線性不等式

Linear Inequality

Vocabulary

Inequality (不等式), Ordered Pair (數對), Respectively (分別地), Plane (平面), Region (區域), Sign (符號), Dashed Line (虛線), Solid Line (實線), Shade (陰影), Entire (全部的), Denote (表示), Vertical (鉛直), Satisfy (滿足), Horizontal (水平).

Illustrations

The Graph of an Inequality

The statements $3x-2y<6$ and $2x^2+3y^2\geq 6$ are inequalities in two variables. An ordered pair (a,b) is a solution of an inequality in x and y when the inequality is true after a and b are substituted for x and y, respectively. The graph of an inequality is the collection of all solutions of the inequality. To sketch the graph of an inequality, begin by sketching the graph of the corresponding equation. The graph of the equation will usually separate the plane into two or more regions.

不等式的圖形

式子$3x-2y<6$與$2x^2+3y^2\geq 6$皆稱為二元不等式。若數對(a,b)分別代入x,y使不等式成立，則數對(a,b)為其解，而不等式的圖解為其所有的解所形成的圖形。若欲畫出不等式的草圖，首先要畫其對應的方程式，其方程式的圖形通常會將平面分割成至少二個區域。

Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign by an equal sign and sketch the graph of the equation. (Use a dashed line for $<$ or $>$ and a solid line for $\leq$ or $\geq$.)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

描繪二元不等式的圖形

1. 以等號代換不等式的符號，並畫出方程式（若是$<$或$>$則用虛線；而$\leq$或$\geq$則用實線）。

2. 在每個區域找一測試點，若其點滿足不等式，將其整個區域上色，因為該區域的每個點皆滿足不等式，。

Examples

Sketch the graph of each linear inequality. (a) $x > -1$ (b) $y \leq 2$.(圖示不等式的解：(a) $x > -1$ (b) $y \leq 2$ 。)

Solution

(a) The graph of the corresponding equation $x = -1$ is a vertical line. The points that satisfy the inequality $x > -1$ are those lying on the right of this line, as shown in Figure 1. (對應不等式的方程式為 $x = -1$ ，其圖形為一條鉛直線，滿足不等式 $x > -1$ 的點在直線的右邊，如圖 1 所示。)

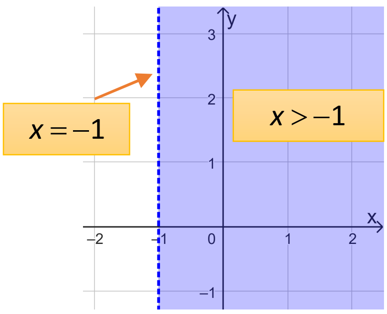


Figure 1

(b) The graph of the corresponding equation $y = 2$ is a horizontal line. The points that satisfy the inequality $y \leq 2$ are those lying below or on this line, as shown in Figure 2. (對應不等式的方程式為 $y = 2$ ，其圖形為一條水平線，滿足不等式 的點在直線的下方，如圖 2 所示。)

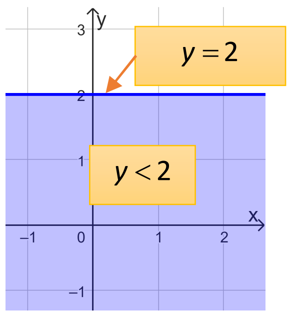


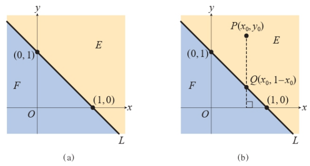
Figure 2

Material

己 二元一次不等式

當常數a,b不全為0時，不等式$ax+by+c>0$, $ax+by+c<0$, $ax+by+c\geq 0$, $ax+by+c\leq 0$稱為**二元一次不等式**，滿足不等式的實數數對(x,y)稱為它的**解**，我們可以在坐標平面上將它的所有解圖示出來。以$x+y-1>0$為例，說明如下：

(1) 在坐標平面上，畫出直線$L:x+y-1=0$的圖形，則L將坐標平面分成L本身及E,F兩個半平面，如圖17(a)所示。



▲圖17

(2) 設$P(x_0,y_0)$是半平面E內的任一點，過P點作x軸的垂線與L交於一點$Q(x_0,1-x_0)$，如圖17(b)所示。因為P點在Q點的上方，所以$y_0>1-x_0$，即

$$x_0+y_0-1>0，$$

因此，E內的每一個點(x,y)都滿足

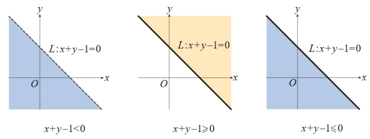
$$x+y-1>0。$$

反之，若$P(x_0,y_0)$是滿足$x+y-1>0$的一個解，即$x_0+y_0-1>0$，則有$y_0>1-x_0$。

也就是說P點在Q點的上方，即P點落在半平面E內。

由(1)(2)的討論知道：不等式$x+y-1>0$的所有解(x,y)所成的圖形，就是半平面E。同理可知：不等式$x+y-1<0$的所有解(x,y)所成的圖形就是半平面F。

另外，將半平面E與直線L合起來，就代表$x+y-1\geq 0$的所有解所成的圖形；而$x+y-1\leq 0$的所有解所形成的圖形為半平面F與直線L。當我們在坐標平面上作不等式的圖解時，若圖解包含直線，則直線以實線表示，否則以虛線表示，如圖18所示。



▲圖18

Vocabulary

Coordinate Plane (坐標平面), Boundary Line (邊界線), Distinct (相異的), Label (標籤), Arbitrary (任意的), Half-plane (半平面), Construct (建構), Perpendicular (垂直), Intersect (相交), Above (上面), Included (包含).

Translations

Linear inequality

Suppose a and b are not both zero. Inequalities $ax+by+c>0$, $ax+by+c<0$, $ax+by+c\geq 0$ and $ax+by+c\leq 0$ are called linear inequalities. The ordered pairs (x,y) that satisfy the inequalities are called the solutions. We show all of the solutions as a graph on the coordinate plane. For example, graph the inequality $x+y-1>0$. (當常數 a,b 不全為 0 時，不等式 $ax+by+c>0$, $ax+by+c<0$, $ax+by+c\geq 0$ 跟 $ax+by+c\leq 0$ 稱為二元一次不等式，滿足不等式的實數數對 (x,y) 稱為它的解，我們可以在坐標平面上將它的所有解圖示出來。以 $x+y-1>0$ 為例。)

(1) Draw the graph of the boundary line $L:x+y-1=0$ on the coordinate plane. The line L separates the coordinate plane into two distinct regions, labeled E and F in Figure 3(a). (在坐標平面上，畫出直線 $L:x+y-1=0$ 的圖形，則 L 將坐標平面分成 L 本身及 E, F 兩個半平面，如圖 3(a) 所示。)

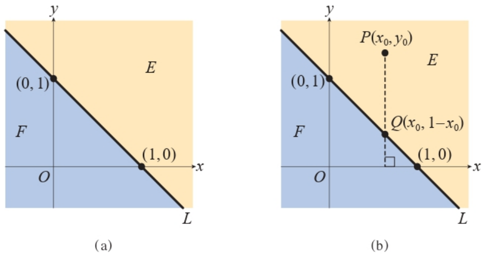


Figure 3

(2) Let point $P(x_0, y_0)$ is an arbitrary point of the half-plane *E*. Construct a perpendicular line of *x*-axis through point *P*, which intersect line *L* at point $Q(x_0, 1-x_0)$, as shown in Figure 3(b). We have $y_0 > 1 - x_0$, because point *P* is above point *Q*.

That is

$$x_0 + y_0 - 1 > 0 .$$

Thus, every point (x, y) of the half-plane *E* satisfies the inequality

$$x + y - 1 > 0 .$$

In other words, if point $P(x_0, y_0)$ is a solution of $x + y - 1 > 0$, which is $x_0 + y_0 - 1 > 0$, then we have $y_0 > 1 - x_0$.

The result shows that point *P* is above point *Q*, so point *P* is in half-plane *E*.

(設 $P(x_0, y_0)$ 是半平面 E 內的任一點，過 P 點作 x 軸的垂線與 L 交於一點 $Q(x_0, 1-x_0)$，如圖 3(b) 所示。因為 *P* 點在 *Q* 點的上方，所以 $y_0 > 1 - x_0$ ，即

$$x_0 + y_0 - 1 > 0$$

因此，E 內的每一個點 (x, y) 都滿足

$$x + y - 1 > 0$$

反之， 若 $P(x_0, y_0)$ 是滿足 $x + y - 1 > 0$ 的一個解，即 $x_0 + y_0 - 1 > 0$ ，則有 $y_0 > 1 - x_0$ 。也就是說 *P* 點在 *Q* 點的上方，即 *P* 點落在半平面 *E* 內。)

From the discussion (1) and (2) above, we know that the graph of all solutions (x, y) to the inequality $x + y - 1 > 0$ is a half-plane *E*. Similarly, the graph of all solutions (x, y) to the inequality $x + y - 1 < 0$ is a half plane *F*. (由 (1) (2) 的討論知道：不等式 $x + y - 1 > 0$ 的所有解 (x, y) 所成的圖形，就是半平面 *E*。同理可知：不等式 $x + y - 1 > 0$ 的所有解 (x, y) 所成的圖形就是半平面 *F*。)

Moreover, if we color the half plane E and line L at the same time, then the graph represents all of the solutions to the inequality $x+y-1\geq 0$; while the graph of the half plane F and line L represents all solutions to the inequality $x+y-1\leq 0$. (另外，將半平面 E 與直線 L 合起來，就代表 $x+y-1\geq 0$ 的所有解所成的圖形；而 $x+y-1\leq 0$ 的所有解所形成的圖形為半平面 F 與直線 L。)

Consequently, we draw a solid line to show that the boundary line is included in the solutions of the inequality. Otherwise, we use dashed line, as shown in Figure 4. (當我們在坐標平面上作不等式的圖解時，若圖解包含直線，則直線以實線表示，否則以虛線表示，如圖 4 所示。)

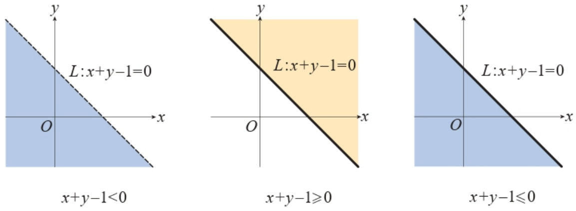


Figure 4

Example

Sketch the graph of $2x-y+4>0$. (圖示二元一次不等式 $2x-y+4>0$ 的解。)

Solution

The graph of the corresponding equation $2x-y+4=0$ is a line, as shown in Figure 5. The origin $(0,0)$ satisfies the inequality $2\times 0-0+4=4>0$, so the graph consists of the half-plane lying below the line. Check a point above the line. Regardless of which point you choose, you will find that it does not satisfy the inequality. (先畫出直線 $L:2x-y+4=0$，如圖 5。此時直線 L 將坐標平面分成包含原點與不包含原點的兩個半平面。再將原點 $(0,0)$ 代入 $2x-y+4$，得 $2\times 0-0+4=4>0$，故不等式的解就是包含原點的下半平面。檢查直線下的點，無論選擇下半面平的哪個點，將會發現每個點皆不滿足不等式。)

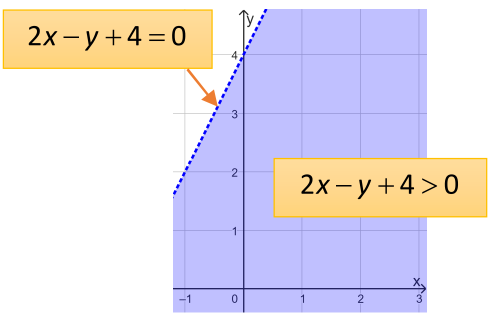
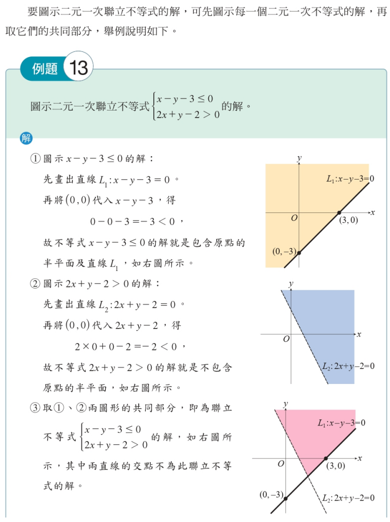


Figure 5

<table>
<tr><th>Material</th><th>Vocabulary</th></tr>
<tr><td>
要圖示二元一次聯立不等式的解，可先圖示每一個二元一次不等式的解，再取它們的共同部分，舉例說明如下。
例題 13
圖示二元一次聯立不等式 $\begin{cases} x-y-3\le 0 \\ 2x+y-2>0 \end{cases}$ 的解。
解
① 圖示 $x-y-3\le 0$ 的解：
先畫出直線 $L_1: x-y-3=0$。
再將 $(0,0)$ 代入 $x-y-3$，得
$0-0-3=-3<0$，
故不等式 $x-y-3\le 0$ 的解就是包含原點的半平面及直線 L_1，如右圖所示。
② 圖示 $2x+y-2>0$ 的解：
先畫出直線 $L_2: 2x+y-2=0$。
再將 $(0,0)$ 代入 $2x+y-2$，得
$2\times 0+0-2=-2<0$，
故不等式 $2x+y-2>0$ 的解就是不包含原點的半平面，如右圖所示。
③ 取①、②兩圖形的共同部分，即為聯立不等式 $\begin{cases} x-y-3\le 0 \\ 2x+y-2>0 \end{cases}$ 的解，如右圖所示，其中兩直線的交點不為此聯立不等式的解。
</td>
<td>Variable (變數), Common (共同), System of Inequalities (聯立不等式), Plug in (代入), Upper Half-plan (上半平面), Lower Half-plan (下半平面), Superimpose (疊加).</td></tr>
<tr><th colspan="2">Translations</th></tr>
<tr><td colspan="2">

To sketch the graph of a system of inequalities in two variables, we can sketch the graph of each individual inequality and then find the region that is common to every graph in the system. (要圖示二元一次聯立不等式的解，可先圖示每一個二元一次不等式的解，再取它們的共同部分。)

Example 13

Sketch the graph of the solution set of the system of inequalities $\begin{cases} x-y-3\le 0 \\ 2x+y-2>0 \end{cases}$. (圖示二元一次聯立不等式 $\begin{cases} x-y-3\le 0 \\ 2x+y-2>0 \end{cases}$ 的解。)

Solution

(1) Graph the solution of inequality $x-y-3\le 0$.

First, sketch line $L_1: x-y-3=0$. By plugging in $(0,0)$, we have $0-0-3=-3<0$ which satisfies the inequality. Thus, the solution region of inequality $x-y-3\le 0$ is upper half-plan and the line L_1 which cover the test point $(0,0)$, as shown in Figure 6. (圖示 $x-y-3\le 0$ 的解：先畫出直線 $L_1: x-y-3=0$。再將 $(0,0)$ 代入 $x-y-3$，得 $0-0-3=-3<0$，故不等式 $x-y-3\le 0$ 的解就是包含原點的半平面及直線 L_1，如圖 6 所示。)

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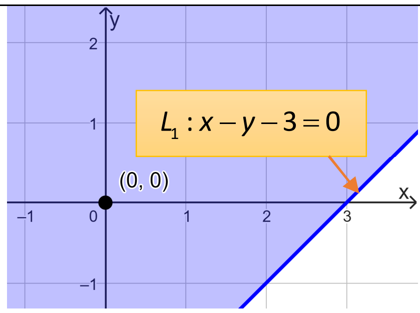

Figure 6

(2) Graph the solution of inequality $2x+y-2>0$.

First, sketch line $L_2: 2x+y-2=0$. By plugging in $(0,0)$, we have $2\times0+0-2=-2<0$ which does not satisfy the inequality. Thus, the solution region of inequality $2x+y-2>0$ is the upper half-plan and the line L_2 which don't cover the test point $(0,0)$, as shown in Figure 7. (圖示 $2x+y-2>0$ 的解：先畫出直線 $L_2: 2x+y-2=0$ 。再將 $(0,0)$ 代入 2x + y - 2，得 $2\times0+0-2=-2<0$ ，故不等式 $2x+y-2>0$ 的解就是不包含原點的半平面，如圖 7 所示。

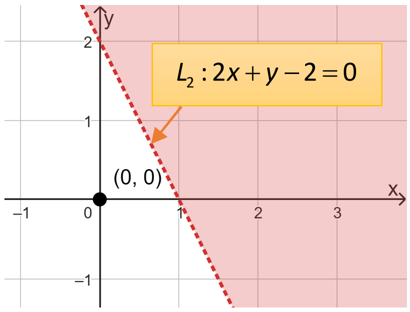

Figure 7

(3) The intersection of the half-planes that are solutions of the inequalities, as shown in Figure 8. (取(1)、(2)兩圖形的共同部分，即為聯立不等式 $\begin{cases} x-y-3\le0 \\ 2x+y-2>0 \end{cases}$ 的解， 如圖 8 所示。)

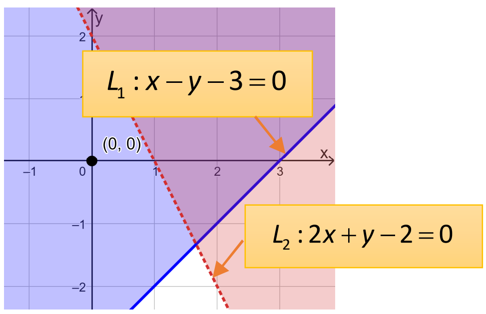

Figure 8

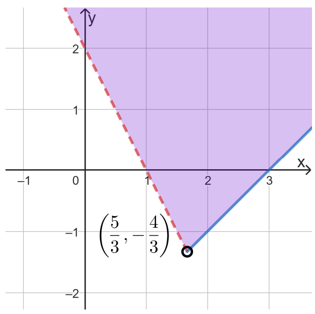

Figure 9

Note in Figure 9 that the vertices of the region are represented by open dots. This means that the vertices are not solutions of the system of inequalities. (圖 9 中，以中空的頂點表兩直線的交點不為此聯立不等式的解。)

Supplementary Materials

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C.

A cup of dietary drink *X* provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C.

A cup of dietary drink *Y* provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C.

Write a system of linear inequalities that describes how many cups of each drink must be consumed each day to meet or exceed the minimum daily requirements for calories and vitamins.

Solution

Begin by letting *x* represent the number of cups of dietary drink *X* and *y* represent the number of cups of dietary drink *Y*. To meet or exceed the minimum daily requirements, the following inequalities must be satisfied.

$$\begin{cases} 60x + 60y \geq 300 & \text{Calories} \\ 12x + 6y \geq 36 & \text{Vitamin A} \\ 10x + 30y \geq 90 & \text{Vitamin C} \\ x \geq 0, y \geq 0 \end{cases}$$

The last two inequalities are included because x and y cannot be negative. The graph of this system of inequalities is shown in Figure 10(a)(b).

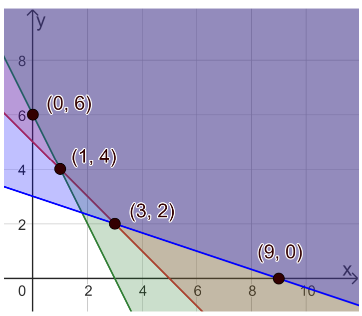

Figure 10(a)

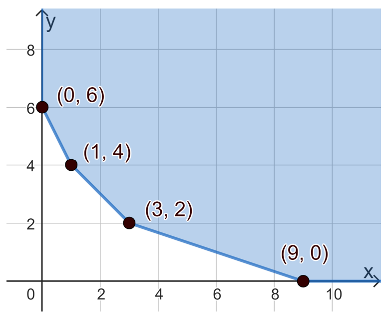

Figure 10(b)

Translations

Vocabulary: Liquid (液體), Portion (部分), Diet (飲食), Calorie (卡路里), Vitamin (維他命), Dietary (規定的飲食), Describe (描述), Consume (消耗), Meet (符合), Exceed (超過), Requirement (需求).

Sentences:

1. Write a system of linear inequalities that describes... (寫出聯立一次不等式描述…)
2. Let x represent the number of cups of dietary drink *X*. (令 *x* 為低卡飲料 *X* 的杯數。)
3. Meet or exceed the minimum daily requirements. (符合或超過每日所需的最小值。)

References

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