組合

Combinations

Material	Vocabulary
組合 從 n 個不同事物中取出 k 個（$0 \le k \le n$）的組合數為 $C_k^n = \frac{n!}{k!(n-k)!}$。 83	1. combination (組合), 2. permutation (排列), 3. column (欄), 4. arrange (安排), 5. convention (常規), 6. reasonable (合理), 7. committee (委員會), 8. membership (會員), 9. applicant (申請人), 10. express (表達), 11. general (一般).

Translations

Combinations[1] of *n* Things Taken *k* at a Time

The number of possible combinations if k items are taken from n ($n \ge k$) items is

$$C_k^n = \frac{n!}{k!(n-k)!}.$$

Note:

(1) $n!$ is read “n factorial.”

(2) C_k^n can also be written as ${}_nC_k$, ${}^nC_k$ or $\binom{n}{k}$; is read as “n choose k.”

(3) P_k^n : The number of permutations[2] as n distinct objects taken k at a time; is read as “n pick k”, “n permutes k” or more precisely “permutation of n elements taken k at a time.”

Illustrations

A Formula for Combinations

In the previous chapter we have learned that the notation P_k^n means the number of permutations of n things taken k at a time. Similarly, the notation C_k^n means the number of combinations of n things taken k at a time. (在先前的章節我們已經學過排列 P_k^n 為 n 個相異物取 k 個，直線排列的計算方式。相似的，C_k^n 代表 n 個相異物取 k 個。)

We can develop a formula for C_k^n by comparing permutations and combinations. Consider the letters A, B, C, and D. The number of permutations of these four letters taken three at a time is $P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$.

比較排列與組合的性質，我們可以發展出 C_k^n 的公式。思考四個字母 A、B、C 與 D，從中取 3 個字母的排列數為 $P_3^4=\frac{4!}{(4-3)!}=\frac{4!}{1!}=24$。

Here are the 24 permutations:

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	DAC	DBC
BCA	BDA	DCA	DCB
CAB	DAB	CAD	CBD
CBA	DBA	CDA	CDB
This column contains only one combination, **ABC**.	This column contains only one combination, **ABD**.	This column contains only one combination, **ACD**.	This column contains only one combination, **BCD**.

We can see that every **column**[3] contains only one combination. The reason is the order of items makes no difference in determining combinations, each column of six permutations represents one combination. There is a total of four combinations:

ABC, ABD, ACD, BCD.

這是 24 種排列數，我們可以發現每一欄只有一種組合，因為組合數不在意字母排列順序滿每一欄的 6 種排列數皆只代表一種組合數，因此只有 4 種組合數。

Thus, $C_3^4=4$: The number of combinations of 4 things taken 3 at a time is 4. With 24 permutations and only four combinations, there are 6, or 3!, times as many permutations as there are combinations.

因此，$C_3^4=4$ ：4 個相異物取 3 個的組合數為 4。4 個相異字母取 3 個，有 24 種排列數但只有 4 種組合數，其排列數為 6 (3!) 倍的組合數。

$$C_3^4=\frac{P_3^4}{3!}=\frac{\frac{4!}{(4-3)!}}{3!}=\frac{4!}{1!3!}=4$$

In general, *k* objects can be chosen from *n* different objects in P_k^n ways, and *k* objects can be **arranged**[4] in *k*! ways. So, the number of combinations of *n* different objects taken *k* at a time is:

$$C_k^n=\frac{P_k^n}{k!}\ (substitute: P_k^n=\frac{n!}{(n-k)!})$$

$$=\frac{\frac{n!}{(n-k)!}}{k!}=\frac{n!}{(n-k)!k!}$$

綜合說明，從 *n* 個相異物中取 *k* 個排列有 P_k^n 種方式，而 *k* 個物品有 *k*!種排列數，所

以，從 n 個相異物中取 k 個的組合數有 $C_k^n = \dfrac{n!}{(n-k)!k!}$ 種方式。

Notes

By **convention**[5], $0!$ is defined to be 1. Thus, $C_0^n = C_n^n = 1$. We also take C_i^n to be equal to 0 when either $i < 0$ or $i > n$.

$0!$ 被定義為 1，因為 $C_0^n = C_n^n = 1$ 。無論 $i < 0$ 或是 $i > n$ ，C_i^n 都被定義為 0。

It might seem **reasonable**[6] that the number of ways to choose 0 things from n is 0 (none). However, there actually is 1 way to choose 0 out of n things.

從 n 個相異物取 0 個為 0 種方法好像很合理，因為選 0 個東西不是一種選擇。然而，應該為 1 種，以下證明：

We can define C_0^n by formula $C_k^n = \dfrac{n!}{k!(n-k)!}$.

$$C_0^n = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

Examples

A four-person **committee**[7] is to be elected from an organization's **membership**[8] of 10 students. How many different committees are possible?

(a) How many ways are there to form a 4-person committee?

(b) How many ways are there to form a 6-person committee?

欲從 10 個會員選出 4 位組成委員會。以下情況有幾種選法：

(a) 選出 4 個申請人

(b)選出 6 位申請人。

Solution

The order of chosen students does not matter. Use the combination formula. (因為沒有順序之分，所以利用組合的公式。)

(a)

$C_k^n = \dfrac{n!}{(n-k)!k!}$ Substitute: n = 10 and k = 4

$$C_4^{10} = \frac{10!}{(10-4)!4!} = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

A committee of 4 students can be chosen in 210 ways.

(b)

$$C_k^n = \frac{n!}{(n-k)!k!} \quad \text{Substitute: } n = 10 \text{ and } k = 6$$

$$C_6^{10} = \frac{10!}{(10-6)!6!} = \frac{10!}{4!6!} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$$

A committee of 6 students can be chosen in 210 ways.

In this example, the answers to parts a and b are the same. This is because the number of ways of choosing 4 students from 10 students is the same as the number of ways of choosing 6 students (that is, not choosing the other 4 students) from 10 students.

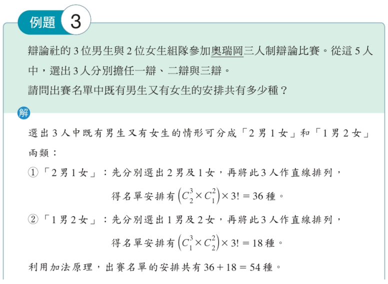
在這個例子中，題目(a)與(b)的答案一樣，這是因為選 4 位學生的方法數，與選 6 位學生(亦即不選 4 位學生)相同。

This relationship can be **expressed**[10] in **general**[11] terms. (此關係可以一般式表達。)

$$C_k^n = \frac{n!}{(n-k)!k!} \quad \text{replace } k \text{ with } n-k$$

$$C_{n-k}^n = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{(n-n+k)!(n-k)!} = \frac{n!}{k!(n-k)!} = C_k^n$$

**The number of ways of choosing *k* objects from a set of *n* objects is the same as the number of ways of not choosing *k* objects from a set of *n* objects. (從 *n* 種相異物選 *k* 種的方法數即等於不選 *k* 種的方法數。)

Material	Vocabulary
 例題 3 辯論社的 3 位男生與 2 位女生組隊參加奧瑞岡三人制辯論比賽。從這 5 人中，選出 3 人分別擔任一辯、二辯與三辯。 請問出賽名單中既有男生又有女生的安排共有多少種？ 解 選出 3 人中既有男生又有女生的情形可分成「2 男 1 女」和「1 男 2 女」兩類： ①「2 男 1 女」：先分別選出 2 男及 1 女，再將此 3 人作直線排列，得名單安排有 $(C_2^3 \times C_1^2) \times 3! = 36$ 種。 ②「1 男 2 女」：先分別選出 1 男及 2 女，再將此 3 人作直線排列，得名單安排有 $(C_1^3 \times C_2^2) \times 3! = 18$ 種。 利用加法原理，出賽名單的安排共有 $36 + 18 = 54$ 種。	12. debate (辯論), 13. elect (選出), 14. counting principle (計數原理), 15. distinguish (區分), 16. separate (分離), 17. a deck of (一副(撲克牌)), 18. A full house (葫蘆).

Translations
A **debate**[12] club consists of 3 boys and 2 girls. They need to **elect**[13] 3 members to be the first speaker, the second speaker and the third speaker of an Oregon-Style Debate. The team has to include at least one boy and one girl. Find the number of ways of selecting a team.

Solution

We have two possibilities to select 3 members with at least 1 boy and 1 girl.

First possibility: 2 boys and 1 girl.

We select 2 boys out of 3 boys and 1 girl out of 2 girls, and arrange 3 people in a row.

The number of ways to arrange the team is: $\left(C_2^3 \times C_1^2\right) \times 3! = 36$.

Second possibility: 1 boy and 2 girls.

We select 1 boy out of 3 boys and all 2 girls out of 2 girls, and arrange 3 people in a row.

The number of ways to arrange the team is: $\left(C_1^3 \times C_2^2\right) \times 3! = 18$.

By the rule of sum, the team can be selected in $36 + 18 = 54$ ways.

Illustrations

When solving problems involving **counting principles**[14], you need to **distinguish**[15] among the various counting principles to determine which is necessary to solve the problem.

1. If the order of the elements matters, then use *permutation*.
2. If the order of the elements doesn't matters, then use *combination*.
3. If the problem involves two or more **separate**[16] events, then use the *fundamental counting principle* first.

若題目須使用基本計數原理時，你要分辨不同的計數原則來決定要如何解題。

1. 若此題目為有順序的元素，則使用排列。
2. 若此題目為無順序的元素，則使用組合。
3. 若此問題包含兩個或多個束件，則一開始就要使用基本計數原理分類。

Examples

You are dealt five cards from **a** standard **deck of**[17] 52 playing cards. In how many ways can you get: (a) **A full house**[18] (b) A five-card combination containing two jacks and three aces?

(你的手牌為從一副 52 張卡牌中取出的 5 張，以下情況有幾種方法數：(a)葫蘆 (b)兩張 J 三張 A。)

Solution

A standard deck of cards has 13 ordinal cards (Ace, 2-10, Jack, Queen, King) - 1 of each ordinal in each of 4 suits (spades, clubs, hearts, diamonds), and so there are $4 \times 13 = 52$ cards.

一副標準撲克牌有 4 種花色－黑桃、梅花、愛心、方塊。每種花色有 13 張牌：A、2-10、J、Q、K。

(a) FULL HOUSE: consists of three of one kind and two of another.

Pick a card type: $C_1^{13}=13$ ways

Pick 3 out of the 4 cards: $C_3^4=4$

Pick a 2nd card type: $C_1^{13-1}=12$ ways

Pick 2 of the 4 cards: $C_2^4=6$

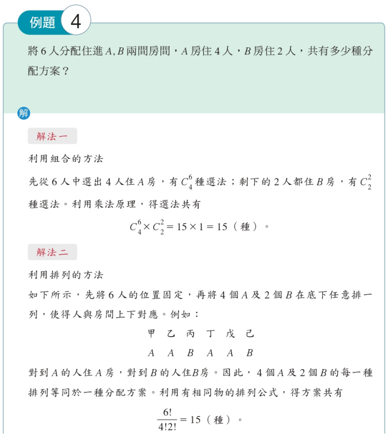
Total $13\times4\times12\times6=3744$ ways.

(b) For the number of hands we can draw getting specifically 2 jacks and 3 aces, we have

Pick 2 jacks out of the 4 cards: $C_2^4=6$

Pick 3 aces of the 4 cards: $C_3^4=4$

Total $6\times4=24$ ways.

Material	Vocabulary
	19. quad room (四人房), 20. double room (雙人房), 21. accommodate (容納), 22. correspond (對應), 23. distinguishable permutations (相同物排列), 24. division (分配), 25. league (聯盟), 26. irrelevant (無關的).

例題 4

將 6 人分配住進 A, B 兩間房間，A 房住 4 人，B 房住 2 人，共有多少種分配方案？

解

解法一

利用組合的方法

先從 6 人中選出 4 人住 A 房，有 C_4^6 種選法；剩下的 2 人都住 B 房，有 C_2^2 種選法。利用乘法原理，得選法共有

$$C_4^6\times C_2^2=15\times1=15\text{（種）。}$$

解法二

利用排列的方法

如下所示，先將 6 人的位置固定，再將 4 個 A 及 2 個 B 在底下任意排一列，使得人與房間上下對應。例如：

甲 乙 丙 丁 戊 己

A A B A A B

對到 A 的人住 A 房，對到 B 的人住 B 房。因此，4 個 A 及 2 個 B 的每一種排列等同於一種分配方案。利用有相同物的排列公式，得方案共有

$$\frac{6!}{4!2!}=15\text{（種）。}$$

Translations

Arrange People in Rooms

How many ways can 6 people be assigned to **quad room**[19] A and **double room**[20] B.

Solution 1: Combination

There are C_4^6 ways of choosing 4 people to live in room A. There are C_2^2 ways of choosing 2 people to live in room B.

By the Fundamental Counting Principle, there are $C_4^6\cdot C_2^2=15\cdot1=15$ ways can 6 people be **accommodated**[21] in 2 rooms.

Solution 2: Permutation

First, we fix the position of 6 people. Arrange 4 "A" and 2 "B" in a row beneath, so people **correspond**[22] to rooms, as figure 1 shown. For instance:

甲　乙　丙　丁　戊　己

A　A　B　A　A　B

Figure 1

The person corresponding to letter “A” stays in room A. The person corresponding to letter “B” stays in room B. Hence, every permutation represents one way to arrange these 6 people. By the ***distinguishable permutation***[23] formula, we have

$$\frac{6!}{2!4!}=15 \text{ ways.}$$

Examples

Division[24] and Distribution of distinct objects

(a) Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one **league**[25] and the B team in another. How many different divisions are possible?

(b) In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

相異物的分組分堆

(a) 10 個小朋友平分至 A、B 兩組，每組 5 人，A、B 兩組分別打不同的聯盟賽，有幾種分配的方式?

(b) 為了要打籃球賽，將他們平分兩組，有幾種分法?

Solution

(a) There are $\frac{10!}{5!5!}=252$ possible divisions.

(b) Note that this example is different from Example (a) because now the order of the two teams is **irrelevant**[26]. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is $\frac{\frac{10!}{5!5!}}{2!}=126$.

解

(a) 有 $\frac{10!}{5!5!}=252$ 可能。

(b) 注意這兩個題目不同，此題不在意組別的順序，就是不分 A、B 組，但要分成兩組。因此答案為 $\frac{\frac{10!}{5!5!}}{2!}=126$ 。

Material	Vocabulary
我們以「9 人任選 4 人」為例，說明兩個性質： (1)從 9 人中選出 4 人，相當於從 9 人中淘汰 5 人，兩者的選法數是一樣的，即 $C_4^9=C_5^9$。 (2)設甲是這 9 人中的一人。依「甲是否被選中」將「9 人任選 4 人」的方法分成以下兩類： **甲被選中**：還要從另 8 人中選出 3 人，選法有 C_3^8 種。 **甲未被選中**：須從另 8 人中選出 4 人，選法有 C_4^8 種。 根據加法原理，組合數 C_4^9 等於上述兩組合數的和，即 $C_4^9=C_3^8+C_4^8$。 一般而言，我們可以推得 **組合數的性質** (1)當 $0\le k\le n$ 時，$C_k^n=C_{n-k}^n$。 (2)當 $1\le k\le n-1$ 時，$C_k^n=C_k^{n-1}+C_{k-1}^{n-1}$。 89	27. scenario (設想), 28. subset (子集), 29.element(元素).

Translations

Taking "choose 4 people out of 9 people" as an example, to illustrate two identities.

(1) The way of choosing 4 people out of 9 people, is the same as not choosing 5 people out of 9 people. That is $C_4^9=C_5^9$.

(2) Let A be 1 of 9 people. We want to know how many ways of choosing 4 people out of 9 people. There are two **scenarios**[27]: A is chosen and A is not chosen.

A is chosen: we can select 3 people out of 8 people, C_3^8 ways.

A is not chosen: we can select 4 people out of 8 people, C_4^8 ways

By the fundamental counting principle, we know $C_4^9=C_3^8+C_4^8$.

In general, we obtained

Combinatorial identity

(1) When $0\le k\le n$, $C_k^n=C_{n-k}^n$.

(2) When $1\le k\le n-1$, $C_k^n=C_{k-1}^{n-1}+C_k^{n-1}$.

illustrations

We begin by asking a question, and answering the question in two ways: How many **subsets**[28] of size *k* are there from a set of size *n* ? "

考慮一個問題，以兩種方法回答此問題：一集合有 *n* 個元素有幾個 *k* 個元素的子集

Answer 1: There are C_k^n subsets. (*n* 個選 *k* 個子集，有 C_k^n 種方法。)

Answer 2: Pick any **element**[29] of the set. That element is either included in a subset, or it is not. (選任意一個在集合內的元素，此元素不是在子集裡就是不在子集裡。)

(i) How many subsets contain this element? We will be picking from the remaining $n-1$ elements. Since we want the subsets to have k elements, but we already have one of them, we have a total of C_{k-1}^{n-1} subsets. (若子集包含此元素，有多少個子集？我們從剩下的 $n-1$ 個元素中選取子集。因為我們想要 k 個元素的子集合，但子集內已經有此元素了，所以我們有 C_{k-1}^{n-1} 種子集合。)

(ii) How many subsets do not contain this element? We will be picking from the remaining $n-1$ elements. Since we want the subsets to have k elements, we have C_{k}^{n-1} such subsets. (若子集包含此元素，有多少個子集？我們從剩下的 $n-1$ 個元素中選取子集合。因為我們想要 k 個元素的子集合，且子集合內沒有此元素了，所以我們有 C_{k}^{n-1} 種子集合。)

Thus, there are $C_{k-1}^{n-1}+C_{k}^{n-1}$ subsets of *k* elements from a set of *n* elements.

Because each answer counted the same objects, but in two different ways, those answers must be the same. Therefore,

$$C_{k}^{n}=C_{k-1}^{n-1}+C_{k}^{n-1}.$$

因此，從 *n* 個元素的集合中有 *k* 個元素的子集合有 $C_{k-1}^{n-1}+C_{k}^{n-1}$ 種。

又因，(i)、(ii)兩種答案是用不同的方法計算，所以方法數一樣。因此 $C_{k}^{n}=C_{k-1}^{n-1}+C_{k}^{n-1}$ 。

Examples

Prove that $C_{r}^{n+m}=C_{0}^{n}C_{r}^{m}+C_{1}^{n}C_{r-1}^{m}+\cdots+C_{r}^{n}C_{0}^{m}$. (Hint: Consider a group of *n* men and *m* women.) How many groups of size *r* are possible?

證明 $C_{r}^{n+m}=C_{0}^{n}C_{r}^{m}+C_{1}^{n}C_{r-1}^{m}+\cdots+C_{r}^{n}C_{0}^{m}$ ，提示:考慮 *n* 個男生 *m* 個女生為一組，若要選 *r* 個人為一組，有多少種可能？

Solution

Consider a group of *n* men and *m* women.

The number of ways in which we can choose a group of *r* people from this group of $n+m$ people is C_{r}^{n+m} ...(1).

考慮 *n* 個男生 *m* 個女生為一組。

從此組共 $n+m$ 個人選出 *r* 個人一方法數為 C_{r}^{n+m} 。

But if we look at it differently, we can choose *k* men and $r-k$ women for every *k* for which $0\le k\le r$. For a fixed *k*, there are

$C_{k}^{n}C_{r-k}^{m}$ possible choices for *k* men and $r-k$ women.

若我們從另一個方式數，我們可以選擇 k 個男生與 $r-k$ 個女生，每個 k 皆在 $0\le k\le r$ 。對任一個 k，選出有 k 個男生與 $r-k$ 個女生 $C_k^nC_{r-k}^m$ 個方法數。

This is obtained by multiplying the number of possible choices for men C_k^n and for women C_{r-k}^m, by the fundamental counting principle.

由基本計數原理知，$C_k^nC_{r-k}^m$ 為選出 k 個男生的方法數 C_k^n 與選出 $r-k$ 個女生的方法數相乘。

So the total number of choices (for every $k=0,1,\cdots,r$) is:

$C_0^nC_r^m+C_1^nC_{r-1}^m+\cdots+C_r^nC_0^m$ ...(2).

As (1) and (2) are the solutions to the same problem, they are equal:

$C_r^{n+m}=C_0^nC_r^m+C_1^nC_{r-1}^m+\cdots+C_r^nC_0^m$.

所以所有的可能(每個 $k=0,1,\cdots,r$)為：

$C_0^nC_r^m+C_1^nC_{r-1}^m+\cdots+C_r^nC_0^m$

因為(1)式及(2)式皆為此問題的答案，所以他們答案相等：

$C_r^{n+m}=C_0^nC_r^m+C_1^nC_{r-1}^m+\cdots+C_r^nC_0^m$

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