Complex Numbers I

I. Key mathematical terms

Terms	Symbol	Chinese translation
Imaginary number		
Complex number		
Polynomial		

II. Complex numbers

The quadratic equation $ax^2+bx+c=0$ has solutions given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

This equation has discriminant $D=b^2-4ac$:

Case1	If $b^2-4ac>0$, there are two distinct real roots.
Case2	If $b^2-4ac=0$, there are two equal real roots.
Case3	If $b^2-4ac<0$, there are no real roots. (Two imaginary roots)

If the expression under the square root is negative, there are no real solutions.
For example:

Solve the equation: $x^2-2x+4=0 \Rightarrow x=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{-3}$

We got two roots $1+\sqrt{-3}$ and $1-\sqrt{-3}$.
Now consider the root $1+\sqrt{-3}$, how can we represent "$\sqrt{-3}$" in a proper way?

We are going to extend the number system to include $\sqrt{-1}$. Since there is no real number that squares to produce -1, the number $\sqrt{-1}$ is called **imaginary number** (   ), and is represented using the letter "i".

Sums of real numbers and imaginary numbers, for example:

$$1-\sqrt{-3}=1-\sqrt{3\times-1}=1-\sqrt{3}\times\sqrt{-1}=1-\sqrt{3}\times i=1-\sqrt{3}i$$

$1-\sqrt{3}i$ is known as a **complex number** (   ).

<key> The “unit” imaginary number

1. $i=\sqrt{-1},\ i^2=-1$ 2.When $a>0,\ \sqrt{-a}=\sqrt{a}i$

<key> The set of all complex numbers is written as $\mathbb{C}$.

For the complex number $z=a+bi \quad a,b\in\mathbb{R}$

Re(z)=a is the real part, Im(z)=b is the imaginary part.

If two complex numbers have the same real part and the same imaginary part, we say these two complex numbers are the same.

For example $a,b,c,d\in\mathbb{R}$, $a+bi=c+di \Leftrightarrow a=c,b=d$.

Example1

Write each of the following in terms of i.

(1) $\sqrt{-5}$ (2) $\sqrt{\frac{-2}{3}}$ (3) $\sqrt{-36}$

Example2

Suppose $a,b\in\mathbb{R}$, if $(a+2)-7i=4+(b+3)i$, find $(a,b)=?$

Complex Number Arithmetic Operation

We've learned the arithmetic of real numbers, when we extend the number system to complex numbers similar arithmetic still holds true. Here are the definition of complex addition, subtraction and multiplication: (We'll talk about the division later.)

Suppose a, b, c, d are real numbers:

(1) Addition: $(a+bi)+(c+di)=(a+c)+(b+d)i$

(2) Subtraction: $(a+bi)-(c+di)=(a-c)+(b-d)i$

(3) Multiplication: $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

<key>

When two complex numbers are multiplied by each other, the multiplication process should be similar to the multiplication of two binomials.

$$\begin{aligned}(a+bi)(c+di) &= a(c+di)+bi(c+di)\\ &= ac+adi+bci+bdi\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

Example3

Simplify each of the following, giving your answer in the form $a+bi,\ a,b\in\mathbb{R}$.

(1) $(2+5i)+(7+3i)$ (2) $(-1+3i)-(2+3i)$ (3) $3(1+2i)-(\frac{10-4i}{2})$

Properties of complex number arithmetic

Suppose z_1, z_2, z_3 are complex numbers, the following holds true:

(1) Addition and multiplication commutative: $z_1+z_2=z_2+z_1,\quad z_1z_2=z_2z_1$

(2) Addition and multiplication associative: $z_1+(z_2+z_3)=(z_1+z_2)+z_3,$
$z_1(z_2z_3)=(z_1z_2)z_3$

(3) Multiplication distributive: $z_1(z_2+z_3)=z_1z_2+z_1z_3$

(4) Addition and multiplication identity: $z_1+0=0+z_1=z_1,\quad z_1\times 1=1\times z_1=z_1$

Example4

Suppose $z_1=-3+2i, z_2=7-4i$, find the following value:

(1) z_1+z_2 (2) z_1-z_2 (3) z_1z_2

Example5

We know that $i=\sqrt{-1}, i^2=-1$, find the following value:

(1) i^n ($n=1,2,3,4,5,6,7,8$)

(2) $1+i+i^2+i^3$

<key>Power of "i": (for $k\in\mathbb{Z}$)

(1) $i^{4k+1}=$ ________ (2) $i^{4k+2}=$ ________ (3) $i^{4k+3}=$ ________ (4) $i^{4k+4}=$ ________

Division of complex numbers

The division of complex numbers makes use of the formula of reciprocal of a complex number. For the two complex numbers $z_1=a+bi, z_2=c+di$ ($ab\neq 0$), we have the division as following:

$$\frac{z_1}{z_2}=(a+bi)\times\frac{1}{(c+di)}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=(\frac{ac+bd}{c^2+d^2})+(\frac{bc-ad}{c^2+d^2})i$$

Example6

Represent the following complex numbers in the form "$a+bi$" for $a,b\in\mathbb{R}$

(1) $\frac{1}{4+3i}$ (2) $\frac{1}{5-2i}$ (3) $\frac{2-3i}{-1+2i}$

III. Solving ~~the~~ real coefficient quadratic equations

As what we've talked in part I, we can use the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ to solve the quadratic equation $ax^2+bx+c=0$. Let's try the following example:

Example7

Solve the e**quati**ons:

(1) $x^2+x-1=0$ (2) $4x^2-12x+9=0$ (3) $2x^2-4x+3=0$

Vieta's Formula ( )

Vieta's formula relates the coefficients of polynomials to the sums and products of their roots, as well as the products of the roots taken in groups.

For example, if there is a quadratic equation $x^2+2x-15=0$, it will have roots of $x=3$ and $x=-5$, because $x^2+2x-15=(x-3)(x+5)$. Vieta's formula can find the sum of the roots and the product of the roots without finding each root directly.

Vieta's Formula (Quadratic Equation)

Given a quadratic equation $ax^2+bx+c=0$, if the equation has roots α,β, then

(1) Sum of the roots: $\alpha+\beta=-\frac{b}{a}$ (2) Product of the roots: $\alpha\beta=\frac{c}{a}$

<key> You can verify the Vieta's formula by the formula mentioned above.

Example8

Suppose the quadratic equation $x^2+2x+5=0$ has roots α,β. Find the following values:

(1) $\frac{1}{\alpha}+\frac{1}{\beta}$ (2) $\alpha^2+\beta^2$ (3) $\alpha^3+\beta^3$

IV. Polynomial Equations

An equation formed by variables, exponents and coefficients is called a polynomial equation. It can have different exponents, where the higher one is called the degree of the equation. We can solve a polynomial by factoring the terms and using the zero factor principle. We'll use these equations to solve the real word problems.

For example:
We have a piece of thin metal sheet with length 20 cm, width 14 cm. How much should we cut from the corner to form a box with volume 200 cm^3?

At first, we can have the volume equation to be:

$$V(x) = x(20-2x)(14-2x) = 4x^3 - 68x^2 + 280x = 200$$

If we can solve this equation, then we can have the x we want.

$$4x^3 - 68x^2 + 280x = 200 \quad \Rightarrow \quad 4x^3 - 68x^2 + 280x - 200 = 0$$

$$(x-5)(x^2 - 12x + 10) = 0 \quad \Rightarrow \quad x = 5 \vee 6 + \sqrt{26} \vee 6 - \sqrt{26}$$

Be careful! The length of the box is $(20-2x)$, and the width of the box is $(14-2x)$. Both length and width of the box should be positive. Hence, we have the answer of x can be: $x = 5 \vee x = 6 - \sqrt{26}$.

What we've done above is called "solving the equation." When the equality holds, the corresponding x is the solution (or root) of the equation. We can use the formula and the properties of complex number to solve various kinds of equations.

Example9

We've known that the equation $2x^3 + x^2 + 8x + k = 0$ has complex root $2i$. Find the value of k.

<資料來源>

1. **Complex Numbers**
   https://www.mathsisfun.com/numbers/complex-numbers.html
   https://www.cuemath.com/numbers/complex-numbers/
   https://byjus.com/maths/complex-numbers/v
   https://brilliant.org/wiki/complex-numbers/

2. **Edexcel as and a level further mathematics core pure mathematics book 1/AS**

3. 南一書局數學甲上冊

製作者：國立臺灣師範大學附屬高級中學　蕭煜修