{4} = \dfrac{3}{4}\left(\pm\dfrac{4}{5}\right) = \pm\dfrac{3}{5}$.

Since the function $f(x, y) = 3x + 4y$ takes the extreme values at the four points $\left(\pm\frac{3}{5}, \frac{4}{5}\right)$ and $\left(\pm\frac{3}{5}, \frac{-4}{5}\right)$, we thus conclude that the maximum and minimum values of f are given by

$$f(\frac{3}{5}, \frac{4}{5}) = \frac{25}{5} = 5 \quad \text{and} \quad f(\frac{-3}{5}, \frac{-4}{5}) = \frac{-25}{5} = -5,$$

respectively, in this case.

Example 4 的示意圖



Consider the constrained optimization problem (約束優化問題)

$$\begin{aligned} &\text{max./min. } f(x, y, z) \\ &\text{subject to } g(x, y, z) = c. \quad \text{(a level surface)} \end{aligned} \tag{P2}$$

Thm (Lagrange's Theorem)

Suppose f and g have conti. first partial derivatives. If the problem (P2) has an extremum at $P_0(x_0, y_0, z_0) \in D = \text{dom}(f)$ with $\nabla g(P_0) \neq \mathbf{0}$, then ∇f and ∇g are parallel at P_0, i.e., $\exists\, \lambda \in \mathbb{R}$ s.t.

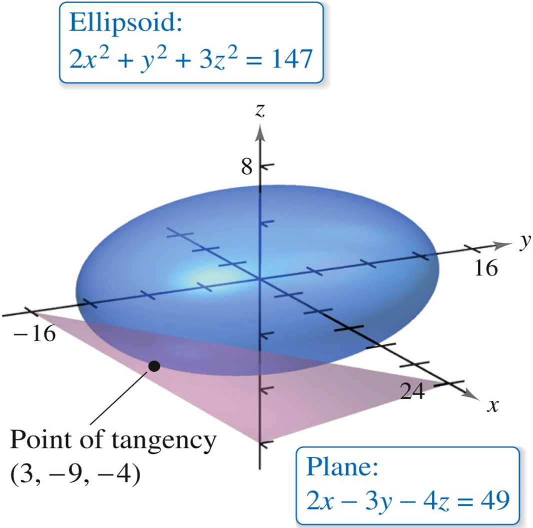
$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

Find the minimum value of the real-valued function

$$f(x, y, z) = 2x^2 + y^2 + 3z^2$$

subject to the plane $g(x, y, z) = 2x - 3y - 4z = 49$.

上述範例的示意圖 (承上頁)



From Lagrange's Thm, we need to solve a system of equations

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = 49 \end{cases} \Longrightarrow \begin{cases} 4x = 2\lambda \\ 2y = -3\lambda \\ 6z = -4\lambda \\ 2x - 3y - 4z = 49. \end{cases}$$

Then $\lambda = 6$, $x = 3$, $y = -9$ and $z = -4$, respectively. (Check!)
So, $f(3, -9, -4) = 147$ is the minimum value relative to the values of f on the plane $g(x,y,z) = 49$.

Type II: Two Constraints (雙限制式)

Consider the constrained optimization problem (約束優化問題)

$$\begin{aligned} &\text{max./min.} && f(x,y,z) \\ &\text{subject to} && g(x,y,z) = c \text{ and } h(x,y,z) = d. \end{aligned} \tag{P3}$$

Thm (Lagrange Multiplier Theorem)

Suppose f, g and h have conti. first partial derivatives. If the problem (P3) has an extremum at $(x_0, y_0, z_0) \in D = \text{dom}(f)$ with $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$ and $\nabla h(x_0, y_0, z_0) \neq \mathbf{0}$, then $\exists\, \lambda, \mu \in \mathbb{R}$ s.t.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$

(總共有五條聯立的非線性方程式!)

Thank you for your attention!