107 下期中考第 1(e) 題的解答

Hung-Yuan Fan

1. (e) Determine the convergence or divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{e^{\sqrt{n}}}$.

Ans: From the Maclaurin series of e^x we see that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq \frac{x^3}{3!} = \frac{x^3}{6} \quad \forall\, x \geq 0,$$

and thus it follows that

$$e^{\sqrt{n}} \geq \frac{(\sqrt{n})^3}{6} = \frac{n^{3/2}}{6} \quad \forall\, n \in \mathbb{N}. \tag{1}$$

Moreover, from (1) we immediately obtain

$$a_n := \frac{\ln n}{e^{\sqrt{n}}} \leq \frac{6n^{1/4}}{n^{3/2}} = \frac{6}{n^{5/4}} := b_n$$

for n sufficiently large. Since $\sum b_n = 6 \sum \frac{1}{n^{5/4}}$ is a convergent p-series with $p = 5/4 > 1$, the given series $\sum a_n$ converges by the Direct Comparison Test.