Equation of a Plane

I. Key mathematical terms

Terms	Symbol	Chinese translation
The general form for the equation of a plane		
Normal vector		
(Cartesian) Coordinate		

II. Equation of a Plane

Before we introduce the equation of a plane, let's review the equation of a line in two-dimensional coordinate. Recall that the general form for the equation of a straight line in two-dimensional coordinate is:

$$L: ax + by + c = 0$$

If $b \neq 0$ this equation also can be written in the form $y = mx + d$, where m is the slope (gradient) and d is the y-intercept. Suppose $P(x_0, y_0)$ is a point that lies on the line. We plug P into the equation and get:

$$c = -(ax_0 + by_0)$$

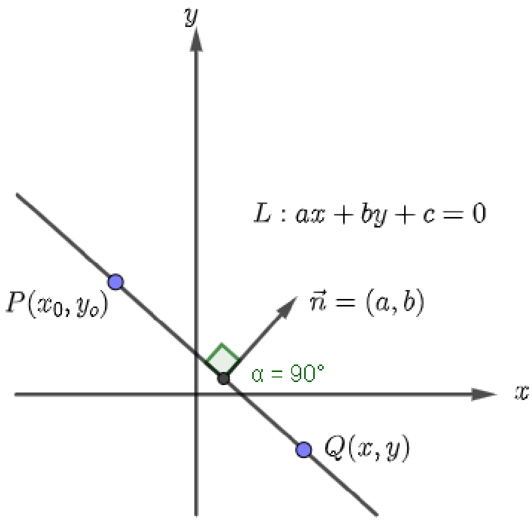
The equation of the line can be written as:

$$ax + by - (ax_0 + by_0) = 0$$
$$a(x - x_0) + b(y - y_0) = 0$$

$Q(x, y)$ is an arbitrary point on the line. $\overrightarrow{PQ} = (x - x_0, y - y_0)$ can be considered as a vector on the line, then we have the following inner product:

$$(a, b) \cdot (x - x_0, y - y_0) = 0$$

Here we call $\vec{n} = (a, b)$ is a normal vector of the line and $\vec{l} = (x - x_0, y - y_0)$ is a direction vector of the line. These two vectors are perpendicular to each other for their inner product is zero. (See figure on the next page.)



P, Q are points on the line. $\vec{n}$ is the normal vector of line L.

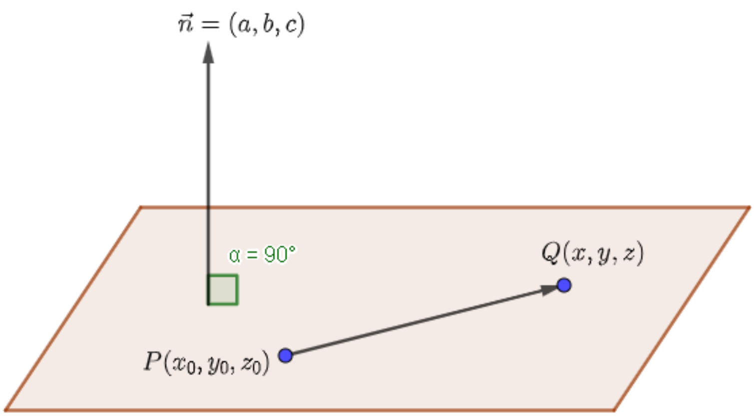
To describe a plane, similarly, we need a fixed point $P(x_0, y_0, z_0)$ on the plane.

Suppose we have a vector $\vec{n} = (a, b, c)$ that is orthogonal (perpendicular) to the plane. This is the normal vector of the plane. Now, assume that $Q(x, y, z)$ is an arbitrary point on the plane. Finally, we can find the equation of this plane with the following inner product:

$$\overrightarrow{PQ} \bullet \overrightarrow{n} = (x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

Hence, we can have the equation of the plane will be:

$$E : a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$\Rightarrow E : ax + by + cz = ax_0 + by_0 + cz_0 \quad (= d)$$
$$\Rightarrow E : ax + by + cz = d$$



P, Q are points on the plane. $\vec{n}$ is the normal vector of plane E.

Equation of a plane (Scalar Form)

Plane E passes through point $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = (a, b, c)$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This is the **scalar form** of the equation of a plane in $\mathbb{R}^3$.

Example#1

Find the equation of plane:

(1) A plane with normal vector $(-2, 3, -4)$ that contains the point $(-2, 3, -4)$.

(2) The equation of xy-plane, yx-plane and zx-plane.

Example#2

Determine the equation of the plane that contains points $P(-1, 2, 3), Q(0, 1, 3)$ and $R(5, 3, 7)$ (Use **scalar form** to represent your answer.)

The general form for the equation of a plane

The **general form** of the equation of a plane in $\mathbb{R}^3$ is:

$$ax + by + cz + d = 0$$

Where a, b, c are the components of the normal vector.

Example#3

Find the normal vector of the following plane:

(1) $2x - 3y + 5z - 10 = 0$

(2) $2x - y + 8 = 0$

(3) $y = 2$

Example#4

Find the equation of the plane that passes through point $A(5,-2,6)$ and parallel to the plane $F: x-y+2z+3=0$. (Use **general form** to represent your answer.)

<key> Two parallel plane will have the same normal vector.

Equation of a plane (Intercept Form)

Plane E intercepts the x, y, z axis at point $A(a,0,0), B(0,b,0), C(0,0,c)$ and $abc \neq 0$.

Then the equation of this plane can be represented by:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

a is the x-intercept, b is the y-intercept and c is the z-intercept.

Example#5

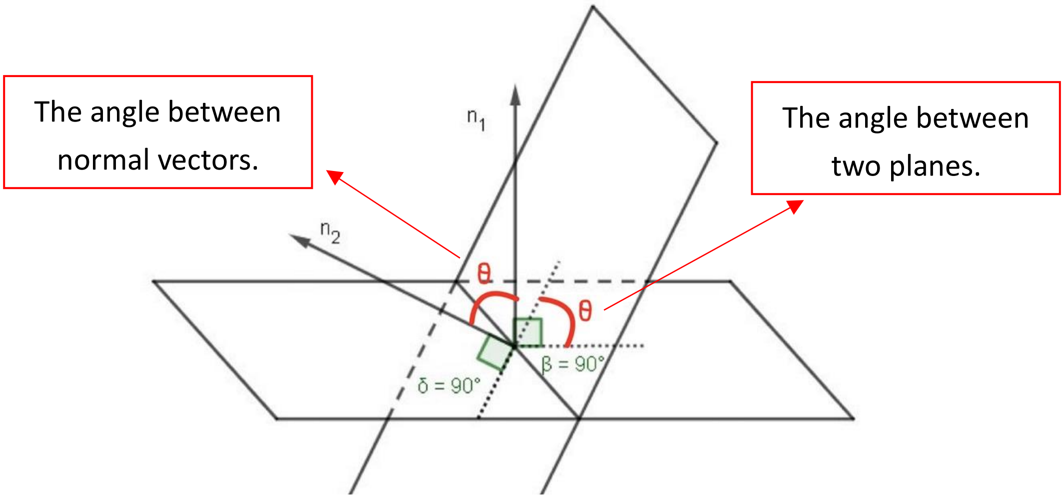
Find the equation of plane whose x,y and z intercepts are 3,-2, 4.

Example#6

Write the equation of plane $E: 16x+2y+8z-32=0$ in intercept form.

III. Angle between two planes

To calculate the angle between two planes, we'll use the normal vectors of two planes. (See figure below.)



Suppose the angle between two planes is θ, then the angle between the normal vectors of the plane will be $[90° - (90° - \theta)] = \theta$.

Now, we can use the inner product formula to find the angle between two planes.

Angle between two planes

Plane $E_1 : a_1x + b_1y + c_1z = d_1$ has normal vector $\overrightarrow{n_1} = (a_1, b_1, c_1)$.

Plane $E_2 : a_2x + b_2y + c_2z = d_2$ has normal vector $\overrightarrow{n_2} = (a_2, b_2, c_2)$.

If the acute angle between plane E_1 and E_2 is θ, then by inner product we have:

$$\cos\theta = \frac{\left|\overrightarrow{n_1}\,\overrightarrow{n_2}\right|}{\left|\overrightarrow{n_1}\right|\left|\overrightarrow{n_2}\right|} = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

<key>
The angle between these two planes can be θ (acute angle) or $180° - \theta$ (obtuse angle).
<key>
If two planes are perpendicular to each other, then their normal vectors' inner product will be zero. ($\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 0$)

Example#7

Determine the angle between the following planes:

1. $E_1: x+2y+3z=7$, $E_2: 2x-3y-z=5$
2. $E_1: x+2y+z=3$, $E_2: x+y=2$

IV. Distance between a point and a plane and Distance between two planes

As what we've learned in the "applications of linear equations". We can easily have the formula of distance between a point and a plane and the formula of distance between two planes.

Distance between a point and a plane

The distance between the point $P(x_0, y_0, z_0)$ and the plane $E: ax+by+cz=d$ is

$$d = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

Distance between two planes

The distance between $E_1: a_1x+b_1y+c_1z=d_1$, $E_2: a_2x+b_2y+c_2z=d_2$ is

$$d(E_1, E_2) = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Example#8

Find the distance between point $P(5,-2,-4)$ and plane $E: 3x+2y+z-21=0$

Example9

Find the distance between $E_1: 3x-2y+z+2=0$, $E_2: 3x-2y+z+5=0$

Example10

The distance between $E_1: x-2y+2z-3=0$, $E_2: x-2y+2z-k=0$ is 2. Find the value of k.

<資料來源>

1. **Equation of a plane**
   https://www.nagwa.com/en/explainers/373101390857/
   https://tutorial.math.lamar.edu/Classes/CalcIII/EqnsOfPlanes.aspx
   https://web.ma.utexas.edu/users/m408m/Display12-5-3.shtml
   https://www.cuemath.com/geometry/equation-of-plane/

2. **Pearson Edexcel AS and A level Mathematics Pure Mathematics Year 2**

3. 南一書局數學 **4A**

製作者：國立臺灣師範大學附屬高級中學 蕭煜修