Chapter 14
Multiple Integrals
(多變量積分)

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本章預定授課範圍

Section 14.1
Double and Iterated Integrals over Rectangles
(矩形區域上的雙重積分與疊積分)

Partitions of a Rectangular Region

Let $f(x, y)$ be defined on a rectangular region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b, c \le y \le d\}.$$

Choose a partition of $\mathcal{R}$ given by

$$\Delta = \{R_i \,|\, R_i \text{ is a small rectangle lying inside } \mathcal{R}, 1 \le i \le n\}.$$

- If d_i = length of the diagonal of R_i for $i = 1, 2, \ldots, n$, the norm of partition Δ is defined by $\|\Delta\| = \max_{1 \le i \le n} d_i$.
- The Riemann sum of f associated with Δ is $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$, where $(x_i, y_i) \in R_i$ and $\Delta A_i := (\Delta x_i)(\Delta y_i)$ for $i = 1, 2, \ldots, n$.

The double integral of $f(x, y)$ over the rectangular region $\mathcal{R}$ is defined by

$$\iint_{\mathcal{R}} f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

for any partition Δ of the plane region $\mathcal{R}$.

Thm 1–Fubini's Theorem (First Form)

If $f(x, y)$ is conti. on a rectangular region given by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\},$$

then the double integral of f over $\mathcal{R}$ can be evaluated by

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

Iterated Integrals of $f(x,y)$

Type 1 : $\displaystyle\int \left[\int f(x,y)\,dx\right] dy = \int G(y)\,dy.$

Type 2 : $\displaystyle\int \left[\int f(x,y)\,dy\right] dx = \int F(x)\,dx.$

[Q]: How to evaluate the functions $G(y)$ and $F(x)$?

- $G(y) = \displaystyle\int f(x,y)\,dx =$ 將 y 視為常數且對 x 積分.
- $F(x) = \displaystyle\int f(x,y)\,dy =$ 將 x 視為常數且對 y 積分.

Calculate the double integral $\iint\limits_{\mathcal{R}} f(x,y)\, dA$ for

$$f(x,y) = 100 - 6x^2 y$$

over the region $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 2, -1 \le y \le 1\}$.

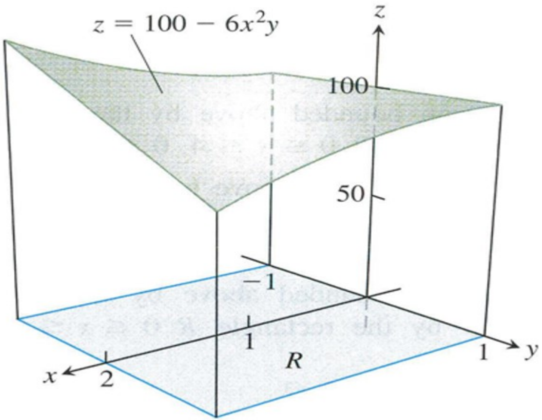


FIGURE 14.6 The double integral $\iint_R f(x, y)\, dA$ gives the volume under this surface over the rectangular region R (Example 1).

Since f is conti. on $\mathcal{R}$, ite follows from Fubini's Thm that

$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{2}(100-6x^2y)\,dx\,dy$$

$$= \int_{-1}^{1}\left(100x - 2x^3y\Big|_{x=0}^{x=2}\right)dy$$

$$= \int_{-1}^{1}(200-16y)\,dy = (200y-8y^2)\Big|_{-1}^{1} = 400.$$

Similarly, changing the orders of integration, we also see that

$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_{0}^{2}\left(\int_{-1}^{1} 100-6x^2y\,dy\right)dx = 400.$$

Section 14.2
Double Integrals over General Regions
(一般區域上的雙重積分)

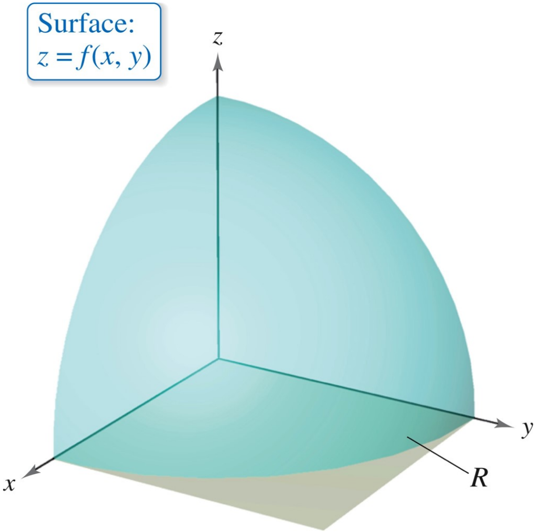
Riemann Sums over a General Region

Let $f(x, y)$ be defined on a closed and bounded plane region $\mathcal{R} \subseteq \mathbb{R}^2$. Choose an inner partition of $\mathcal{R}$ as

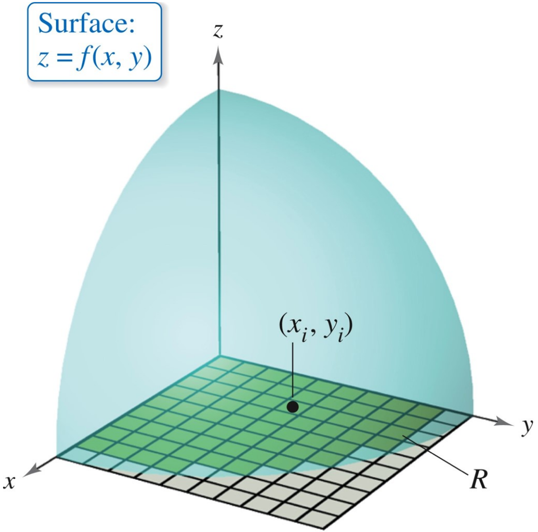
$$\Delta = \{R_i \,|\, R_i \text{ is a small rectangle lying inside } \mathcal{R}, 1 \leq i \leq n\}.$$

- If d_i = length of the diagonal of R_i for $i = 1, 2, \ldots, n$, the norm of partition Δ is defined by $\|\Delta\| = \max_{1 \leq i \leq n} d_i$.
- The Riemann sum of f associated with Δ is $\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$, where $(x_i, y_i) \in R_i$ and $\Delta A_i = \text{area}(R_i)$ for $i = 1, 2, \ldots, n$.

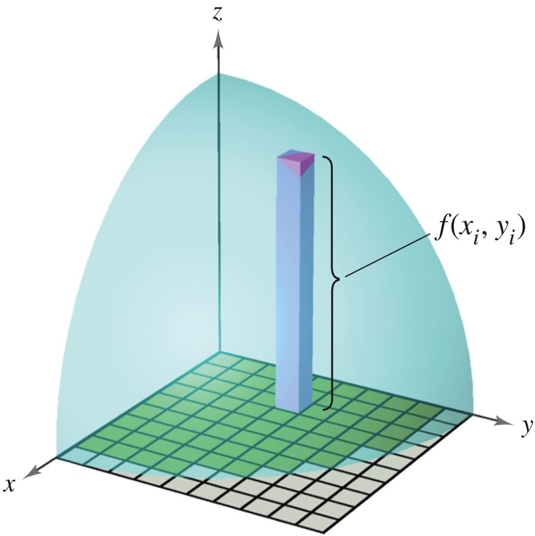
雙重積分的示意圖 (1/4)



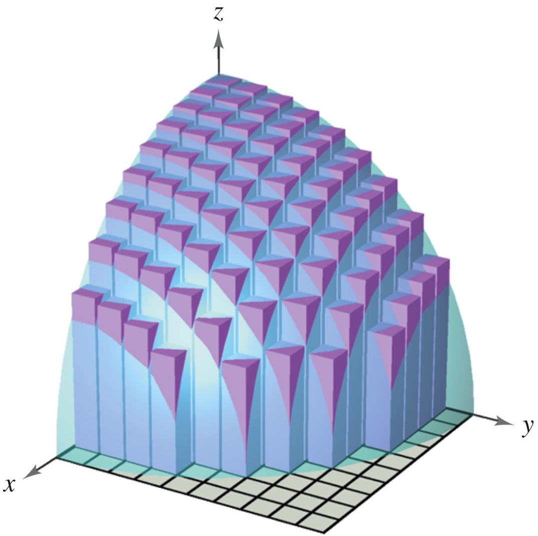
雙重積分的示意圖 (2/4)



雙重積分的示意圖 (3/4)



雙重積分的示意圖 (4/4)



Def (雙重積分的定義)

Let $\mathcal{R}$ be a closed and bounded region in $\mathbb{R}^2$.

(1) The double integral of $f(x, y)$ over the plane region $\mathcal{R}$ is

$$\iint_{\mathcal{R}} f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

for any inner partition Δ of $\mathcal{R}$.

(2) We say that f is integrable over $\mathcal{R}$ if $\iint_{\mathcal{R}} f(x, y)\, dA \quad \exists$.

Thm (雙重積分存在性的充分條件)

If $f(x, y)$ is conti. on a closed and bounded plane region $\mathcal{R} \subseteq \mathbb{R}^2$, then f is integrable over $\mathcal{R}$, i.e., the double integral

$$\iint_{\mathcal{R}} f(x, y)\, dA \quad \exists.$$

The Volume of a Solid Region (實心區域)

Def (實心區域的體積)

If $f(x, y)$ is integrable over a closed and bounded region $\mathcal{R}$, and $f(x, y) \geq 0 \quad \forall\, (x, y) \in \mathcal{R}$, then the volume of the solid region that lies over $\mathcal{R}$ and below the graph of f is

$$V = \iint\limits_{\mathcal{R}} f(x, y)\, dA \geq 0.$$

Some Questions

- What is the relationship between double integrals and iterated integrals?
- Are these two integrals *always* equal for each function $f(x, y)$?
- How to evaluate its double integral quickly when f is integrable over $\mathcal{R}$?
- Fubini's theorem will answer above questions partly!

Thm 2 (Fubini's Theorem; 1/2)

Let $f(x, y)$ be conti. on a plane region $\mathcal{R} \subseteq \mathbb{R}^2$.

(1) If $g_1(x), g_2(x)$ are conti. on $[a, b]$ and the region is defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

then the double integral of f over $\mathcal{R}$ is given by

$$\iint\limits_{\mathcal{R}} f(x, y)\, dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \Big)\, dx.$$

Thm 2 (Fubini's Theorem; 2/2)

(2) If $h_1(y), h_2(y)$ are conti. on $[c, d]$ and the region is defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\},$$

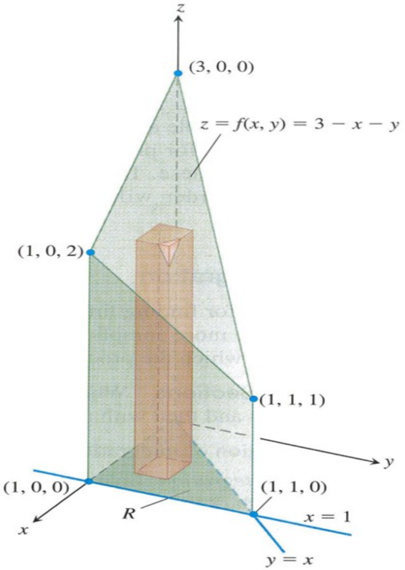
then the double integral of f over $\mathcal{R}$ is given by

$$\iint_{\mathcal{R}} f(x, y)\, dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \Big)\, dy.$$

Find the volume of a prism (角柱體) whose base is the triangle in the xy-plane bounded by the x-axis, $x = 1$ and $y = x$, and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Example 1 的示意圖

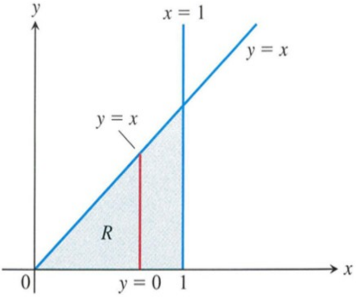


(a)

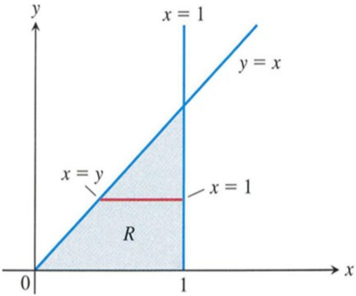
Note that $f(x, y) = 3 - x - y$ is conti. and positive on the region

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le x\}$$
$$= \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le y \le 1, y \le x \le 1\}.$$

平面區域 $\mathcal{R}$ 的示意圖



(b)



(c)

So, it follows from Thm 2 that the volume of the solid is

$$\begin{aligned} V &= \iint_{\mathcal{R}} f(x,y)\,dA = \int_0^1 \int_0^x (3-x-y)\,dy\,dx \\ &= \int_0^1 \left(3y - xy - \frac{y^2}{2}\right)\Big|_{y=0}^{y=x} dx = \int_0^1 \left(3x - \frac{3}{2}x^2\right) dx \\ &= \left(\frac{3}{2}x^2 - \frac{x^3}{2}\right)\Big|_0^1 = \frac{3}{2} - \frac{1}{2} = 1. \end{aligned}$$

Similarly, it follows from Thm 2 again that

$$V = \iint_{\mathcal{R}} f(x,y)\,dA = \int_0^1 \int_y^1 (3 - x - y)\,dx\,dy$$
$$= \int_0^1 \left(3x - \frac{x^2}{2} - xy\right)\Big|_{x=y}^{x=1} dy = \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy$$
$$= \left(\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right)\Big|_0^1 = \frac{5}{2} - 2 + \frac{1}{2} = 1.$$

Calculate the following double integral

$$\iint_{\mathcal{R}} \frac{\sin x}{x}\, dA,$$

where $\mathcal{R}$ is the triangle bounded by the x-axis, $y = x$ and $x = 1$.

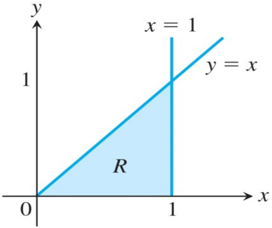


FIGURE 14.13 The region of integration in Example 2.

If $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le x\}$, we then see that

$$\iint\limits_{\mathcal{R}} \frac{\sin x}{x}\, dA = \int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(\frac{y \sin x}{x} \Big|_{y=0}^{y=x} \right) dx$$

$$= \int_0^1 \sin x \, dx = -\cos x \Big|_0^1 = 1 - \cos(1) \approx 0.46.$$

Remark (改變積分順序後 ...)

On the other hand, if we rewrite the plane region $\mathcal{R}$ as

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1, y \le x \le 1\},$$

then it is not easy to get the antiderivative of the iterated integral

$$\iint_{\mathcal{R}} \frac{\sin x}{x}\, dA = \int_0^1 \left(\int_y^1 \frac{\sin x}{x}\, dx \right) dy.$$

Thm (Properties of Double Intergrals; 1/2)

Suppose f and g are integrable over a closed, bounded region $\mathcal{R}$.

(1) $$\iint\limits_{\mathcal{R}} [c \cdot f(x,y)]\, dA = c \cdot \left[\iint\limits_{\mathcal{R}} f(x,y)\, dA\right] \quad \forall\, c \in \mathbb{R}.$$

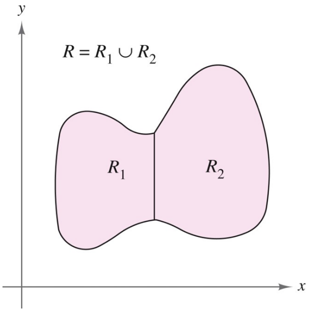
(2) $$\iint\limits_{\mathcal{R}} [f(x,y) \pm g(x,y)]\, dA = \left[\iint\limits_{\mathcal{R}} f(x,y)\, dA\right] \pm \left[\iint\limits_{\mathcal{R}} g(x,y)\, dA\right].$$

(3) $$\iint\limits_{\mathcal{R}} f(x,y)\, dA \geq 0 \text{ if } f(x,y) \geq 0 \quad \forall\, (x,y) \in \mathcal{R}.$$

(4) $$\iint\limits_{\mathcal{R}} f(x,y)\, dA \geq \iint\limits_{\mathcal{R}} g(x,y)\, dA \text{ if } f(x,y) \geq g(x,y) \quad \forall\, (x,y) \in \mathcal{R}.$$

Thm (Properties of Double Integrals; 2/2)

(5) $\displaystyle\iint_{\mathcal{R}} f(x,y)\,dA = \iint_{\mathcal{R}_1} f(x,y)\,dA + \iint_{\mathcal{R}_2} f(x,y)\,dA$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are nonoverlapping subregions (非重疊子區域) of $\mathcal{R}$ with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.

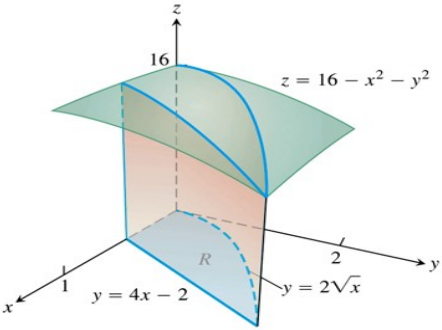


Find the volume of the solid theat lies beneath the surface

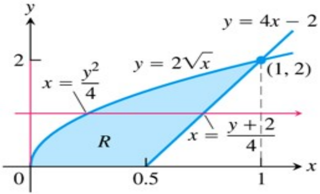
$$z = f(x, y) = 16 - x^2 - y^2$$

and above the region $\mathcal{R}$ bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$ and the x-axis.

Example 4 的示意圖



(a)



(b)

Solution of Example 4 (1/2)

We first note that $y = 2\sqrt{x}$ and $y = 4x - 2$ intersect at $(1, 2)$, since

$$2\sqrt{x} = y = 4x-2 \Longrightarrow 4x^2-5x+1 = (4x-1)(x-1) = 0 \Longrightarrow x = 1.$$

So, the plane region $\mathcal{R}$ can be denoted by the following set

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 2,\ \frac{y^2}{4} \leq x \leq \frac{y+2}{4}\}.$$

From the Fubini's Thm, the volume of the solid is given by

$$V = \iint\limits_{\mathcal{R}} f(x, y)\, dA = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dy$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3 \cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3 \cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680}.$$

Section 14.3
Area by Double Integration
(以雙重積分求面積)

Def (以雙重積分定義平面域的面積)

The area of a closed, bounded plane region $\mathcal{R}$ is defined by

$$A = \iint\limits_{\mathcal{R}} dA = \iint\limits_{\mathcal{R}} 1\, dA,$$

where the integrand $f(x, y) \equiv 1 \quad \forall\, (x, y) \in \mathcal{R}$.

Type I: 第一型平面區域面積

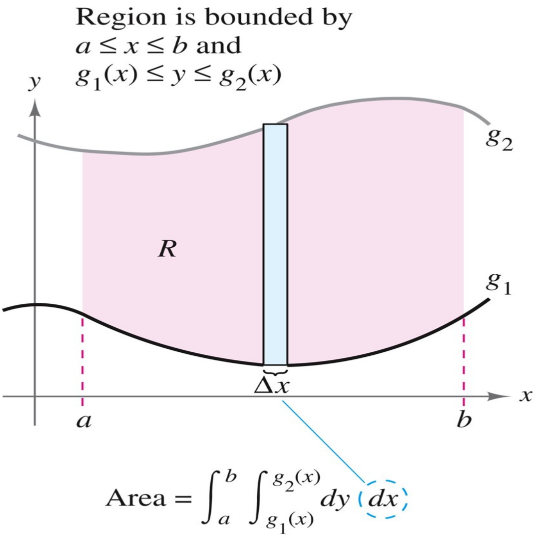
If $g_1(x), g_2(x)$ are conti. on $[a, b]$ and let

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

then the area of the plane region $\mathcal{R}$ is given by

$$A = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy \right) dx$$
$$= \int_a^b \Big[g_2(x) - g_1(x) \Big] \, dx. \quad \text{(上學期學過的結果!)}$$

Type I 的示意圖


Region is bounded by
$a \le x \le b$ and
$g_1(x) \le y \le g_2(x)$
y
R
g_2
g_1
Δx
x
a
b
Area $= \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$

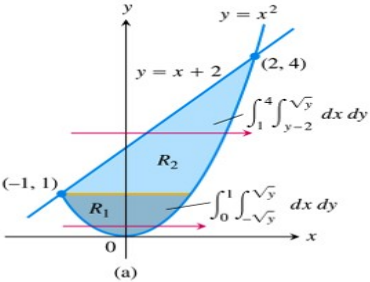
Example 2 (第一型區域面積的例子)

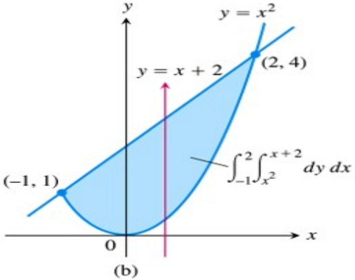
Find the area of the plane region $\mathcal{R}$ enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = x + 2,$$

using the Figure (b) below.

Example 2 的示意圖





Solution of Example 2 (1/2)

We first solve the scalar equation $x^2 = y = x + 2$ for x:

$$x^2 = x + 2 \Longrightarrow x^2 - x - 2 = (x+1)(x-2) = 0 \Longrightarrow x = -1, 2.$$

So, the given curves intersect aat the points $(-1, 1)$ and $(2, 4)$, respectively, and the plane region can be written as

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 2,\ x^2 \leq y \leq x + 2\},$$

with $g_1(x) \equiv x^2 \leq x + 2 \equiv g_2(x) \quad \forall\, x \in [-1, 2]$.

Then the area of the region $\mathcal{R}$ is given by

$$\begin{aligned} A &= \iint_{\mathcal{R}} dA = \int_{-1}^{2} \left(\int_{x^2}^{x+2} 1\, dy \right) dx \\ &= \int_{-1}^{2} (x + 2 - x^2)\, dx = \left(\frac{x^2}{2} + 2x - \frac{x^3}{3} \right) \Big|_{-1}^{2} \\ &= (2 + 4 - \frac{8}{3}) - (\frac{1}{2} - 2 + \frac{1}{3}) = \frac{10}{3} - (\frac{-7}{6}) = \frac{27}{6} = \frac{9}{2}. \end{aligned}$$

Type II: 第二型平面區域面積

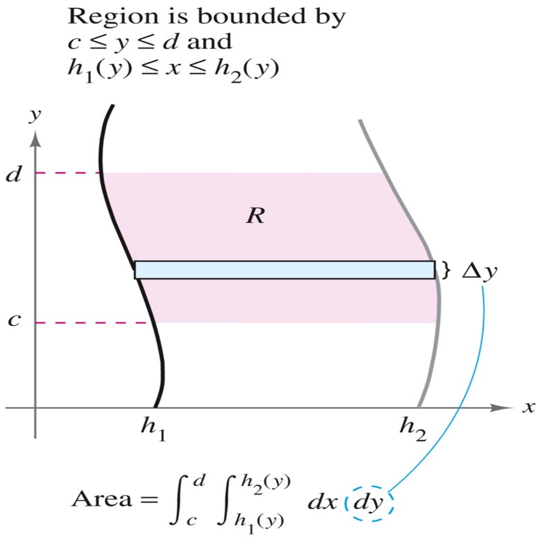
If $h_1(y), h_2(y)$ are conti. on $[c, d]$ and let

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d, h_1(y) \le x \le h_2(y)\},$$

then the area of the plane region $\mathcal{R}$ is given by

$$\begin{aligned} A &= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} dx \right) dy \\ &= \int_c^d \Big[h_2(y) - h_1(y) \Big] \, dy. \quad \text{(上學期學過的結果!)} \end{aligned}$$

Type II 的示意圖



Region is bounded by
$c \leq y \leq d$ and
$h_1(y) \leq x \leq h_2(y)$
y
x
d
c
R
Δy
h_1
h_2
Area $= \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy$

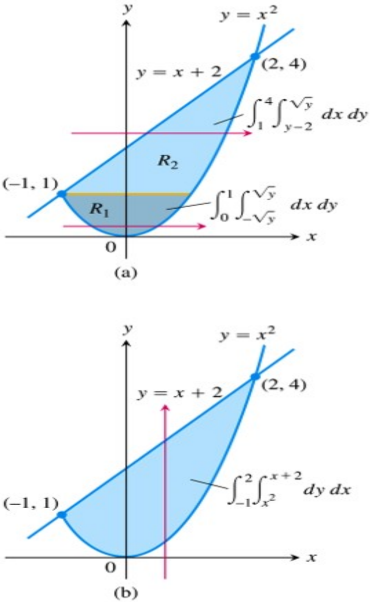
Example 2 (使用第二型區域面積的公式)

Find the area of the plane region $\mathcal{R}$ enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = x + 2,$$

using the Figure (a) below.

Example 2 的示意圖



Since the region $\mathcal{R}$ is the union of two nonoverlapping subregions

$$\mathcal{R}_1 = \{(x,y) \in \mathbb{R}^2 \,|\, 0 \le y \le 1,\ -\sqrt{y} \le x \le \sqrt{y}\} \text{ and}$$
$$\mathcal{R}_2 = \{(x,y) \in \mathbb{R}^2 \,|\, 1 \le y \le 4,\ y-2 \le x \le \sqrt{y}\},$$

it follows from the Fubini's Thm that the area of $\mathcal{R}$ is

$$\begin{aligned} A = \iint_{\mathcal{R}} dA &= \iint_{\mathcal{R}_1} dA + \iint_{\mathcal{R}_2} dA \\ &= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy \\ &= \cdots = \frac{9}{2}. \qquad \text{(Check!)} \end{aligned}$$

The Average Value of $f(x, y)$

Def (雙自變量函數的平均值)

If $f(x, y)$ is integrable over a closed and bounded region $\mathcal{R}$, then the average value of f over $\mathcal{R}$ is

$$\mathrm{av}(f) \equiv \frac{1}{\mathrm{area}(\mathcal{R})} \iint_{\mathcal{R}} f(x, y)\, dA.$$

Recall (單自變量函數的平均值)

The average value of an integrable function $y = f(x)$ on $[a, b]$ is

$$\mathrm{av}(f) \equiv \frac{1}{b-a} \int_a^b f(x)\, dx.$$

Find the average value of the real-valued function

$$f(x, y) = x\cos(xy)$$

over the rectangular region defined by

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq \pi,\ 0 \leq y \leq 1\}.$$

Since area$(\mathcal{R}) = \pi$ and from Fubini's Thm we see that

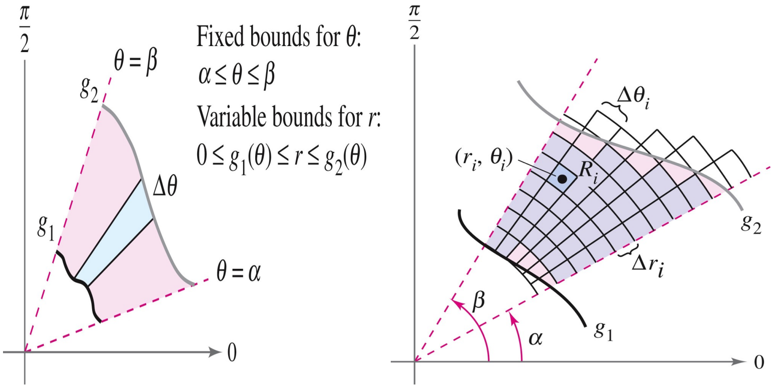
$$\iint\limits_{\mathcal{R}} f(x,y)\,dA = \int_0^{\pi}\int_0^{1} x\cos(xy)\,dy\,dx = \int_0^{\pi}\Big[\sin(xy)\Big]_{y=0}^{y=1} dx$$

$$= \int_0^{\pi} \sin x\,dx = (-\cos x)\Big|_0^{\pi} = 1+1 = 2,$$

the average value of f over $\mathcal{R}$ is $\text{av}(f) = \frac{2}{\pi}$.

Section 14.4
Double Integrals in Polar Form
(雙重積分的極坐標形式)

平面極坐標區域的分割

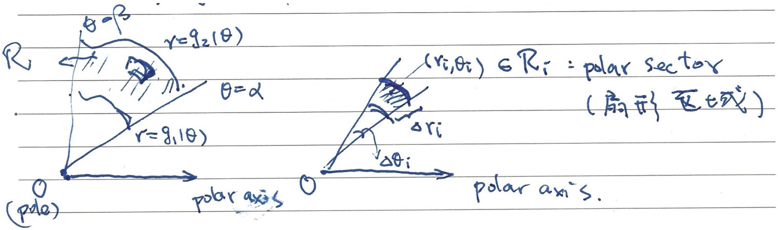

$\frac{\pi}{2}$
$\theta=\beta$
g_2
Fixed bounds for θ:
$\alpha \le \theta \le \beta$
Variable bounds for r:
$0 \le g_1(\theta) \le r \le g_2(\theta)$
$\Delta\theta$
g_1
$\theta=\alpha$
0
$\frac{\pi}{2}$
$\Delta\theta_i$
(r_i, θ_i)
R_i
g_2
Δr_i
β
α
g_1
0

Let $f(x, y)$ be conti. on a closed and bounded region $\mathcal{R} \subseteq \mathbb{R}^2$. If we can rewrite $\mathcal{R}$ as a polar region defined by

$$\mathcal{R} = \{(r, \theta) \mid \alpha \le \theta \le \beta, 0 \le g_1(\theta) \le r \le g_2(\theta)\},$$

then consider an inner partition of $\mathcal{R}$ as

$$\Delta = \{R_i \mid R_i \text{ is a polar sector lying insise } \mathcal{R}, 1 \le i \le n\}.$$



- For each $i = 1, 2, \ldots, n$ and any $(r_i, \theta_i) \in R_i$, the area of R_i is

$$\Delta A_i = \frac{1}{2}\left(r_i + \frac{\Delta r_i}{2}\right)^2 \Delta\theta_i - \frac{1}{2}\left(r_i - \frac{\Delta r_i}{2}\right)^2 \Delta\theta_i$$
$$= \frac{1}{2}(2r_i)(\Delta r_i)\Delta\theta_i = r_i \cdot \Delta r_i \cdot \Delta\theta_i.$$

- The Riemann sum of f associated with Δ is given by

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \cdot \Delta r_i \cdot \Delta\theta_i.$$

If $g_1(\theta), g_2(\theta)$ are conti. on $[\alpha, \beta]$ with $0 \le \beta - \alpha < 2\pi$, and $f(x, y)$ is conti. on the polar region

$$\mathcal{R} = \{(r, \theta) \mid \alpha \le \theta \le \beta, 0 \le g_1(\theta) \le r \le g_2(\theta)\},$$

then the double integral of f over $\mathcal{R}$ is given by

$$\begin{aligned}
\iint_{\mathcal{R}} f(x, y)\, dA &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f(r_i \cos\theta_i, r_i \sin\theta_i) r_i \Delta r_i \Delta\theta_i \\
&= \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.
\end{aligned}$$

Example 3 (在半圓上求雙重積分)

Evaluate the following double integral

$$\iint\limits_{\mathcal{R}} e^{x^2+y^2}\, dA = \iint\limits_{\mathcal{R}} e^{x^2+y^2}\, dy\, dx,$$

where $\mathcal{R}$ is the semicircular region (半圓區域) bounded by the x-axis and the curve $y = \sqrt{1-x^2}$.

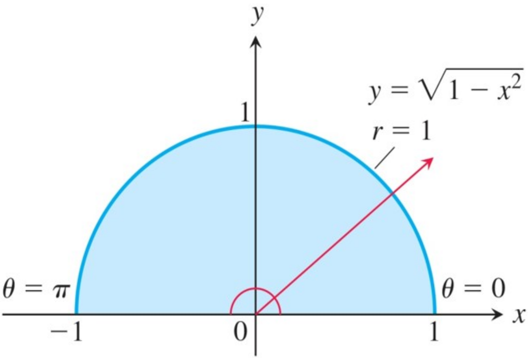


FIGURE 14.27 The semicircular region in Example 3 is the region

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi.$$

Firstly, we rewrite the polar region $\mathcal{R}$ as

$$\mathcal{R} = \{(r,\theta)\,|\,0 \le \theta \le \pi,\ 0 \le r \le 1\}.$$

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then $x^2 + y^2 = r^2$ and we thus have

$$\iint_{\mathcal{R}} e^{x^2+y^2}\,dy\,dx = \int_0^\pi \left(\int_0^1 e^{r^2} r\,dr\right) d\theta = \int_0^\pi \left[\frac{e^{r^2}}{2}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^\pi \frac{e-1}{2}\,d\theta = \frac{\pi}{2}(e-1).$$

Evaluate the iterated integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \iint_{\mathcal{R}} (x^2 + y^2)\, dA,$$

where the region of integration in Cartesian coordinates

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}\}$$

is the interior of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

If we let $x = r\cos\theta$ and $y = r\sin\theta$ for $(r, \theta) \in \mathcal{R}$, where the polar region $\mathcal{R}$ is defined by

$$\mathcal{R} = \{(r, \theta) \,|\, 0 \le \theta \le \pi/2,\ 0 \le r \le 1\},$$

then $x^2 + y^2 = r^2$ and hence the iterated integral becomes

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 r^2 \cdot r\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 r^3\, dr\, d\theta$$
$$= \int_0^{\pi/2} \left(\frac{r^4}{4} \Big|_0^1 \right) d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Find the volume of the solid bounded above by the paraboloid

$$z = f(x, y) = 9 - x^2 - y^2$$

and below by the unit circle in the xy-plane.

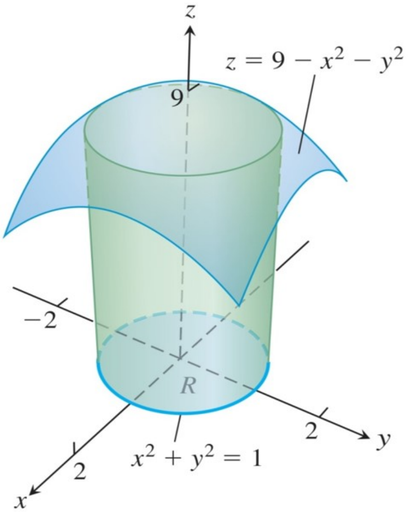


FIGURE 14.28 The solid region in Example 5.

Let $x = r\cos\theta$ and $y = r\sin\theta$ for $(r, \theta) \in \mathcal{R}$, where the region is

$$\mathcal{R} = \{(r, \theta) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}.$$

Then $x^2 + y^2 = r^2$ and hence the volume of the solid is given by

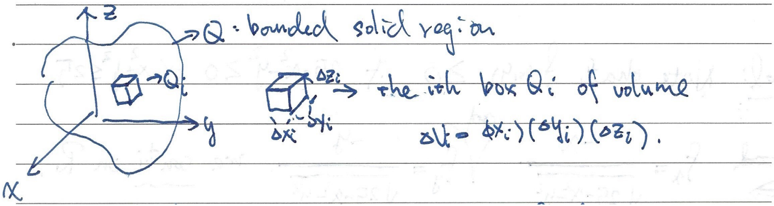
$$\begin{aligned} V &= \iint_{\mathcal{R}} (9 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (9 - r^2) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (9r - r^3)\, dr\, d\theta = \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{17}{4}\, d\theta = \frac{17}{4} \cdot 2\pi = \frac{17}{2}\pi. \end{aligned}$$

Section 14.5
Triple Integrals in Rectangular Coordinates
(直角坐標上的三重積分)

Let $f(x, y, z)$ be defined on a bounded solid region $Q \subseteq \mathbb{R}^3$. Choose an inner partition Δ of Q defined as

$$\Delta = \{Q_i \,|\, Q_i \text{ is a small box lying entirely within } Q, 1 \leq i \leq n\},$$

where each box Q_i is of volume $\Delta V_i = \Delta x_i \cdot \Delta y_i \cdot \Delta z_i \quad \forall\, i.$



- If d_i is the length of the diagonal of Q_i for $i = 1, 2, \ldots, n$, the norm of Δ is defined by $\|\Delta\| \equiv \max_{1 \le i \le n} d_i > 0$.
- The Riemann sum of $f(x, y, z)$ associated with Δ is

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta x_i \cdot \Delta y_i \cdot \Delta z_i,$$

where $(x_i, y_i, z_i) \in Q_i$ is selected arbitrarily for $i = 1, 2, \ldots, n$.

Def (三重積分的定義)

Let $f(x, y, z)$ be defined on a bounded solid region $Q \subseteq \mathbb{R}^3$.

(1) f is integrable over Q, if the triple integral of f over Q

$$\iiint_Q f(x, y, z)\, dV \equiv \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i \quad \exists$$

for any inner partition Δ of Q.

(2) The volume of Q is defined by $V = \iiint_Q 1\, dV$.

Existence of Triple Integrals

Thm (三重積分存在性的充分條件)

If $f(x, y, z)$ is conti. on a bounded solid region $Q \subseteq \mathbb{R}^3$, then f is integrable over Q, i.e., the triple integral

$$\iiint_Q f(x, y, z)\, dV \quad \exists.$$

Thm (Properties of Triple Integrals)

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are integrable over a bounded solid region $Q \subseteq \mathbb{R}^3$.

(1) $\displaystyle\iiint_Q [c \cdot f(x, y, z)]\, dV = c \cdot \iiint_Q f(x, y, z)\, dV \quad \forall\, c \in \mathbb{R}.$

(2) $\displaystyle\iiint_Q [f(x, y, z) \pm g(x, y, z)]\, dV = \Big(\iiint_Q f(x, y, z)\, dV \Big) \pm \Big(\iiint_Q g(x, y, z)\, dV \Big).$

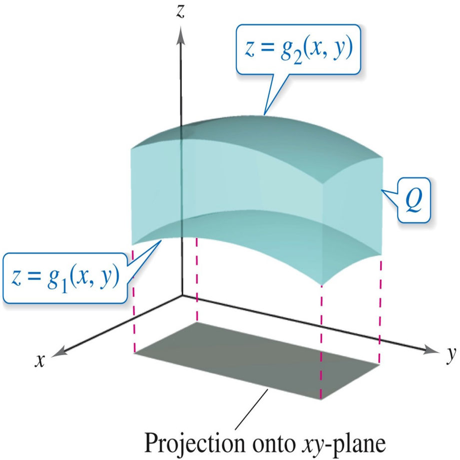
(3) $\displaystyle\iiint_Q f(x, y, z)\, dV \geq 0$ if $f(x, y, z) \geq 0 \quad \forall\, (x, y, z) \in Q.$

(4) $\displaystyle\iiint_Q f(x, y, z)\, dV \geq \iiint_Q g(x, y, z)\, dV$ if $f(x, y, z) \geq g(x, y, z) \quad \forall\, (x, y, z) \in Q.$

(5) $\displaystyle\iiint_Q f(x, y, z)\, dV = \iiint_{Q_1} f(x, y, z)\, dV + \iiint_{Q_2} f(x, y, z)\, dV$, where Q_1 and Q_2 are nonoverlapping solid subregions of Q with $Q = Q_1 \cup Q_2$.

Main Question

How to evaluate $\iiint_Q f(x, y, z)\, dV$ via some iterated integrals?



Fubini's Theorem for Triple Integrals

Suppose that $f(x, y, z)$ is conti. on a bounded solid region Q, and that h_1, h_2, g_1, g_2 are conti. functions.

(1) If $Q = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b, h_1(x) \le y \le h_2(x), g_1(x, y) \le z \le g_2(x, y)\}$, then the triple integral of f over Q is given by

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

(2) If $Q = \{(x, y, z) \in \mathbb{R}^3 \mid c \le y \le d, h_1(y) \le x \le h_2(y), g_1(x, y) \le z \le g_2(x, y)\}$, then the triple integral of f over Q is given by

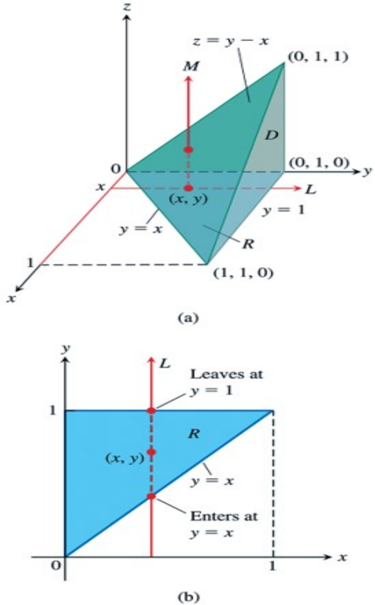
$$\iiint_Q f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\, dx\, dy.$$

Example 2 (建立三重疊積分)

Set up an iterated integral, using the order of integration $dz\,dy\,dx$, for evaluating the triple integral

$$\iiint_D F(x, y, z)\, dV,$$

where D is the tetrahedron (四面體) whose vertices (頂點) are $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 1, 1)$, respectively.


z
z = y − x
(0, 1, 1)
M
D
0
(0, 1, 0)
y
x
L
(x, y)
y = 1
y = x
R
1
(1, 1, 0)
x
(a)
y
L
Leaves at
y = 1
1
R
(x, y)
y = x
Enters at
y = x
0
1
x
(b)

Since the plane passing through $(0,0,0)$, $(1,1,0)$ and $(0,1,1)$ is $x-y+z=0$ or $z=y-x$, the tetrahedron can be represented by

$$D=\{(x,y,z)\in\mathbb{R}^3\,|\,0\le x\le 1,\ x\le y\le 1,\ 0\le z\le y-x\}.$$

So, from Fubini's Thm the given triple integral is evaluated by

$$\iiint\limits_D F(x,y,z)\,dV=\int_0^1\int_x^1\int_0^{y-x}F(x,y,z)\,dz\,dy\,dx.$$

Find the volume of the tetrahedron D in Example 2 for evaluating

$$V = \iiint_D 1 \, dV$$

using the orders $dz\,dy\,dx$ and $dy\,dz\,dx$, respectively.

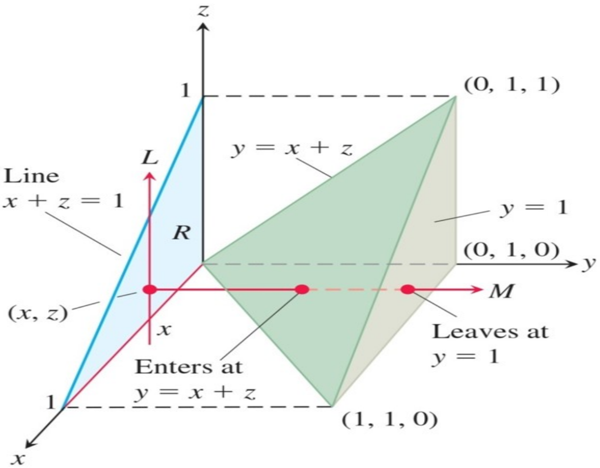


FIGURE 14.34 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Example 3).

Solution of Example 3 (1/2)

Taking $F(x, y, z) \equiv 1$ in Example 2, the volume of D is given by

$$V = \iiint\limits_D 1\, dV = \int_0^1 \int_x^1 \int_0^{y-x} 1\, dz\, dy\, dx = \int_0^1 \int_x^1 (y - x)\, dy\, dx$$

$$= \int_0^1 \left[\frac{y^2}{2} - xy\right]_{y=x}^{y=1} dx = \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2}\right) dx$$

$$= \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{2} + \frac{1}{6} = \frac{1}{6}.$$

For the order of integration $dy\,dz\,dx$, we consider the tetrahedron

$$D = \{(x, y, z) \in \mathbb{R}^3 \,|\, 0 \le x \le 1,\ 0 \le z \le 1 - x,\ x + z \le y \le 1\}.$$

Thus, from Fubini's Thm the volume of D is given by

$$V = \iiint\limits_D 1\,dV = \int_0^1 ! \int_0^{1-x} \int_{x+z}^1 1\,dy\,dz\,dx = \int_0^1 \int_0^{1-x} (1 - x - z)\,dz\,dx$$

$$= \int_0^1 \left[(1-x)z - \frac{z^2}{2}\right]_{z=0}^{z=1-x} dx = \int_0^1 \left[(1-x)^2 - \frac{(1-x)^2}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 (1-x)^2\,dx = \frac{-1}{6}(1-x)^3\Big|_0^1 = 0 - (\frac{-1}{6}) = \frac{1}{6}.$$

Find the volume of the solid region D enclosed the surfaces

$$z = g_1(x, y) = x^2 + 3y^2$$

and

$$z = g_2(x, y) = 8 - x^2 - y^2.$$

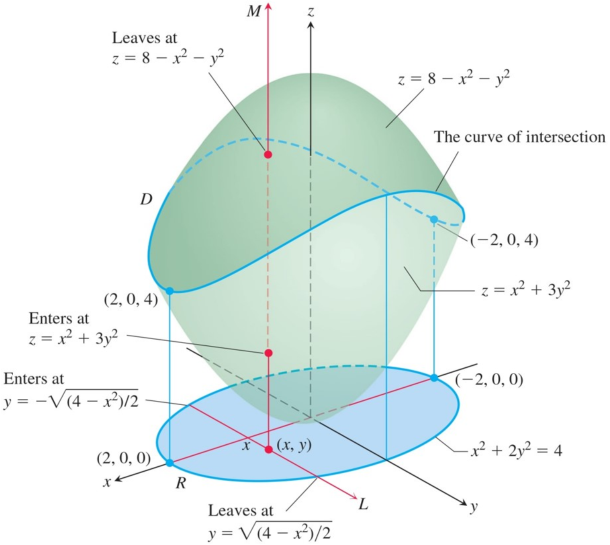


FIGURE 14.35 The volume of the region enclosed by two paraboloids, calculated in Example 4.

We first note that these two surfaces intersect when

$$x^2 + 3y^2 = z = 8 - x^2 - y^2 \implies x^2 + 2y^2 = 4.$$

So, the projection of D onto the xy-plane is the shadow region

$$\mathcal{R} = \left\{(x, y) \in \mathbb{R}^2 \mid -2 \le x \le 2, -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}\right\},$$

and we see that $g_1(x, y) \le g_2(x, y) \quad \forall\, (x, y) \in \mathcal{R}$.

If we write $D = \{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \mathcal{R}, g_1(x, y) \le z \le g_2(x, y)\}$, then the volume of the solid region D is given by

$$
\begin{aligned}
V &= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx \\
&= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx \\
&= 2\int_{-2}^{2} \left[(4 - x^2)y - \frac{2y^3}{3}\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx \\
&= \frac{32\sqrt{2}}{3} \cdot 4 \int_{0}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \left(\frac{32\sqrt{2}}{3}\right)\left(\frac{3\pi}{4}\right) = 8\sqrt{2}\pi.
\end{aligned}
$$

Section 14.7
Triple Integrals in Cylindrical and Spherical Coordinates
(在柱面與球面坐標上的三重積分)

Main Goal

In this section, we will transform the triple integral $\iiint\limits_Q f(x, y, z)\, dV$ in the rectangular coordinates to some iterated integral in other coordinate systems.

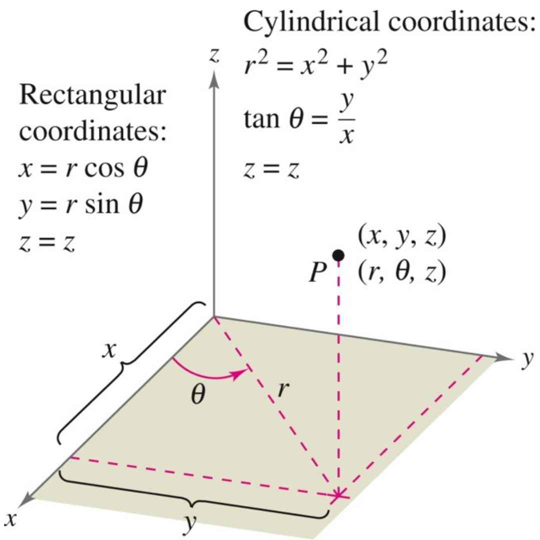
Type I: Change of Variables in Cylindrical Coordinates

Type II: Change of Variables in Spherical Coordinates

Def (柱面坐標系統)

In the cylindrical coordinate system, a point $P(x, y, z) \in \mathbb{R}^3$ is represented by an ordered triple (r, θ, z) with

- (r, θ) is the polar coordinates of the (orthogonal) projection $P_0(x, y, 0)$ of P in the xy-plane.
- z is the directed distance from P_0 to P.



Cylindrical coordinates:
$r^2 = x^2 + y^2$
$\tan\theta = \dfrac{y}{x}$
$z = z$
Rectangular coordinates:
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
(x, y, z)
(r, θ, z)
P
z
x
y
θ
r
x
y

Cylindrical Coordinates vs. Cartesian Coordinates

- Cylindrical to Cartesian $(r, \theta, z) \mapsto (x, y, z)$:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

- Cartesian to Cylindrical $(x, y, z) \mapsto (r, \theta, z)$:

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(\frac{y}{x}), \quad z = z.$$

Thm (柱面坐標的變數變換公式)

If $f(x,y,z)$ is conti. on a solid region defined, in the cylindrical coordinates, as

$$Q=\{(r,\theta,z)\,|\,\theta_1\le\theta\le\theta_2, 0\le g_1(\theta)\le r\le g_2(\theta), h_1(r,\theta)\le z\le h_2(r,\theta)\},$$

where g_1, g_2, h_1, h_2 are conti. functions, then

$$\iiint_Q f(x,y,z)\,dV=\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta.$$

Find the limits of integration in cylindrical coordinates for integrating the triple integral

$$\iiint_D f(r, \theta, z)\, dV,$$

where D is the solid region bounded below by $z = 0$, laterally (側邊地) by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$, respectively.

Example 1 的示意圖

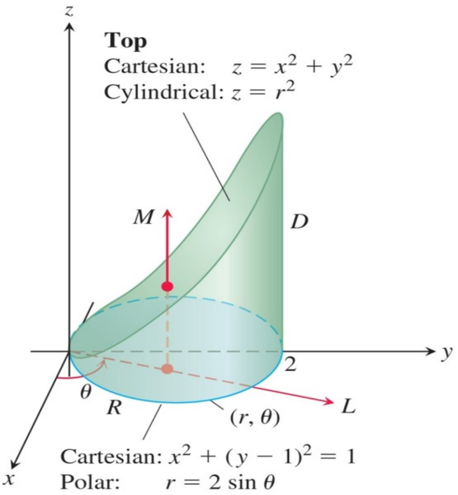


FIGURE 14.48 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then the given cylinder becomes

$$x^2 + (y-1)^2 = 1 \Longrightarrow x^2 + y^2 - 2y + 1 = 1 \Longrightarrow x^2 + y^2 - 2y = 0$$
$$\Longrightarrow r^2 - 2r\sin\theta = r(r - 2\sin\theta) = 0 \Longrightarrow r = 2\sin\theta$$

for $0 \le \theta \le \pi$. So, the solid region D can be written in the cylindrical coordinates as

$$D = \{(r, \theta, z) \mid 0 \le \theta \le \pi, 0 \le r \le 2\sin\theta, 0 \le z \le r^2\},$$

and thus the given integral is transformed to

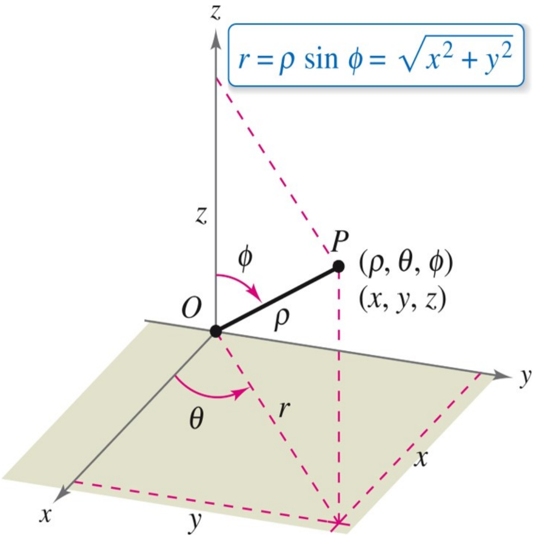
$$\iint_D f(r, \theta, z)\, dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_0^{r^2} f(r, \theta, z) \cdot r\, dz\, dr\, d\theta.$$

Def (球面坐標系統)

In the spherical coordinate system, a point $P(x, y, z) \in \mathbb{R}^3$ is represented by an ordered triple (ρ, ϕ, θ) or (ρ, θ, ϕ) with

- $\rho = |\overline{OP}| = \sqrt{x^2 + y^2 + z^2} \geq 0$.
- ϕ is the directed angle from the positive z-axis to $\overline{OP}$. $(0 \leq \phi \leq \pi)$
- θ is the directed angle from the positive x-axis to $\overline{OP_0}$, where $P_0(x, y, 0)$ is the (orthogonal) projection of P in the xy -plane.

球面坐標的示意圖 (承上頁)



- Spherical to Cartesian $(\rho, \phi, \theta) \mapsto (x, y, z)$:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

- Cartesian to Spherical $(x, y, z) \mapsto (\rho, \phi, \theta)$:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Example 3 (單位球的球面坐標方程式)

Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1$$

or the Cartesian coordiante equation (直角坐標方程式)

$$x^2 + y^2 + z^2 - 2z = 0.$$

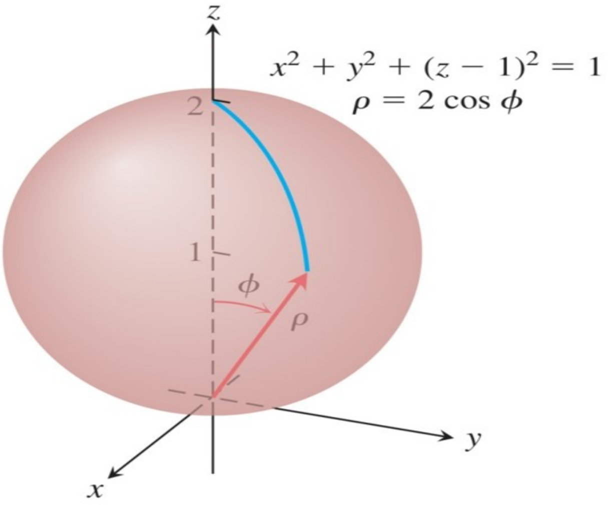


FIGURE 14.52 The sphere in Example 3.

If we let

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

then $\rho^2 = x^2 + y^2 + z^2 \geq 0$ and hence

$$x^2 + y^2 + z^2 - 2z = 0 \Longrightarrow \rho^2 - 2\rho\cos\phi = \rho(\rho - 2\cos\phi) = 0.$$

Thus, the spherical coordinate equation of the sphere is given by

$$\rho - 2\cos\phi = 0 \quad \text{or} \quad \rho = 2\cos\phi$$

for $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$.

Example 4 (上半錐面的球面坐標方程式)

Find a spherical coordinate equation for the cone

$$z = \sqrt{x^2 + y^2} \geq 0$$

for all $(x, y) \in \mathbb{R}^2$.

Example 4 的示意圖

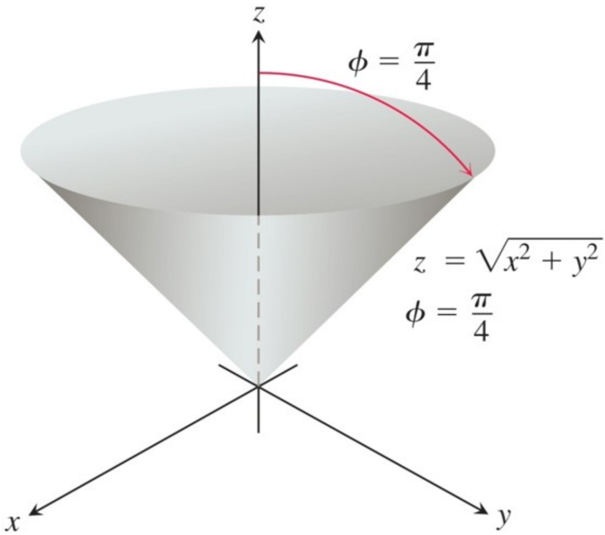


FIGURE 14.53 The cone in Example 4.

If we let

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi,$$

then $x^2 + y^2 = \rho^2 \sin^2\phi$ and hence

$$z = \sqrt{x^2 + y^2} \Longrightarrow \rho \cos\phi = \sqrt{\rho^2 \sin^2\phi} = \rho \sin\phi,$$

since $\rho \geq 0$ and $\sin\phi \geq 0$ for $0 \leq \phi \leq \pi$. Thus, the spherical coordinate equation of the cone is given by

$$\cos\phi = \sin\phi \quad \text{or} \quad \phi = \frac{\pi}{4},$$

and $0 \leq \theta \leq 2\pi$.

Thm (球面坐標的變數變換公式)

If $f(x, y, z)$ is conti. on a solid region defined, in the spherical coordinates, as

$$Q = \{(\rho, \phi, \theta) \mid \rho_1 \le \rho \le \rho_2, \phi_1 \le \phi \le \phi_2, \theta_1 \le \theta \le \theta_2\},$$

where $\rho_1 \ge 0$, $0 \le \phi_1 \le \phi_2 \le \pi$ and $0 \le \theta_2 - \theta_1 \le 2\pi$, then

$$\iiint_Q f(x, y, z)\, dV = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$

Example 5 (計算冰淇淋錐面的體積)

Find the volume of the "ice cream cone" (冰淇淋錐面) D bouned above by the spere $\rho = 1$ and bounded blow by the cone $\phi = \frac{\pi}{3}$.

Example 5 的示意圖

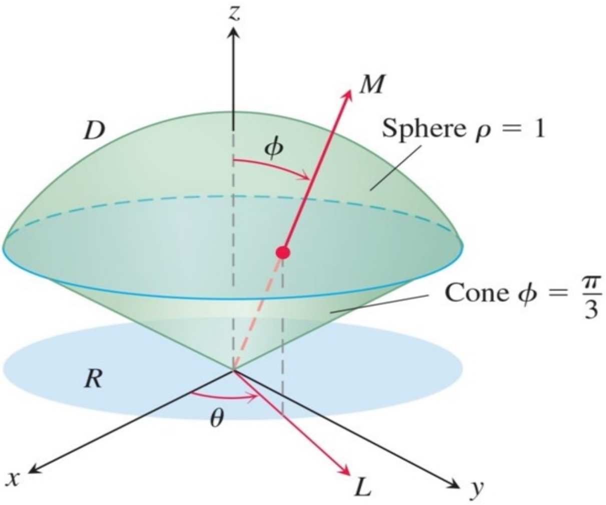


FIGURE 14.55 The ice cream cone in Example 5.

In spherical coordinates, the solid region D can be denoted by

$$D = \{(\rho, \phi, \theta) \,|\, 0 \le \rho \le 1,\ 0 \le \phi \le \frac{\pi}{3},\ 0 \le \theta \le 2\pi\}.$$

Therefore, the volume of D is given by

$$v = \iiint_D 1\, dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/3} d\theta = (\frac{1}{3})(\frac{-1}{2} + 1)(2\pi) = \frac{\pi}{3}.$$

Section 14.8
Substitutions in Multiple Integrals
(多重積分的代換法)

Def (Jacobian 行列式的定義; 1/2)

(1) If $g(u,v)$ and $h(u,v)$ have conti. first partial derivatives, the Jacobian determinant or Jacobian of the coordinate transformation (坐標變換)

$$x = g(u,v), \quad y = h(u,v)$$

is defined by

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{(二階行列式值)}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Def (Jacobian 行列式的定義; 2/2)

(2) If $g(u, v, w)$, $h(u, v, w)$ and $k(u, v, w)$ have conti. first partial derivatives, the Jacobian of the coordinate transformation

$$x = g(u, v, w), \quad y = h(u, v, w), \quad z = k(u, v, w)$$

is defined by

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}. \quad \text{(三階行列式值)}$$

Thm 3 (雙重積分的代換法)

Suppose that $T(u, v) = (x, y) = (g(u, v), h(u, v))$ is one-to-one on G, where g and h have conti. first partial derivatives. If $f(x, y)$ is conti. on $\mathcal{R}$ and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \quad \forall\, (u, v) \in G$, then

$$\iint\limits_{\mathcal{R}} f(x, y)\, dx\, dy = \iint\limits_{G} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

Example 1 (極坐標變換的 Jacobian 行列式)

Find the Jacobian of the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and write the Cartesian integral $\iint_R f(x, y)\, dx\, dy$ as a polar integral.

Solution of Example 1

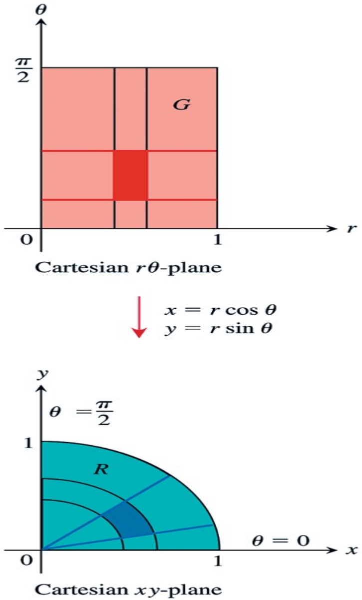
The Jacobian of the polar coordinate transformation is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Since we assume $r \geq 0$ when integrating in polar coordinates, $|J(r,\theta)| = |r| = r$ and it follows from Thm 3 that

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_G f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$$

Example 1 的示意圖



Evaluate the following iterated integral

$$\int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2}\, dx\, dy$$

by applying the coordinate transformation

$$u = \frac{2x-y}{2} = x - \frac{y}{2}, \quad v = \frac{y}{2}. \qquad (*)$$

From Eq. (*), we see that x and y are expressed by

$$x = u + v, \quad y = 2v. \qquad (**)$$

Let the region R of integration in the xy-plane be denoted by

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le y \le 4, \frac{y}{2} \le x \le \frac{y}{2} + 1\}.$$

Then the equations in (*) transform R into a region in the uv-plene

$$G = \{(u, v) \,|\, 0 \le u \le 1, 0 \le v \le 2\},$$

where the boundaries of G are determined by the following table:

xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G	Simplified uv-equations
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

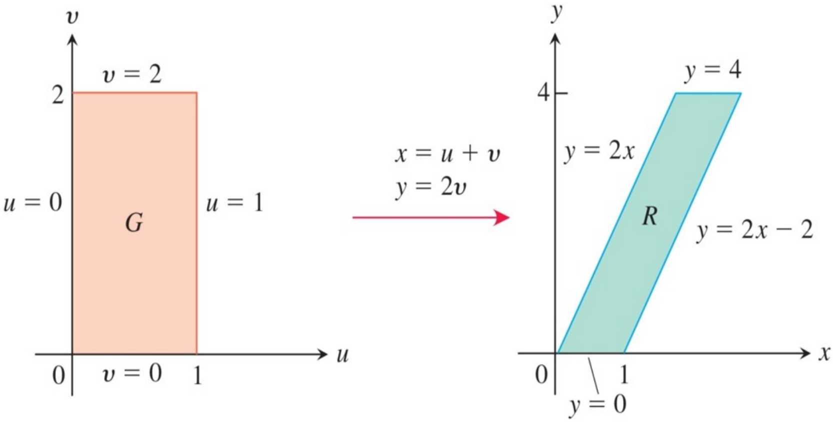


FIGURE 14.58 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 2).

Since the Jacobian of the transformation (**) is given by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2,$$

it follows from Thm 3 that the iterated integral becomes

$$\int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2}\,dx\,dy = \int_0^2 \int_0^1 u \cdot |J(u,v)|\,du\,dv = 2\int_0^2 \int_0^1 u\,du\,dv$$

$$= 2\int_0^2 \left[\frac{u^2}{2}\right]_0^1 dv = (2)(\frac{1}{2})(2) = 2.$$

Example 3 (Thm 3 的例子)

Evaluate the following iterated integral

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx$$

by applying the coordinate transformation

$$u = x + y, \quad v = y - 2x. \qquad (*)$$

From Eq. (*), we see that x and y are expressed by

$$x = \frac{1}{3}u - \frac{1}{3}v, \quad y = \frac{2}{3}u + \frac{1}{3}v. \qquad (**)$$

Let the region R of integration in the xy-plane be denoted by

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}.$$

Then the equations in (*) transform R into a region in the uv-plene

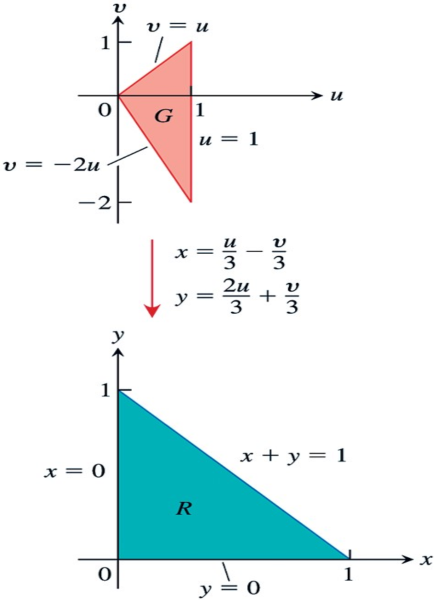
$$G = \{(u, v) \,|\, 0 \leq u \leq 1, -2u \leq v \leq u\},$$

where the boundaries of G are determined by the following table:

Solution of Example 3 (2/4)

xy-equations for the boundary of *R*	Corresponding *uv*-equations for the boundary of *G*	Simplified *uv*-equations
$x + y = 1$	$\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u}{3} + \frac{v}{3} = 0$	$v = -2u$

Solution of Example 3 (3/4)



Solution of Example 3 (4/4)

Since the Jacobian of the transformation (**) is given by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{vmatrix} = \frac{1}{3},$$

it follows from Thm 3 that the iterated integral becomes

$$\begin{aligned}\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx &= \frac{1}{3}\int_0^1 \int_{-2u}^{u} u^{1/2} v^2\,dv\,du \\ &= \frac{1}{3}\int_0^1 u^{1/2}\left[\frac{v^3}{3}\right]_{v=-2u}^{v=u} du \\ &= \frac{1}{3}\int_0^1 u^{1/2}(3u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9}.\end{aligned}$$

Thm (三重積分的代換法)

Suppose $T(u, v, w) = (x, y, z) = (g(u, v, w), h(u, v, w), k(u, v, w))$ is one-to-one on G, where g, h and k have conti. first partial derivatives. If $f(x, y, z)$ is conti. on Q and the Jacobian of T satisfies $\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0 \quad \forall\, (u, v, w) \in G$, then

$$\iiint\limits_{Q} f(x, y, z)\, dx\, dy\, dz = \iiint\limits_{G} f(g(u, v, w), h(u, v, w), k(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

Triple Integrals in Cylindrical Coordinates, Revisited

It is easily verified that the Jacobian of the cylindrical coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

If we assume $r \geq 0$, the triple integral of f over $Q \subseteq \mathbb{R}^3$ becomes

$$\begin{aligned} \iiint\limits_{Q} f(x,y,z)\,dx\,dy\,dz &= \iiint\limits_{G} F(r,\theta,z)\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| dr\,d\theta\,dz \\ &= \iiint\limits_{G} F(r,\theta,z)\,r\,dr\,d\theta\,dz. \end{aligned}$$

Triple Integrals in Spherical Coordinates, Revisited

It is easily verified that the Jacobian of the spherical coordinate transformation $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\phi$ is

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2 \sin\phi \geq 0 \quad \forall\, \phi \in [0, \pi].$$

So, the triple integral of f over $Q \subseteq \mathbb{R}^3$ becomes

$$\iiint_Q f(x,y,z)\, dx\, dy\, dz = \iiint_G F(\rho,\phi,\theta) \left| \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} \right| d\rho\, d\phi\, d\theta$$

$$= \iiint_G F(\rho,\phi,\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Example: (三重積分的變數變換)

Evaluate $\displaystyle\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(\frac{2x-y}{2}+\frac{z}{3}\right)dx\,dy\,dz$

Sol: Let $u=\dfrac{2x-y}{2}$, $v=\dfrac{y}{2}$ and $w=\dfrac{z}{3}$. — (*)

$\Rightarrow$ $x=u+v$, $y=2v$ and $z=3w$ — (**).

From (*) $\Rightarrow$

(x,y,z)-space	(u,v,w)-space
$x=\frac{y}{2}$	$u=0$
$x=y/2+1$	$u=1$
$y=0$	$v=0$
$y=4$	$v=2$
$z=0$	$w=0$
$z=3$	$w=1$

From (**) $\Rightarrow \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$

So, it follows from Thm 12.5 that

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\,dy\,dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w) \cdot 6 \, du\,dv\,dw = 6 \int_0^1 \int_0^2 \left(\frac{u^2}{2} + wu \right) \Big|_{u=0}^{u=1} dv\,dw.$$

$$= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\,dw = (12) \int_0^1 \left(\frac{1}{2} + w \right) dw$$

$$= 12 \left(\frac{1}{2} w + \frac{1}{2} w^2 \right) \Big|_0^1 = 12.$$

#

Thank you for your attention!