排列組合 - 直線排列

Permutation and Combination - Permutation in Linear Order

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1st Period</th></tr>
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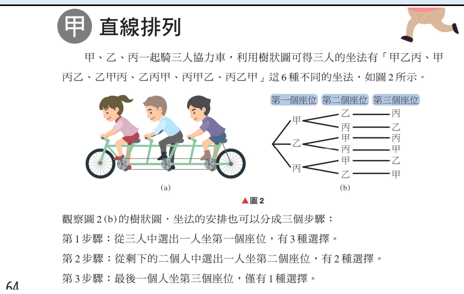

甲 直線排列

甲、乙、丙一起騎三人協力車，利用樹狀圖可得三人的坐法有「甲乙丙、甲丙乙、乙甲丙、乙丙甲、丙甲乙、丙乙甲」這 6 種不同的坐法，如圖 2 所示。

▲圖 2

觀察圖 2 (b) 的樹狀圖，坐法的安排也可以分成三個步驟：
第 1 步驟：從三人中選出一人坐第一個座位，有 3 種選擇。
第 2 步驟：從剩下的二個人中選出一人坐第二個座位，有 2 種選擇。
第 3 步驟：最後一個人坐第三個座位，僅有 1 種選擇。

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Word：Permutation (排列), Tree diagram (數狀圖), Arrangement (安排的情況), Scenario (情況).

Sentence：

1. If we seat them in order, how many different people could sit in the 1^{st} seat? (如果將他們按順序排成一列，第一個位子有幾個人可入座？)
2. For each of the 3 scenarios, after the first seat is taken, 2 different people could be put in the 2^{nd} seat. (在這 3 種情況下，第 2 張椅子有 2 個人可以坐。)
3. We can draw a tree diagram, the first section has 3 branches with person A, B, or C. (使用樹狀圖的話，第一個分支即有 3 個人 A，B 或 C。)
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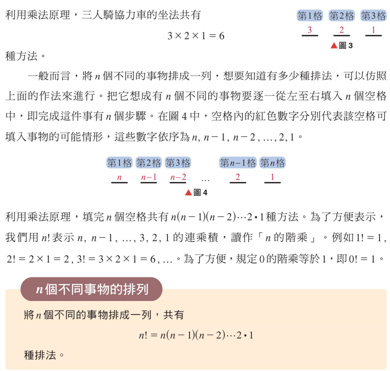

利用乘法原理，三人騎協力車的坐法共有

$$3\times2\times1=6$$

種方法。

▲圖 3

一般而言，將 n 個不同的事物排成一列，想要知道有多少種排法，可以仿照上面的作法來進行。把它想成有 n 個不同的事物要逐一從左至右填入 n 個空格中，即完成這件事有 n 個步驟。在圖 4 中，空格內的紅色數字分別代表該空格可填入事物的可能情形，這些數字依序為 $n, n-1, n-2, \dots, 2, 1$。

▲圖 4

利用乘法原理，填完 n 個空格共有 $n(n-1)(n-2)\cdots2\cdot1$ 種方法。為了方便表示，我們用 $n!$ 表示 $n, n-1, \dots, 3, 2, 1$ 的連乘積，讀作「n 的階乘」。例如 $1!=1$, $2!=2\times1=2$, $3!=3\times2\times1=6$, …。為了方便，規定 0 的階乘等於 1，即 $0!=1$。

n 個不同事物的排列

將 n 個不同的事物排成一列，共有

$$n!=n(n-1)(n-2)\cdots2\cdot1$$

種排法。
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Word：Factorial (階乘)

Sentence：

The number of permutations for seating these n people in n steats is n times n-1 times n-2 times to 1, which we call it n factorial. (n 個人坐 n 張椅子的排列數是，n 乘 n-1 乘 n-2 乘到 1，我們稱為「n 的階乘」。)
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<td>例題 1

學校獨唱比賽共有 6 位同學報名參加，出場順序由抽籤決定。共有多少種可能的抽籤結果？

解
抽籤的結果可視作將 6 位參賽者排成一列，其中排在最左邊代表第 1 位出場，其後依次為第 2, 3, 4, 5, 6 位出場。因為 6 位參賽者排成一列共有
$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
種排法，所以抽籤結果也有 720 種。

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<td>Translation:

In this question, we have 6 people needed to fit in 6 places. So the permutations for this question is 6 factorial, which is 6 times 5 times 4 times 3 times 2 times 1. And it is equal to 720 arrangements.</td>
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<td>接下來，探討從 n 個不同的事物中任選 k 個（$1 \le k \le n$）排成一列的排列數。先看這個例子：從 7 人中任選 3 人排成一列，共有多少種排法？仿照前面填空格的方式，把它想成有 7 個不同的事物要逐一從左至右填入 3 個空格中：

第1格 第2格 第3格
7 6 5
▲圖 5

如圖 5，利用乘法原理，排法共有
$7 \times 6 \times 5 = 210$
種。利用階乘的符號將 $7 \times 6 \times 5$ 表示成
$\overbrace{7 \times 6 \times 5}^{3\text{ 個}} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{7!}{4!} = \frac{7!}{(7-3)!}$。

一般而言，利用填空格的方式，可以推得：從 n 個不同的事物中任選 k 個（$1 \le k \le n$）排成一列，排法共有 $\frac{n!}{(n-k)!}$ 種，我們將這個排列數記作 P^n_k（讀作 Pn 取 k）。因為取 0 物排列只有「不取」一種方法，所以定義 $P^n_0 = 1$。這時前述的公式也會正確，因為 $P^n_0 = 1 = \frac{n!}{(n-0)!}$。將這個結論整理如下。</td>
<td>Sentence:

1. How can I relate factorial to this problem? (要如何連結至階乘呢？)

2. It looks like we kind of did factorial, but we stopped. We didn't go times 4 times 3 times 2 times 1. (可以用另一種想法，我們其實是使用了 7 的階乘，但沒有乘 4 到 1。)

3. We can write it in terms of factorial. We could write this as 7 factorial over 4 factorial. (我們就可以用階乘把算式寫成 7 的階乘除以 4 的階乘。)</td>
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<td>直線排列

從 n 個不同事物中任選 k 個（$0 \le k \le n$）排成一列，共有
$P^n_k = \frac{n!}{(n-k)!}$
種排法。</td>
<td>Suggested Instruction:

In conclusion, we have a notation P^n_k for the number of permutations where we put n people in k chairs is going to be n factorial over n minus k factorial.

Note:

P^n_k can be written in ${}^nP_k$, ${}_nP_k$, nPk or $P(n,k)$.</td>
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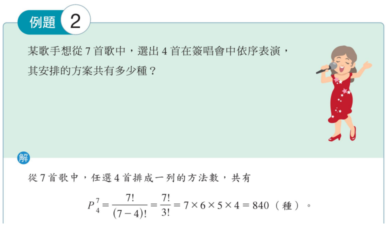
例題 2

某歌手想從 7 首歌中，選出 4 首在簽唱會中依序表演，其安排的方案共有多少種？

解

從 7 首歌中，任選 4 首排成一列的方法數，共有

$$P_4^7=\frac{7!}{(7-4)!}=\frac{7!}{3!}=7\times6\times5\times4=840\text{（種）}。$$

Translation:

In this question, there are 7 songs, but there are only 4 shows to perform. Therefore, we have 7 times 6 times 5 possible scenarios to give a performance, which is equal to P_4^7, also is 210.

補充題

Material

Find the number of different 8-letter arrangements that can be made from the letters of the word DAUGHTER so that

(i) All vowels occur together

(ii) No vowels occur together

Solution:

(i) There are 8 different letters in the word DAUGHTER. There are 3 vowels, namely, A, U and E. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). This single object together with 5 remaining letters (objects) will be counted as 6 objects. Then we count permutations of these 6 objects taken all at a time. This number would be 6P6 = 6!. Corresponding to each of these permutations, we shall have 3! permutations of the three vowels A, U, E taken all at a time. Hence, by the multiplication principle the required number of permutations = 6 ! × 3 ! = 4320.

(ii) If we have to count those permutations in which no vowels can be together, we first have to find all possible arrangements of 8 letters taken all at a time, which can be done in 8! ways. Then, we have to subtract from this number, the number of permutations in which the vowels are always together. Therefore, the required number 8 ! – 6 ! × 3 ! = 6 ! (7×8 – 6) = 2 × 6 ! (28 – 3) = 50 × 6 ! = 50 × 720 = 36000

Note
Word: Vowel (母音). **Sentence:** 1. All vowels occur together. (母音完全相鄰) 2. No vowels occur together. (母音不完全相鄰) 3. Since the vowels have to occur together, we can for the time being, assume them as a single object (AUE). (因為要將母音排在一起，我們可以將 AUE 先視為一體。) 4. We count permutations of these 6 objects taken all at a time. (我們先數 6 個物品的排列數。)

參考資料
References

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