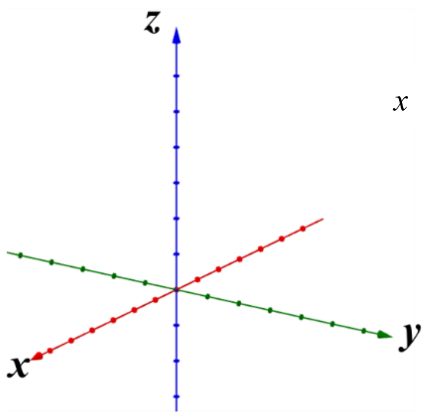
Vectors in Space and Vector Products

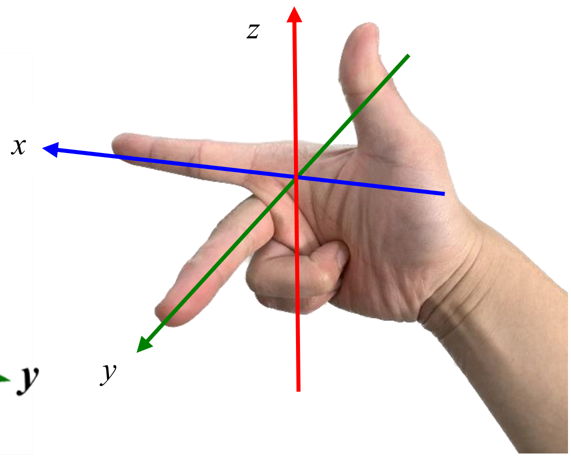
I. Key mathematical terms

Terms	Symbol	Chinese translation
Three-Dimensional Coordinate Systems		
Dot Product/ Inner Product		
Cross Product/ Outer Product		

II. Vectors in Space

We have learned about vectors in two-dimensional space, while most of our daily life happens in three dimensions space. It's necessary to create a framework for describing three-dimensional space. We use the horizontal number line x-axis and the vertical number line y-axis to form the coordinate plane. (two-dimensional space) Now we add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional (rectangular) coordinate system.





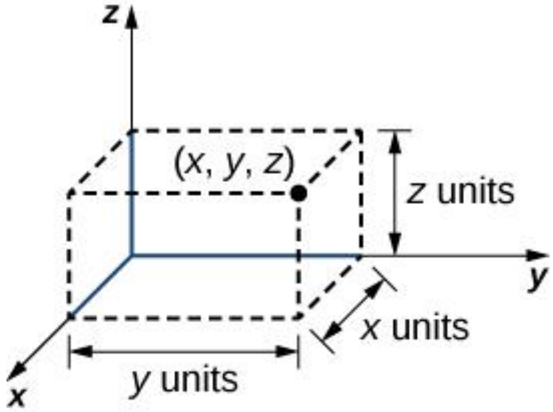
The coordinate system on the left is known as the right-hand coordinate system.

The index finger on the right gives the positive direction of the x-axis.

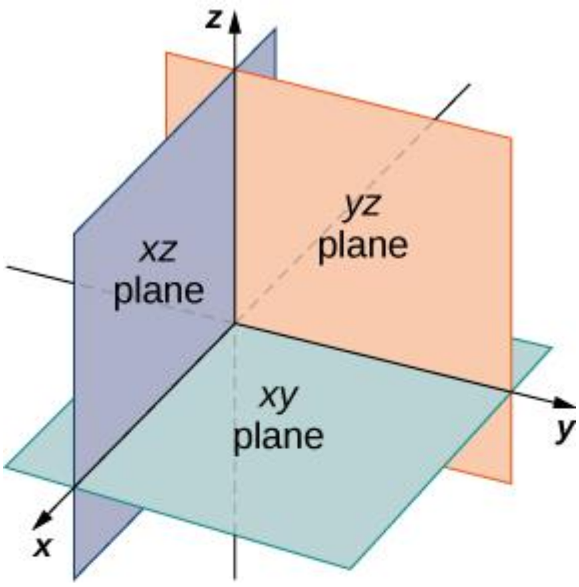
The middle finger on the right gives the positive direction of the y-axis.

The thumb on the right gives the positive direction of the x-axis.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the coordinate axis. In three dimensions, the new coordinate z is appended to indicate alignment with the z-axis. For example, a point (x, y, z), goes x units along the x-axis, y units along the y-axis, and z units in the direction of the z-axis. (See figure below.)



After we know the three axes of the pace, now we'll introduce the three coordinate planes: the xy-plane(The plane containing the x and y axes is the xy-plane.)
, the xz-plane, and the yz-plane. (See figure below)



The distance and vector formula between two points in space is similar to the formula on a plane.

The distance between two points in space

Suppose point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are points in the space. Then the distance between P and Q will be:

$$\overline{PQ} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Position Vector Formula in space

Suppose point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are points in the space.

1. The vector from P to Q is: $\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

2. The vector from Q to P is: $\overrightarrow{QP} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$

3. Magnitude of vector PQ is:

$$\left|\overrightarrow{PQ}\right| = |(x_2 - x_1, y_2 - y_1, z_2 - z_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

<key>

For vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ have same properties of vector addition and scalar multiplication. (See handout “Vectors”.)

Example1

Consider the points $A(1, 2, 7)$, $B(-1, 3, 0)$, $C(-2, 0, 5)$. Find:

(1) $\overrightarrow{AB}$.

(2) $\overrightarrow{BA}$.

(3) $\overrightarrow{BC}$.

(4) The distance from the origin to point C.

(5) $|\overrightarrow{CA}|$

Example2

Consider points $A(5, 0, 3)$, $B(4, 2, k)$ and $\overline{AB} = 3$. Find the possible value of k.

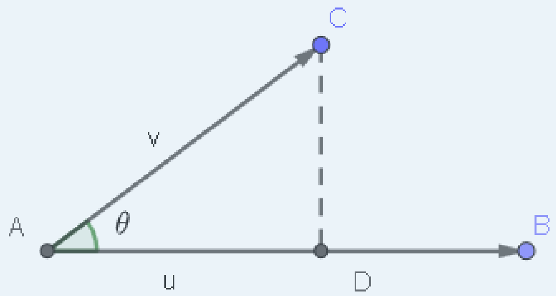
III. Vector Products

We've learned that a vector tis a quantity that contains both magnitude and direction, also we've learned two basic vector operations: vector addition (subtraction) and scalar multiplication. How about vector multiplication?
The multiplication of vectors can be done in two ways, i.e. dot product and cross product.

Dot Product (Inner Product)

The definition of the dot product (inner product) can be given in two ways. First, geometrically, the dot product is the product of two vectors' magnitudes and the cosine of the angle between them. Second, algebraically, the dot product is defined as the sum of the products of the corresponding entries of the two sequences of numbers.

Definition of the Dot Product (Geometrically)



The dot product of vector u and v

$= \vec{u}\cdot\vec{v} = \overrightarrow{AB}\cdot\overrightarrow{AC}$

$= |\vec{u}|\cdot|\vec{v}|\cdot\cos\theta = |\overrightarrow{AB}|\cdot|\overrightarrow{AC}|\cdot\cos\theta$

$= \overline{AD}\cdot\overline{AB}$

Definition of the Dot Product (Algebraically)

Dot product in $\boldsymbol{R}^2$: $\vec{u}=(u_1,u_2), \vec{v}=(v_1,v_2) \Rightarrow \vec{u}\cdot\vec{v}=u_1v_1+u_2v_2$

Dot product in $\boldsymbol{R}^3$: $\vec{u}=(u_1,u_2,u_3), \vec{v}=(v_1,v_2,v_3) \Rightarrow \vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$

Dot product in $\boldsymbol{R}^n$: $\vec{u}=(u_1,u_2,u_3,\ldots,u_n), \vec{v}=(v_1,v_2,v_3,\ldots,v_n)$

$$\Rightarrow \vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3+\ldots+u_nv_n$$

Example3

Find the following dot products:

(1) $(2,3)\cdot(-1,5)$

(2) $(-1,a,b)\cdot(2,5,0)$

(3) $|\vec{u}|=3, |\vec{v}|=5$, angle between vector u and v equals $30°$.

Properties of the Dot Product

Let $\boldsymbol{u}$,$\boldsymbol{v}$ and **w** are vectors in the plane or in the space and let $\boldsymbol{c}$ be a scalar.

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

2. $0\cdot\vec{u} = \vec{u}\cdot 0 = \vec{u}\cdot\vec{0} = 0$

3. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

4. $\vec{u}\cdot\vec{u} = \left\|\vec{u}\right\|^2$

5. $c(\vec{u}\cdot\vec{v}) = c\,\vec{u}\cdot\vec{v} = \vec{u}\cdot c\,\vec{v}$

Example4

Let $\vec{u} = (-1,3,5), \vec{v} = (0,1,-2), \vec{w} = (2,5,7)$, find the following dot products:

(1) $(\vec{u}+\vec{v})\cdot\vec{w}$

(2) $\vec{u}\cdot(-2)\vec{v}$

(3) $\left\|\vec{u}\right\|$

Cross Product (Outer Product)

Cross product is a binary operation on two vectors **<u>only in three-dimensional space</u>**. It results in a vector that is perpendicular to both vectors. We'll denote the cross product of vector u and v by "$u \times v$". This vector is perpendicular to both vector u and vector v. To calculate the cross product, we'll use the right-hand rule and formula to help us.

Cross Product Formula

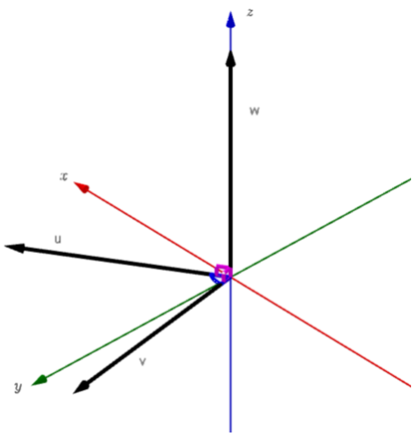
If θ is the angle between the given two vectors ***u*** and ***v***, and $\vec{u} = (u_1, u_2, u_3)$,

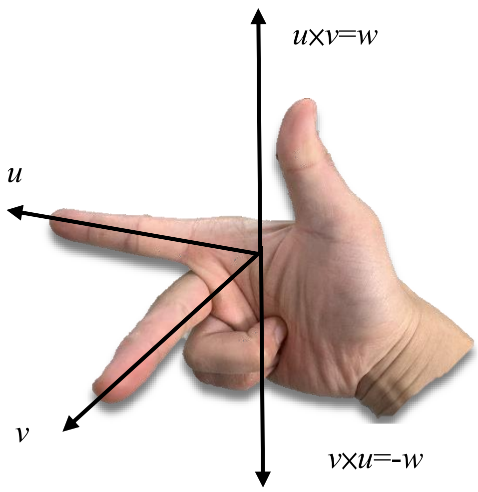
$\vec{v} = (v_1, v_2, v_3)$, then the formula for the cross product of vectors is given by:

$$\vec{u}\times\vec{v} = \left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}, \begin{vmatrix}u_3 & u_1\\ v_3 & v_1\end{vmatrix}, \begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right), \left|\vec{u}\times\vec{v}\right| = \left|\vec{u}\right|\left|\vec{v}\right|\sin\theta$$

<key> Right-Hand Rule Cross Product

We can find the direction of the unit vector with the help of the right-hand rule. For the following graph we have $u\times v = w$ and u, v are perpendicular to w.





Example5

Let $\vec{v} = (1,2,3), \vec{w} = (2,-1,2)$, find the following:

(1) $\vec{v}\times\vec{w}$ and $\vec{w}\times\vec{v}$

(2) $\vec{v}\cdot(\vec{w}\times\vec{v})$ and $\vec{w}\cdot(\vec{w}\times\vec{v})$

Properties of the Cross Product

Let $\boldsymbol{u}$,$\boldsymbol{v}$ and **w** are vectors in the space and let $\boldsymbol{c}$ be a scalar.

1. $(k\,\vec{u})\times\vec{v} = k(\vec{u}\times\vec{v}) = \vec{u}\times(k\,\vec{v})$

2. $\vec{u}\times\vec{v} = -\,\vec{v}\times\vec{u}$

3. $(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w}+\vec{v}\times\vec{w}\,,\quad \vec{w}\times(\vec{u}+\vec{v}) = \vec{w}\times\vec{u}+\vec{w}\times\vec{v}$

4. $\vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0$

5. $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$

Example6

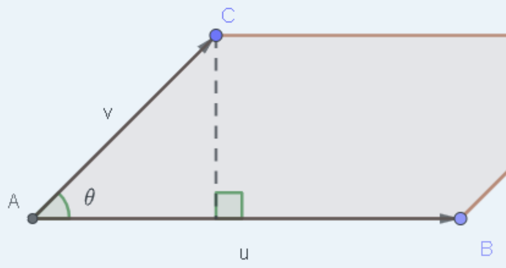
Vector $\vec{n}$ is perpendicular to both $\vec{v} = (1,3,2)$ and $\vec{w} = (1,1,1)$, also $|\,\vec{n}\,| = 2\sqrt{6}$.

Find the vector $\vec{n}$.

xaCross Product and Area

Vector $\vec{u}=(u_1,u_2,u_3), \vec{v}=(v_1,v_2,v_3)$ are nonzero and nonparallel. The magnitude of

vector $\vec{u}\times\vec{v}$ is the area of the parallelogram spanned by vector u and v.



The area of the parallelogram ABCD

$=\overline{AB}\cdot h=\overline{AB}\cdot\overline{AC}\cdot\sin\theta$

$=|\vec{u}||\vec{v}|\sin\theta$

$=\sqrt{|\vec{u}|^2|\vec{v}|^2(1-\cos^2\theta)}$

$=\sqrt{|\vec{u}|^2|\vec{v}|^2-|\vec{u}|^2|\vec{v}|^2\cos^2\theta}$

$=\sqrt{|\vec{u}|^2|\vec{v}|^2-(\vec{u}\cdot\vec{v})^2}$

$=\sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2}$

$=\sqrt{(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2}$

$=|\vec{u}\times\vec{v}|$

Example7

Three points $A(0,3,2), B(-1,3,4), C(-1,6,2)$, find the area of the parallelogram spanned by vector AB and vector AC.

<資料來源>

1. **Three-Dimensional Coordinate Systems**
   11.2: Vectors in Space - Mathematics LibreTexts
   12.1: Three-Dimensional Coordinate Systems - Mathematics LibreTexts
2. **Dot Product**
   https://byjus.com/maths/dot-product-of-two-vectors/
   https://www.mathsisfun.com/algebra/vectors-dot-product.html
3. **Cross Product**
   https://byjus.com/maths/cross-product/
   https://www.mathsisfun.com/algebra/vectors-cross-product.html
4. **Ron Larson, Precalculus, International Metric Edition**
5. **Pearson Edexcel AS and A level Mathematics Pure Mathematics Year 2**

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