Rewrite 12
雙語教學主題(國中九年級上學期教材): 三角形的內心
Topic: introducing incenter of a triangle

The teaching materials for introducing circles in our textbooks have been changed quite a bit due to the **108 syllabi.** Therefore, the content here is based on the official textbooks-NANI, KANG HSUAN and HANLIN.
由於 108 新綱教材大改，所以這個單元參考 108 新課綱及南一、康軒及翰林版國中數學課本第五冊

Vocabulary
Incenter, angle bisector, incircle, vertex, vertices, congruent, equidistant,

Before we start introducing this new lesson, we definitely need to review some of the related geometry properties.

Review :
Property of angle bisector:

In figure 1, $\overrightarrow{BP}$ is an angle bisector of $\angle ABC$,

point D is any point on $\overrightarrow{BP}$, $\angle ABD=\angle CBD$.

$\overline{DE} \perp \overline{BA}$ and $\overline{DE}$ intersects $\overline{BA}$ at point E,

$\overline{DF} \perp \overline{BC}$ and $\overline{DF}$ intersects $\overline{BC}$ at point F. Then $\overline{DE} = \overline{DF}$

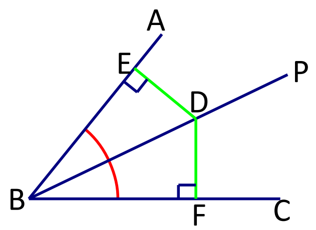


Figure 1

As usual, review what you have learned before, and do the proofs on your own. (hint: triangle congruency is one of the methods.)

The inverse of **Property of angle bisector** is true, too.

That is:In figure 2, point D is inside $\angle ABC$, $\overline{DE} = \overline{DF}$

$\overline{DE} \perp \overline{AB}$ and $\overline{DE}$ intersects $\overline{AB}$ at point E,

$\overline{DF} \perp \overline{BC}$ and $\overline{DF}$ intersects $\overline{BC}$ at point F.

Then $\angle ABD=\angle CBD$

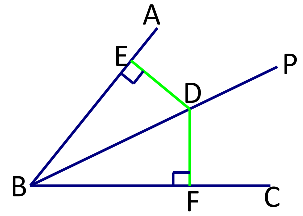


Figure 2

We don’t do the proof here. Students can think of different ways to prove it. (hint: symmetry or triangle congruency)

Incenter:

As shown in figure 1,

$\overrightarrow{BD}$ is an angle bisector of the interior angle $\angle ABC$ and $\overrightarrow{CE}$ is an angle bisector of the interior angle $\angle ACB$ in ΔABC. $\overrightarrow{BD}$ and $\overrightarrow{CE}$ intersect at point I.

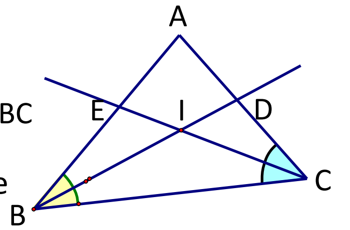


Figure 1

In figure 2, construct $\overline{IP} \perp \overline{AB}$ and $\overline{IQ} \perp \overline{BC}$.

From review 2, we know that $\overline{IP} = \overline{IQ}$,

As shown in figure 2.

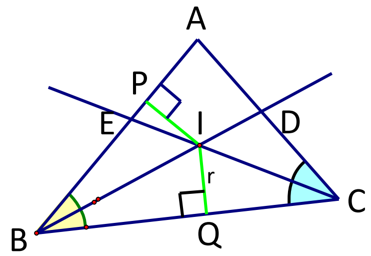


Figure 2

Will the three angle bisectors of three interior angles in ΔABC intersect at the same point?

The answer is yes, too.

In figure 3,

Construct $\overline{IR} \perp \overline{AC}$, we can easily get the conclusion that $\overline{IR} = \overline{IP} = \overline{IQ}$,

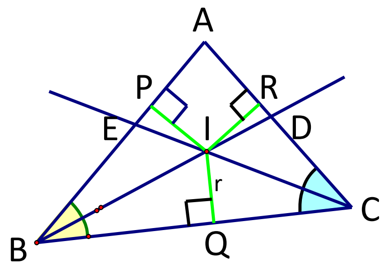


Figure 3

so point I is also on the angle bisector of $\angle BAC$. It means three angle bisectors intersect at the same point I, and point I is the **incenter** of ΔABC.

Fact 1:

Since point I is the intersection point of three interior angle bisectors, point I always stays **inside** ΔABC. No matter what kind of triangles it is. See figure 4.

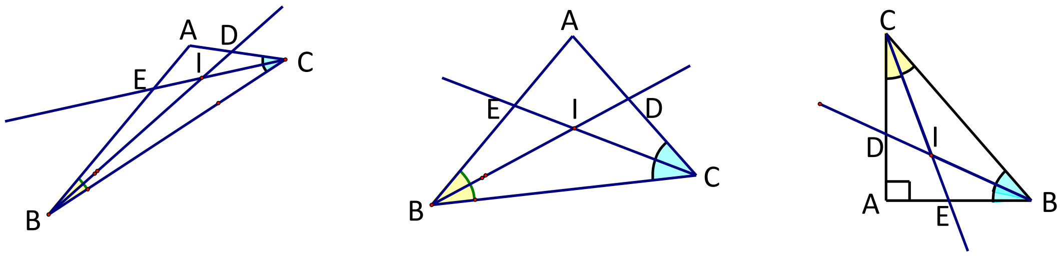


Figure 4

Fact 2:

In figure 3, we get $\overline{IR}=\overline{IP}=\overline{IQ}$. We can then construct a circle inside the ΔABC where point I is its center and $\overline{IR}=\overline{IP}=\overline{IQ}=r$ is the radius.

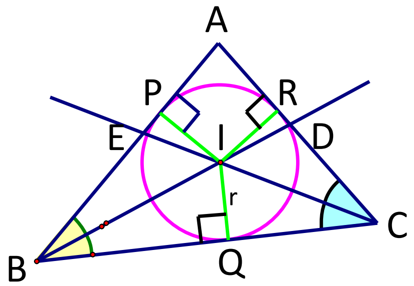


Figure 5

As shown in figure 5. Circle I is the incircle of ΔABC, and intersects each side of the triangle at only one point, point P, point Q, and point R respectively. Circle I is an inscribed circle of ΔABC and the three sides are tangent to the incircle.

Fact 3:

in figure 6, since $\overline{IP}\perp\overline{AB}$, $\overline{IQ}\perp\overline{BC}$, $\overline{IR}\perp\overline{AC}$, and $r=\overline{IR}=\overline{IP}=\overline{IQ}$.

The area of $\Delta ABC=\Delta AIB+\Delta BIC+\Delta CIA$

$$=\frac{1}{2}\overline{IP}\cdot\overline{AB}+\frac{1}{2}\overline{IQ}\cdot\overline{BC}+\frac{1}{2}\overline{IR}\cdot\overline{AC}$$

$$=\frac{1}{2}r\cdot\overline{AB}+\frac{1}{2}r\cdot\overline{BC}+\frac{1}{2}r\cdot\overline{AC}$$

$$=\frac{1}{2}r(\overline{AB}+\overline{BC}+\overline{AC})$$

=rs(let s be half of the perimeter of ΔABC)

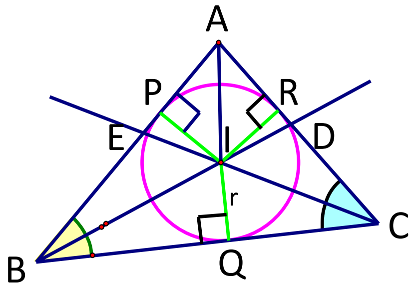


Figure 6

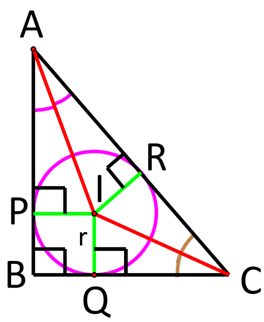
We learned that right triangles have special properties concerning circumcenters. Same thing happens here.
There are some very important properties of incenters in right triangles.

In the figure on the right side, point I is the incenter of ΔABC and $\overline{AB}\perp\overline{BC}$.

$\overline{IP}\perp\overline{AB}$, $\overline{IQ}\perp\overline{BC}$., and $\overline{IR}\perp\overline{AC}$.

Then (1) $\overline{AR}=\overline{AP}$, $\overline{BP}=\overline{BQ}$, and $\overline{CR}=\overline{CQ}$.

(2) the inradius $\overline{IP}=\frac{\overline{AB}+\overline{BC}-\overline{AC}}{2}$



Pf:

(1)we can easily get the result we learned before: by the tangent property of a circle that from any exterior point of a circle, the length of tangent segment is always equal.

So $\overline{AR} = \overline{AP}$, $\overline{BP} = \overline{BQ}$, and $\overline{CR} = \overline{CQ}$

Attention: Please notice that this is not only true in right triangles, it's true in all kind of triangles. It's a very powerful property.

(2)since the four interior angles of quadrilateral IPBQ are right angles, plus $\overline{IP} = \overline{IQ}$ (same radius), we get that quadrilateral IPBQ is a square. That tells us that

$$\overline{IP} = \overline{IQ} = \overline{BP} = \overline{BQ}$$

2 times of inradius $2\overline{IP} = \overline{BP} + \overline{BQ}$

$$= \overline{AB} - \overline{AP} + \overline{BC} - \overline{CQ}$$

$$= \overline{AB} - \overline{AR} + \overline{BC} - \overline{CR}$$

$$= \overline{AB} + \overline{BC} - \overline{AR} - \overline{CR}$$

$$= \overline{AB} + \overline{BC} - (\overline{AR} + \overline{CR})$$

$$= \overline{AB} + \overline{BC} - \overline{AC}$$

Then $\overline{IP} = \frac{1}{2}(\overline{AB} + \overline{BC} - \overline{AC})_{\#}$

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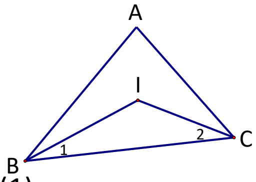
Please review these properties over and over again till you can recall them easily whenever you need to use them.

Let’s do some examples together.

Ex 1:

See the figure on the right side, point I is the incenter of ΔABC. $\angle BAC=70°$, find the measure of $\angle BIC$



Sol:

Set $\angle IBC=\angle 1$, $\angle ICB=\angle 2$, then $\angle 1=\frac{1}{2}\angle ABC$, $\angle 2=\frac{1}{2}\angle ACB$……(1)

In ΔABC, $\angle ABC+\angle ACB=180°-\angle BAC$

$=180°-70°$,

$=110°$……(2)

And in ΔBIC, $\angle BIC=180°-(\angle 1+\angle 2)$

$=180°-(\frac{1}{2}\angle ABC,+\ \frac{1}{2}\angle ACB)$ ……from (1)

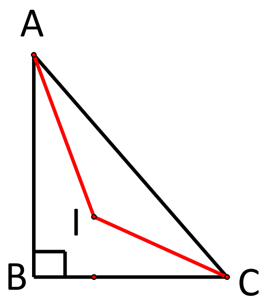
$=180°-\frac{1}{2}(\angle ABC+\angle ACB)$

$=180°-\frac{1}{2}\cdot 110°$

$=125°$#

Ex 2:

In the figure on the right side, point I is the incenter of the right triangle ABC, $\angle B=90°$. If $\overline{AB}=8$, $\overline{BC}=6$, please find out



(1)the length of segment AC ($\overline{AC}$)

(2)the measure of angle AIC($\angle AIC$)

(3)the area of $\triangle ABC$

Sol:

(1)by the Pythagorean theorem: $\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$

$=8^2+6^2$

$=100$

Then $\overline{AC}=10$

(2) $\angle AIC=180°-(\frac{1}{2}\angle BAC+\frac{1}{2}\angle ACB)$

$=180°-\frac{1}{2}(\angle BAC+\angle ACB)$

$=180°-\frac{1}{2}(180°-\angle ABC,)$

$=180°-\frac{1}{2}(180°-90°)$

$=135°$

(3)the inradius $=\frac{1}{2}(\overline{AB}+\overline{BC}-\overline{AC})=\frac{1}{2}(8+6-10)$

$=2$

The area of $\triangle ABC=\frac{1}{2}\cdot 2\cdot(8+6+10)$

$=24$#

Get more familiar with all the properties, they are quite useful in the coming days.

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