Chapter 5
Initial-Value Problems for Ordinary Differential Equations

Hung-Yuan Fan (范洪源)

Department of Mathematics,
National Taiwan Normal University, Taiwan

Spring 2016

Section 5.1
The Elementary Theory of Initial-Value Problems
(初值問題的基本理論)

Objectives

- Develop numerical methods for approximating the solution to **initial-value problem (IVP)**

$$\begin{cases} \frac{dy}{dt} = f(t,y), & a \le t \le b, \\ y(a) = \alpha, \end{cases} \tag{1}$$

  where $y(t)$ is the **unique solution** to IVP (1) on $[a,b]$.
- Error analysis for these numerical methods.

Note:

1. The first equation in (1) is an **ordinary differential equation (ODE; 常微分方程式)**.
2. $y(a) = \alpha$ is called an **initial condition (IC; 初值條件)**.

Def 5.1, p. 261

A function $f(t, y)$ satisfies a **Lipschitz condition** in y on a set $D \subseteq \mathbb{R}^2$ if $\exists$ a **Lipschitz constant** $L > 0$ s.t.

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|,$$

whenever $(t, y_1) \in D$ and $(t, y_2) \in D$.

Thm 5.4 (IVP 解的唯一性)

Suppose that $f(t, y)$ is conti. on $D = \{(t, y) \mid a \leq t \leq b \text{ and } y \in \mathbb{R}\}$. If f satisfies a Lipschitz condition in y on D, then the IVP (1) has a **unique solution** $y(t)$ for $a \leq t \leq b$.

Corollary of Thm 5.4

Suppose that $f(t, y)$ is conti. on $D = \{(t, y) \mid a \leq t \leq b \text{ and } y \in \mathbb{R}\}$. If $\exists$ a Lipschitz constant $L > 0$ with

$$\left| \frac{\partial f}{\partial y}(t, y) \right| \leq L \qquad \forall\, (t, y) \in D,$$

then $f(t, y)$ satisfies a Lipschitz condition in y on D, and therefore, the IVP (1) has a **unique solution** $y(t)$ for $a \leq t \leq b$.

Def 5.5, p. 263 (Well-Posedness of IVP)

The IVP

$$\begin{cases} \frac{dy}{dt} = f(t, y), & a \le t \le b, \\ y(a) = \alpha \end{cases}$$

is said to be **a well-posed problem** if

- A **unique solution** $y(t)$ exists on $[a, b]$, and
- $\exists\ \varepsilon_0$ and $k > 0$ s.t. for any $0 < \varepsilon < \varepsilon_0$, whenever $\delta(t) \in C[a, b]$ with $|\delta(t)| < \varepsilon$ for $t \in [a, b]$ and $|\delta_0| < \epsilon$, the **perturbed** IVP

$$\begin{cases} \frac{dz}{dt} = f(t, z) + \delta(t), & a \le t \le b, \\ z(a) = \alpha + \delta_0 \end{cases}$$

  has a **unique solution** $z(t)$ satisfying

$$|y(t) - z(t)| < k\varepsilon \quad \forall\, t \in [a, b].$$

Thm 5.6 (初值問題是 Well-Posed 的充分條件)

Suppose that $D = \{(t, y) \,|\, a \le t \le b \text{ and } -\infty < y < \infty\} \subseteq \mathbb{R}^2$. If $f \in C(D)$ satisfies a Lipschitz condition in y on D, then the IVP (1) is **well-posed**.

Remarks

- Because any round-off error introduced in the representation perturbs the original IVP (1), numerical methods will always be connected with solving **a perturbed IVP**.
- If **the original IVP is well-posed**, the numerical solution to a perturbed problem will accurately approximate the unique solution to the original problem!

Section 5.1 勾選習題

1, 3

Section 5.2
Euler's Method
(尤拉法或歐拉法)

- Assume that the following IVP (1)

$$\begin{cases} \frac{dy}{dt} = f(t, y), & a \leq t \leq b, \\ y(a) = \alpha \end{cases}$$

is **well-posed** and $y(t)$ is the **unique sol.** to IVP (1) on $[a, b]$.

- Choose the equally-distributed mesh points (網格點) on $[a, b]$

$$t_i = a + i \cdot h, \quad i = 0, 1, 2, \ldots, N, \tag{2}$$

where $N \in \mathbb{N}$ and $h = (b - a)/N$ is the **step size**. (步長)

Note

$t_0 = a,\ t_{i+1} - t_i = h$ for $i = 0, 1, \ldots, N - 1$ and $t_N = b$.

The graph of the unique solution $y(t)$ evaluated at each mesh point is shown below.

- If $y(t) \in C^2[a,b]$, it follows from Taylor's Thm that for each $i = 0, 1, \ldots, N-1$, $\exists \xi_i \in (t_i, t_{i+1})$ s.t.

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1}-t_i)^2}{2} y''(\xi_i) \\ &= y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i) \end{aligned}$$

where h is the step size and t_i is chosen as in (2).

- Deleting the remainder term $\Longrightarrow$ **Euler's method** constructs $w_i \approx y(t_i)$ for $i = 0, 1, \ldots, N$ by

$$w_0 = \alpha, \quad w_{i+1} = w_i + hf(t_i, w_i) \text{ for } i = 0, 1, \ldots, N-1. \quad (3)$$

The fist step of Euler's method is shown below.

After N steps of Euler's method defined as in (3), the differences between $y(t_i)$ and w_i $(i = 1, 2, \ldots, N)$ are shown below.

Algorithm 5.1: Euler's Method

INPUT endpoints a, b; positive integer N; initial condition α.

OUTPUT approximation w to y at the $(N+1)$ values of t.

Step 1 Set $h = (b-a)/N$; $t = a$; $w = \alpha$;
OUTPUT$(t,\ w)$.

Step 2 For $i = 1, 2, \ldots, N$ do **Steps 3–4**.

Step 3 Set $w = w + h \cdot f(t, w)$; (Compute w_i.)
$t = a + i \cdot h$. (Compute t_i.)

Step 4 OUTPUT$(t,\ w)$.

Step 5 **STOP**.

Example 1, p. 268

Consider the following IVP

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

(a) Show that $y(t) = (t+1)^2 - 0.5e^t$ is the unique solution to above IVP.

(b) Apply Euler's method (Alg. 5.1) with $h = 0.2$ and $N = 10$ to obtain approximations w_i, and compare these with the actual values of $y(t_i)$ for $i = 1, 2, \ldots, N$.

Solution

(a) Note that $y(0) = (1+0)^2 - 0.5e^0 = 0.5$ and it is easily seen that $y(t)$ satisfies the given ODE by direct computation!

(b) From the Euler's method in (3), we have

$$\begin{aligned} w_0 &= 0.5, \quad t_0 = 0, \\ w_{i+1} &= w_i + 0.2(w_i + t_i^2 - 1) = w_i + 0.2[w_i - (0.2i)^2 + 1] \\ &= \mathbf{1.2}w_i - \mathbf{0.008}i^2 + \mathbf{0.2}, \quad i = 0, 1, \ldots, 9, \end{aligned}$$

where $t_i = 0 + 0.2i = 0.2i$ for $i = 0, 1, 2, \ldots, 10 = N$.

The numerical results of Part (b) are shown in the following table.

Thm 5.9 (Theoretical Error Bound)

Suppose that f is conti. and satisfies a Lipschitz condition with constant $L > 0$ on $D = \{(t, y) \in \mathbb{R}^2 \mid a \leq t \leq b, \ -\infty < y < \infty\}$, and that $\exists\ M > 0$ with

$$|y''(t)| \leq M \quad \forall\, t \in [a, b],$$

where $y(t)$ is the unique sol. to the IVP (1). If $w_0, w_1, \cdots, w_N$ are the approximations generated by Euler's method for some $N \in \mathbb{N}$, then for each $i = 0, 1, \ldots, N$, we have

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(t_i - a)} - 1\right]$$

with t_i being the grid points and h being the step size.

Comments on Thm 5.9

- In practice, it is difficult to verify the boundedness of $y''(t)$!
- Instead, we may check for the boundednes of

$$
\begin{aligned}
y''(t) &= \frac{dy'(t)}{dt} = \frac{d}{dt} f(t, y(t)) \\
&= \frac{\partial f}{\partial t}(t, y(t)) + \frac{\partial f}{\partial y}(t, y(t)) \cdot f(t, y(t))
\end{aligned}
$$

without explicitly knowing the unique solution $y(t)$.

Example 2, p. 272 (驗證 Thm 5.9 的誤差上界)

As in **Example 1**, Euler's method with $h = 0.2$ is applied for computing the approximations w_i $(i = 0, 1, \ldots, N)$ of the unique solution $y(t) = (t+1)^2 - 0.5e^t$ to the IVP

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Compare the error bounds given in Thm 5.9 to the actual errors $|y(t_i) - w_i|$ for $i = 0, 1, \ldots, N$.

Solution (1/2)

Let $f(t, y) = y - t^2 + 1$ be a real-valued function defined on the set

$$D = \{(t, y) \in \mathbb{R}^2 \,|\, 0 \leq t \leq 2, \quad -\infty < y < \infty\}.$$

Then $f \in C(D)$ satisfies a Lipschitz condition in y on D with $L = 1$, since

$$\frac{\partial f}{\partial y}(t, y) = 1 \quad \text{or} \quad \left|\frac{\partial f}{\partial y}(t, y)\right| \leq 1 \quad \forall\, (t, y) \in D.$$

Moreover, since the unique sol. is $y(t) = (t+1)^2 - 0.5e^t$, we have $y''(t) = 2 - 0.5e^t$ and hence

$$|y''(t)| \leq 0.5e^2 - 2 \equiv M \quad \forall\, t \in [0, 2].$$

Solution (2/2)

So, it follows from Thm 5.9 that the error bounds for Euler's method are given by

$$\begin{aligned} |y(t_i) - w_i| &\le \frac{hM}{2L}\left[e^{L(t_i-a)} - 1\right] \\ &= (0.1)\cdot(0.5e^2 - 2)\cdot(e^{t_i} - 1), \end{aligned}$$

where the approx. w_i computed by Euler's method are

$$w_0 = 0.5, \quad w_{i+1} = 1.2w_i - 0.008i^2 + 0.2 \text{ for } i = 0, 1, \ldots, 9,$$

and the mesh points are $t_i = 0.2i$ for $i = 0, 1, 2, \ldots, 10 = N$.

The numerical comparison between actual errors and error bounds is shown in the following table.

Rate of Convergence for Euler's Method

$$|y(t_i) - w_i| = O(h) \quad \text{for each } i = 0, 1, \ldots, N.$$

Finite-Digit Approximations to $y(t_i)$

If $h = \dfrac{b-a}{N}$ and $t_i = a + ih$ for $i = 0, 1, \ldots, N$, then note that Euler's method is performed

- in the **exact arithmetic**:

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hf(t_i, w_i) \quad \text{for } i = 0, 1, \ldots, N-1.$$

- in the **finite-digit arithmetic**:

$$u_0 = \alpha + \delta_0,$$
$$u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1} \quad \text{for } i = 0, 1, \ldots, N-1, \qquad (4)$$

where δ_i denotes the round-off error associated with u_i for each $i = 0, 1, \ldots, N$.

Thm 5.10 (Error Bound in Finite-Digit Arithmetic)

Let $y(t)$ be the unique solution to the IVP (1) and $u_0, u_1, \cdots, u_N$ be finite-digit approximations obtained using (4). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the sufficient conditions of Thm 5.9 hold, then

$$|y(t_i) - u_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(t_i - a)} - 1\right] + |\delta_0| e^{L(t_i - a)} \tag{5}$$

for each $i = 0, 1, \ldots, N$.

Comments on Thm 5.10

- Since it is easily seen that

$$\lim_{h \to 0^+} \left(\frac{hM}{2} + \frac{\delta}{h} \right) = \infty,$$

deceasing h tends to increase the total error in the approximation.

- If we let

$$E(h) = \frac{hM}{2} + \frac{\delta}{h} \quad \text{for } h > 0,$$

then $E'(h) = \frac{M}{2} - \frac{\delta}{h^2}$, and therefore, it follows from Calculus that $E(h)$ is minimized at $h^* = \sqrt{\frac{2\delta}{M}}$. In fact, we know that

$$E'(h) < 0 \quad \text{or} \quad E(h) \text{ is decreasing for } 0 < h < h^*,$$
$$E'(h) > 0 \quad \text{or} \quad E(h) \text{ is increasing for } h > h^*.$$

Section 5.2 勾選習題

1, 3

Section 5.3
High er-Order Taylor Methods
(高階泰勒法)

Let $y(t)$ be the unique solution to the IVP

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \tag{6}$$

Def 5.11 (局部截斷誤差)

The difference method (差分方法) for solving the IVP (6)

$$w_0 = \alpha, \quad w_{i+1} = w_i + h\phi(t_i, w_i) \quad \text{for } i = 0, 1, \ldots, N-1,$$

has **local truncation error (簡稱 LTE)**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

where $y_{i+1} = y(t_{i+1})$ and $y_i = y(t_i)$ for each $i = 0, 1, \ldots, N-1$.

The LET of Euler's Method

- If $y(t) \in C^2[a,b]$, it follows from Taylor's Thm that for each $i = 0, 1, \ldots, N-1$, $\exists \xi_i \in (t_i, t_{i+1})$ s.t.

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} y''(\xi_i), \tag{7}$$

where h is the step size and t_i is chosen as in (2).

- From (7), LTE of Euler's method at the ith step is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i) = \frac{h}{2} y''(\xi_i).$$

So, we see that

$$\tau_i(h) = O(h) \quad \text{for each } i = 1, 2, \ldots, N,$$

since y'' is bounded on $[a,b]$.

- If the sol. $y(t)$ is smooth enough, say, $y \in C^{n+1}[a, b]$, then $\exists\ \xi_i \in (t_i, t_{i+1})$ s.t.

$$y(t_{i+1}) = y(t_i) + \sum_{k=1}^{n} \frac{h^k}{k!} y^{(k)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i) \qquad (8)$$

for each $i = 0, 1, \ldots, N-1$.

- Successive differentiation gives that

$$y'(t) = f(t, y(t)), y''(t) = f'(t, y(t)), \cdots, y^{(n+1)}(t) = f^{(n)}(t, y(t)).$$

- Then Eq. (8) can be rewritten as

$$y(t_{i+1}) = y(t_i) + h \sum_{k=1}^{n} \frac{h^{k-1}}{k!} f^{(k-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \qquad (9)$$

for each $i = 0, 1, \ldots, N-1$ and $f^{(0)} \equiv f$.

Taylor's Method of Order $n \in \mathbb{N}$

The approximations w_i to $y(t_i)$ $(i = 0, 1, \ldots, N)$ are computed by

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i) \quad \text{for each } i = 0, 1, \ldots, N-1, \qquad (10)$$

where h is the step size and

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

for each $i = 0, 1, \ldots, N-1$.

Note: Euler's method is just the Taylor's method of order one!

Thm 5.12 (高階泰勒法的局部截斷誤差)

Let $y(t)$ be the unique solution to the IVP (6) on $[a, b]$. If $y \in C^{n+1}[a, b]$, then the LTEs of Taylor's method of order n defined in (10) satisfy

$$\tau_i(h) = O(h^n) \quad \text{for each } i = 1, 2, \ldots, N,$$

where n and N are some positive integers.

Recall from Eq. (9)

For each $i = 0, 1, \ldots, N-1$, $\exists \xi_i \in (t_i, t_{i+1})$ s.t.

$$y_{i+1} = y_i + h \cdot \sum_{k=1}^{n} \frac{h^{k-1}}{k!} f^{(k-1)}(t_i, y_i) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)).$$

Proof

- From Taylor's Thm and (9), we obtain

$$\frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i))$$

for some $\xi_i \in (t_i, t_{i+1})$.

- Sine $y^{(n+1)} = f^{(n)} \in C[a, b]$, $|f^{(n)}(t, y(t))| < M \quad \forall\, t \in [a, b]$ and hence the LTE at the ith step satisfies

$$|\tau_{i+1}(h)| \leq \frac{M}{(n+1)!} h^n \quad \text{for } h > 0,$$

i.e., $\tau_{i+1}(h) = O(h^n)$ for each $i = 0, 1, \ldots, N-1$.

Example 1, p. 278

Apply Taylor's method of orders **(a) two** and **(b) four** with $N = 10$ to compute the approximations w_i $(i = 0, 1, \ldots, N)$ of the unique solution

$$y(t) = (t+1)^2 - 0.5e^t$$

to the IVP

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

- Let f be a real-valued function defined by

$$f(t, yt)) = y(t) - t^2 + 1 \quad \forall\, t \in [0, 2].$$

- $N = 10 \Rightarrow h = 0.2$ and $t_i = 0.2i$ for $i = 0, 1, \ldots, N$.

Solution of Part (a): Taylor's Method of Order 2

- Since $f'(t,y) = y' - 2t = y - t^2 + 1 - 2t$, we have

$$\begin{aligned} T^{(2)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) \\ &= (1 + \frac{h}{2})(w_i - t_i^2 + 1) - ht_i \end{aligned}$$

for each $i = 0, 1, \ldots, N-1$.

- Taylor's method of order 2 is defined by

$$\begin{aligned} w_0 &= 0.5, \\ w_{i+1} &= w_i + hT^{(2)}(t_i, w_i) \\ &= w_i + (0.2)\left[(1 + \frac{0.2}{2})(w_i - (0.2i)^2 + 1) - (0.2)(0.2i)\right] \\ &= \mathbf{1.22} w_i - \mathbf{0.0088} i^2 - \mathbf{0.008} i + \mathbf{0.22} \end{aligned}$$

for each $i = 0, 1, \ldots, 9$.

The numerical results of Part (a) are shown in the following table.

Solution of Part (b): Taylor's Method of Order 4 (1/2)

- Similarly, successive differentiation w.r.t. t gives

$$f''(t,y) = y - t^2 - 2t - 1 = f^{(3)}(t,y).$$

- The function $T^{(4)}(t_i, w_i)$ is defined by

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) \\ &+ \frac{h^3}{24} f^{(3)}(t_i, w_i) \\ &= (1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24})(w_i - t_i^2) - (1 + \frac{h}{3} + \frac{h^2}{12})(ht_i) \\ &+ 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}. \end{aligned}$$

Solution of Part (b): Taylor's Method of Order 4 (2/2)

- Substituting $h = 0.2$ and $t_i = 0.2i$ into $T^{(4)}$, we thus obtain Taylor's method of order 4 as

$$\begin{aligned} w_0 &= 0.5, \\ w_{i+1} &= w_i + hT^{(4)}(t_i, w_i) \\ &= \mathbf{1.2214} w_i - \mathbf{0.008856} i^2 - \mathbf{0.00856} i + \mathbf{0.2186} \end{aligned}$$

  for each $i = 0, 1, \ldots, 9$.

The numerical results of Part (b) are shown in the following table.

Section 5.3 勾選習題

5, 7

Section 5.4
Runge-Kutta Methods
(R-K 法)

Taylor's Methods v.s. Runge-Kutta Methods

1. Taylor's Methods
    - **Advantage**: They have **hogh-order** LTE at each step.
    - **Disadvantage**: They require computation and evaluation of the high-order derivatives of $f(t, y(t))$ w.r.t. the variable t.
2. Runge-Kutta Methods
    - They also have the high-order LTE at each step.
    - But eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

Thm 5.13 (Taylor's Thm for Functions of Two Variables)

Suppose $f(t, y)$ and all its partial derivatives of order $\leq n+1$ are conti. on $D = \{(t, y) \mid a \leq t \leq b,\ c \leq y \leq d\}$, and let $Q_0(t_0, y_0) \in D$. Then for every $(t, y) \in D$, $\exists\, \xi$ between t and t_0 and $\exists\, \mu$ between y and y_0 s.t.

$$f(t, y) = P_n(t, y) + R_n(t, y),$$

where

$$\begin{aligned} P_n(t, y) &= f(t_0, y_0) + \Big[(t - t_0) f_t(t_0, y_0) + (y - y_0) f_y(t_0, y_0)\Big] \\ &+ \left[\frac{(t - t_0)^2}{2} f_{tt}(Q_0) + (t - t_0)(y - y_0) f_{yt}(Q_0) + \frac{(y - y_0)^2}{2} f_{yy}(Q_0)\right] \\ &+ \cdots + \left[\frac{1}{n!}\sum_{j=0}^{n} \binom{n}{j} (t - t_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0)\right] \end{aligned} \tag{11}$$

Thm 5.13–Conti'd

and

$$R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu). \tag{12}$$

In this case, $P_n(t,y)$ is called the *n***th Taylor polynomial in two variables for the function** f **about** $Q_0(t_0,y_0)$, and $R_n(t,y)$ is the **remainder term associated with** $P_n(t,y)$.

A Class of R-K Methods

1. Runge-Kutta Methods of Order 2 (二階 R-K 法)
   - Midpoint method (中點法)
   - Modified Euler method (修正歐拉法)
   - LTE at each step = $O(h^2)$
2. Runge-Kutta Methods of Order 3 (三階 R-K 法)
   - Heun's method
   - LTE at each step = $O(h^3)$
3. Runge-Kutta Methods of Order 4 with LTE $O(h^4)$

The Midpoint Method (1/2)

- For the **Midpoint Method**, we try to determine the values for a_1, α_1 and β_1 s.t. $a_1 f(t+\alpha_1, y+\beta_1)$ approximates

$$\begin{aligned} T^{(2)}(t,y) &= f(t,y) + \frac{h}{2} f'(t,y) \\ &= f(t,y) + \frac{h}{2}\Big[f_t(t,y) + f_y(t,y)\cdot y'(t)\Big] \\ &= f(t,y) + \frac{h}{2} f_t(t,y) + \frac{h}{2} f_y(t,y)\cdot f(t,y) \qquad (13) \end{aligned}$$

- From Thm 5.13 with (11)–(12) $\Rightarrow \exists\,\xi$ between $t+\alpha_1$ and t and $\exists\,\mu$ between $y+\beta_1$ and y s.t.

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + (a_1\alpha_1) f_t(t,y) + (a_1\beta_1) f_y(t,y) + R_1, \qquad (14)$$

where the first-order remainder term is

$$R_1 \equiv \frac{\alpha_1^2}{2} f_{tt}(\xi,\mu) + \alpha_1\beta_1 f_{yt}(\xi,\mu) + \frac{\beta_1^2}{2} f_{yy}(\xi,\mu). \qquad (15)$$

- Comparing the coefficients in (13) and (14) $\Rightarrow$

$$a_1 = 1, \quad \alpha_1 = \frac{h}{2}, \quad \beta_1 = \frac{h}{2} f(t, y).$$

- So, we further obtain

$$T^{(2)}(t, y) = f(t + \frac{h}{2}, y + \frac{h}{2} f(t, y)) - R_1$$

with $R_1 = \frac{h^2}{8} f_{tt}(\xi, \mu) + \frac{h^2}{4} f(t, y) f_{yt}(\xi, \mu) + \frac{h^2}{8} [f(t, y)]^2 f_{yy}(\xi, \mu)$.

- This means that the ith LTE of **Midpoint Method** satisfies

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y_{i+1} - y_i}{h} - f(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)) \\ &= \frac{y_{i+1} - y_i}{h} - T^{(2)}(t_i, y_i) + R_1 = O(h^2), \end{aligned}$$

since $R_1 = O(h^2)$ if all second-order partial derivatives of f are bounded on $[a, b]$.

R-K Methods of Order Two

1. **Midpoint Method:**

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right),$$

for each $i = 0, 1, \ldots, N-1$.

2. **Modified Euler Method:**

$$w_0 = \alpha,$$
$$w_{i+1} = w_i + \frac{h}{2}\Big[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\Big],$$

for each $i = 0, 1, \ldots, N-1$.

Comments on R-K Methods of Order 2

- **Two function evaluations of** f are required per step.
- For the **Modified Euler Method**, we use the function

$$a_1 f(t,y) + a_2 f(t + \alpha_2, y + \delta_2 f(t,y))$$

to approximate $T^{(3)}(t,y)$. The desired parameters are given by

$$a_1 = a_2 = \frac{1}{2}, \quad \alpha_2 = \delta_2 = h.$$

- The LTE at each step is $O(h^2)$.

Example 2, p. 286

Apply **(a) Midpoint Method** and **(b) Modified Euler Method** with $N = 10$ to compute the approximations w_i $(i = 0, 1, \ldots, N)$ of the unique solution

$$y(t) = (t+1)^2 - 0.5e^t$$

to the IVP

$$y' = f(t, y) = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

The numerical results of Parts (a) and (b) are shown in the following table.

The function $T^{(3)}(t, y)$ can be approximated by

$$f\Big(t+\alpha_1, y+\delta_1 f\Big(t+\alpha_2, y+\delta_2 f(t,y)\Big)\Big),$$

involving 4 parameters to be determined.

Heun's Method (R-K Method of Order 3)

$$\begin{aligned} w_0 =& \alpha, \\ w_{i+1} =& w_i + \frac{h}{4}\Big[f(t_i, w_i) + \\ & 3f\Big(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\Big(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\Big)\Big)\Big], \end{aligned}$$

for each $i = 0, 1, \ldots, N-1$.

Algorithm: Heun's Method (Order Three)

$$
\begin{aligned}
w_0 &= \alpha, \\
k_1 &= h \cdot f(t_i, w_i), \\
k_2 &= h \cdot f\left(t_i + \frac{h}{3}, w_i + \frac{1}{3}k_1\right), \\
k_3 &= h \cdot f\left(t_i + \frac{2h}{3}, w_i + \frac{2}{3}k_2\right), \\
w_{i+1} &= w_i + \frac{1}{4}(k_1 + 3k_3),
\end{aligned}
$$

for each $i = 0, 1, \ldots, N-1$.

Comments on R-K Methods of Order 3

- **Three function evaluations of** f are required per step.
- The LTE at each step is $O(h^3)$.
- R-K methods of order 3 are NOT generally used in practice!

Example for Heun's Method

Apply **Heun's Method** with $N = 10$ to compute the approximations w_i $(i = 0, 1, \ldots, N)$ of the unique solution

$$y(t) = (t+1)^2 - 0.5e^t$$

to the IVP

$$y' = f(t, y) = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

The numerical results of Heun's method with $N = 10$ and $h = 0.2$ are shown in the following table.

Algorithm 5.2: Runge-Kutta (Order Four)

$$
\begin{aligned}
w_0 &= \alpha, \\
k_1 &= h \cdot f(t_i, w_i), \\
k_2 &= h \cdot f\Big(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\Big), \\
k_3 &= h \cdot f\Big(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\Big), \\
k_4 &= h \cdot f(t_{i+1}, w_i + k_3), \\
w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),
\end{aligned}
$$

for each $i = 0, 1, \ldots, N-1$.

Comments on R-K Methods of Order 4

- **Four function evaluations of** f are required per step.
- The LTE at each step is $O(h^4)$.
- These methods are the most commonly used for solving the IVPs in practice!

Example 3, p. 289

Apply **Runge-Kutta Method of Order 4** with $N = 10$ to compute the approximations w_i $(i = 0, 1, \ldots, N)$ of the unique solution

$$y(t) = (t+1)^2 - 0.5e^t$$

to the IVP

$$y' = f(t, y) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5.$$

Numerical results for the R-K method of order 4 are shown in the following table.

For R-K methods, the relationship between **the number of (function) evaluations per step** and **the order of LTE** is shown in the following table.

Reference

J. C. Butcher, *The non-existence of ten-stage eight-order explicit Runge-Kutta methods*, **BIT**, Vol. 25, pp. 521–542, 1985.

Comparisons Between R-K Methods

- If the R-K method of order 4 is to be superior to Euler's method, it should give more accuracy with step size h than Euler's method with step size $h/4$.
- If the R-K method of order 4 is to be superior to the R-K method of order 2, it should give more accuracy with step size h than second-order method with step size $h/2$.

Illustration

For the same IVP as before

$$y' = f(t, y) = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5,$$

we apply **Euler's method** with $h = 0.1/4 = 0.025$, **Modified Euler method** with $h = 0.1/2 = 0.05$, and **R-K method of order 4** with $h = 0.1$. The numerical results are given as follows.

Section 5.4 勾選習題

1, 5, 9, 13, 30

Thank you for your attention!