**Topic: Solving exponential equations and inequalities**

1. Review

 Compare the following numbers **in ascending form**(from the least to the greatest)by using the graph of $y = 2^x$. (Hint: Rewrite the numbers in base 2)

$A = 2$, $B = 4^{-0.25}$,$C = (\frac{1}{2})^{-2}$,$D = (\sqrt{2})^{\sqrt{2}}$

2. One–to–One Property

Notice that the graph of an exponential function is always increasing or decreasing, so an exponential function is a one–to–one function. Therefore, exponential functions have the following property.
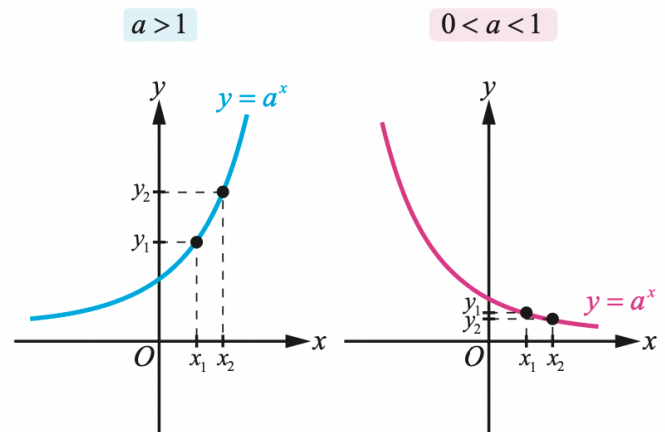
One–to–One Property

For $a > 0$ and $a \neq 0$, $a^x = a^y$ if and only if $x = y$.

---

3. Exercise

Use the One–to–One Property to solve the equations below.

a. $3^{x+1} = 9$

b. $(0.5)^x = 2^{-2x+3}$

4. Advanced Exercise

Use the One–to–One Property to solve the equations below.

a. $4^x - 2^{x+1} = 8$

b. $9^x + 3^{x+1} - 18 = 0$

5. Property of Inequalities for an exponential function

The graph of an exponential function always increases or decreases, so we have the following properties.

> • For $a > 1$, $a^x > a^y$ if $x > y$
>
> • For $0 < a < 1$, $a^x > a^y$ if $x < y$

6. Exercise

Solving the following exponential inequalities.

a. $3^{2x+1} > \dfrac{1}{3}$

b. $(0.7)^{x^2} > (0.49)^x$

c. $4^x - 2^{x+1} - 8 > 0$

d. $4^x - 3 \times 2^x + 2 < 0$

7. Challenge

Solve the equation and approximate the result to **three decimal places**(小數點後三位). (Hint: Use the logarithm)

$3(2^x) + 5 = 56$

Topic: Solve the exponential equations and inequalities使用建議

內容	1. Review …以下省略
使用建議	[教學活動安排] 利用將各數字比大小來複習指數函數圖形的遞增、遞減性質 [可參考的英文問句/提問/開場] Today we're going to learn how to solve the exponential equations and inequalities. It's an application of graphs of exponential functions. We learned some properties of the graphs at the last class. Now let's do the review. Try to make the connection with the graph of $y = 2^x$. We'll check it in two minutes. [參考解答] Follow the hint, let's rewrite the numbers in base two. $A = 2^1, B = (2^2)^{-0.25} = 2^{2\times(-0.25)} = 2^{-0.5}, C = (2^{-1})^{-2} = 2^{(-1)\times(-2)} = 2^2,$ $D = (2^{\frac{1}{2}})^{\sqrt{2}} = 2^{\frac{\sqrt{2}}{2}}$ Since the graph of $y = 2^x$ is increasing, which means y increases when x increases, so we can just compare the exponents. $-0.5 < \dfrac{\sqrt{2}}{2} < 1 < 2$,so $B < D < A < C$
內容	2. One-to-One Property

使用建議	[教學活動安排] 介紹1對1性質 [可參考的英文問句/提問/開場] Since the graphs of exponential functions are increasing or decreasing, there is another property which is One–to–one property. Let's look closely at the graphs, we can see that for every y–value corresponds to exactly one x–value. Therefore we have one–to–one property, $a^x = a^y$ if and only if $x = y$.
內容	3. Exercises Use the One–to–One Property to solve the equations below. a. $3^{x+1} = 9$ b. $(0.5)^x = 2^{-2x+3}$

使用建議	[教學活動安排]
	利用1對1性質解指數方程式
	
	[可參考的英文問句/提問/開場]
	Now we can start to solve the exponential equations.
	Let's see how to use the one–to–one property to slove them.
	a. $3^{x+1} = 9$, we write both side into base of three.
	  => $3^{x+1} = 3^2$
	  By one–to–one property, we know that $x + 1 = 2$, $x = 1$.
	Now, it's your turn. Slove the equation b. We will check it in two minutes.
	b. $(0.5)^x = 2^{-2x+3}$
	=>$2^{-x} = 2^{-2x+3}$ we write both side into base of two.
	=>$-x = -2x + 3$
	=>$x = 3$

內容	4. Advanced exercises Use the One–to–One Property to solve the equations below. a. a. $4^x - 2^{x+1} = 8$ b. $9^x + 3^{x+1} - 18 = 0$

| 使用建議 | [教學活動安排]
含變數變換的指數方程式
a. $4^x - 2^{x+1} = 8$

b. $9^x + 3^{x+1} - 18 = 0$

[a解答]
a. Let $A = 2^x$, $4^x = A^2$, $2^{x+1} = 2A$
$A^2 - 2A - 8 = 0$
$=>(A + 2)(A - 4) = 0$
$=>A = -2$ or $A = 4$
Since $A = 2^x > 0$, so $A = 4$, $2^x = 4$, $x = 2$

[a.可參考的英文解說]
Let's do the advanced exercises.
These equations are not just exponential equations, they are also quadratic equations. Let's do the substitution.
Let $A = 2^x$, then the term $4^x = A^2$, why?
And how can we rewrite the term 2^{x+1} ?
Brainstorming, does anyone have an idea?
$4^x = (2^2)^x = (2^x)^2 = A^2$ (four to the power of x equals two squared to the power of x ,and equals two to the power of x squared, then two to the power of x substituted into A ,so we get A squared.)
For 2^{x+1}, $2^{x+1} = 2^x \times 2 = 2A$(two to the power of x plus one equals two to the power of x times two, then two to the power of x substituted into A ,so we get 2A.)
So the equation can be written into $A^2 - 2A - 8 = 0$ it's a quadratic equation in variable A.
$A^2 - 2A - 8 = 0$
$=> (A + 2)(A - 4) = 0$ (Factorise the equation.)
$=>A = -2$ or $A = 4$
Since $A = 2^x$ is always positive, so $A = 4$
And, we get $2^x = 4$,$x = 2$
Now it's your turn. Go and get a try for b. We will check it in five minutes.

[b.解答]
b. Let $A = 3^x$, $9^x = A^2$, $3^{x+1} = 3A$
$A^2 + 3A - 18 = 0$
$=>(A + 6)(A - 3) = 0$
$=>A = -6$ or $A = 3$
Since $A = 3^x > 0$, so $A = 3$, $3^x = 3$, $x = 1$ |

內容	5. Property of Inequalities for an exponentail function
使用建議	[教學活動安排] 介紹指數函數圖形的遞增、遞減性質用來解指數不等式 [可參考的英文解說] Let's go through another property of the graphs of exponentail functions. We can apply this property to slove exponential inequalities. Anyone wants to read the content of this property for us?
使用建議	[教學活動安排] 介紹指數函數圖形的遞增、遞減性質用來解指數不等式 [可參考的英文解說]

內容	6. Exercises a and b Solve the following exponential inequalities. a. $3^{2x+1} > \dfrac{1}{3}$ b. $(0.7)^{x^2} > (0.49)^x$
使用建議	[教學活動安排] 解指數不等式 a 及b [可參考的英文解說] Let's do the exercise a together. Does anyone know the first thing we need to do to solve this exponential inequality? Good. Change into the same base three. Therefore we get $3^{2x+1} > 3^{-1}$(three to the power of two x plus one is greater than three to the power of negative one.) Then, we get $2x + 1 > -1$ (two x plus one is greater than negative one) Isolate x , we get the answer $x > -1$ (x is greater than negative one) Ok,it's your turn. Solve the inequality of b. And notice that the base 0.7 is less than one. We'll check it in two minutes. For the question b, we change both side into the same base 0.7. We get $0.7^{x^2} > 0.7^{2x}$. Since $0 < 0.7 < 1$, we get $x^2 < 2x$. Isolate x , $x^2 - 2x < 0$. factorize the inequality we get $x(x - 2) < 0$. we get $0 < x < 2$
內容	6. Exercises Solve the exponential inequalities c. $4^x - 2^{x+1} - 8 > 0$ d. $4^x - 3 \times 2^x + 2 < 0$

| 使用建議 | [教學活動安排]
解指數不等式 c 及d
c. $4^x - 2^{x+1} - 8 > 0$

d. $4^x - 3 \times 2^x + 2 < 0$

[c.解答]
c. Let $A = 2^x$, $4^x = A^2$, $2^{x+1} = 2A$
$A^2 - 2A - 8 > 0$
=>$(A + 2)(A - 4) > 0$
=>$A < -2$ or $A > 4$
Since $A = 2^x > 0$, so $A > 4$, $2^x > 4$, $x > 2$

[c.參考解說]
Let's solve exercise c together.
You might find this inequality familiar. Yes, we solve the equation
version at the advanced exercise before.
So, it is almost the same as we did before.
Does anyone remember what we did before?
Yes, good.
Let's do the substitution.
Let $A = 2^x$, then how can we rewrite the term 4^x and 2^{x+1}?
$4^x = A^2$, $2^{x+1} = 2A$
So the inequality can be written into $A^2 - 2A - 8 > 0$ it's a quadratic
inequality in variable A.
$A^2 - 2A - 8 > 0$
=> $(A + 2)(A - 4) > 0$ (Factorise the inequality.)
=>$A < -2$ or $A > 4$
Since $A = 2^x$ is always positive, so $A > 4$
We get
$2^x > 4$
=>$2^x > 2^2$
=>$x > 2$
Now it's your turn. Go try for d. We will check it in five minutes.

[d.解答]
Let $A = 2^x$, $4^x = A^2$
$A^2 - 3A + 2 < 0$
=>$(A - 1)(A - 2) < 0$
=>$1 < A < 2$
So, $1 < 2^x < 2$ => $0 < x < 1$ |

內容	7. Challenge
使用建議	[教學活動安排] 解需使用取對數的指數方程式 [可參考的英文解說] $3(2^x) + 5 = 56$ Isolate 2^x, we get $2^x = 17$ $=> x = log_2 17$ (Take log base two of each side.) $=> x = \dfrac{log 17}{log 2}$ (Use change base formula.) $=> x = 4.08746... \approx 4.087$ (Use a calculator.)

製作者：台北市立育成高中 教師林玉惇