計數原理

Counting Methods

Material	Vocabulary
「且」與「或」 (1)當命題 P 與 Q 皆正確時，「P 且 Q」才是正確的命題，否則就是錯誤的命題。 (2)當命題 P 與 Q 至少有一為正確時，「P 或 Q」才是正確的命題，否則就是錯誤的命題。 **邏輯的笛摩根定律** (1)命題「P 且 Q」的否定命題為「$\sim P$ 或 $\sim Q$」。 (2)命題「P 或 Q」的否定命題為「$\sim P$ 且 $\sim Q$」。	1. atomic (原子的/極微的), 2. declarative (陳述性的), 3. statement (敘述), 4. logical (合乎邏輯的), 5. symbol (符號), 6. connective (連接的), 7. determine (決定).

Illustrations

Introduction to Logic

An **atomic**[1](simple) statement which cannot be broken down into smaller statements is a **declarative**[2] **statement**[3] without **logical**[4] connectives that has a truth value. A logical connective is a word or symbol that joins two atomic statements to form a larger logical statement. Here are two declarative statements that are atomic statements:

P = It is snowing.

Q = I am cold.

簡單敘述句或命題是一種沒有邏輯連接詞的直述句，能夠以一致標準判斷對錯的語句。邏輯的連接詞或符號可以將多個敘述合併成一個大的敘述句。這裡有兩個命題：P = 正在下雪；Q = 我覺得冷。

And and Or statements

Consider the atomic statement P joined with the atomic statement Q. The following sentence can be written using the **symbol**[5] “ ∨ ” for the logical connective “or”.

It is snowing or I am cold.

$$P \vee Q$$

Next, consider the following statement that uses the **connective**[6] “and”. The following sentence can be written using the symbol “ ∧ ” for the logical connective “and”.

It is snowing and I am cold.

$$P \wedge Q$$

考慮用符號“∨”“或”來連接命題 P 與 Q：「正在下雨或我覺得冷；$P \vee Q$。」

接著考慮用符號“∧”“且”來連接命題 P 與 Q：「正在下雨且我覺得冷；$P \wedge Q$。」

Empty Set (Null Set)

A set that does not contain any element is called an empty set or a null set. An empty set is denoted using the symbol " $\varnothing$ " or represented as $\{\ \}$. It is read as "phi."

不包含任何元素的集合稱為空集合，記作$\varnothing$或$\{\ \}$，讀作 phi。

Subset

If all elements of set A are present in set B then we say that set A is a subset of set B The set notation to represent a set A as a subset of set B is written as $A \subseteq B$. For example,

$$\{1,2,3\} \subseteq \{1,2,3,4,5\}.$$

Note that every set is a subset of itself and also the empty set $\varnothing$ is also a subset of every set.

當集合 A 中的每一個元素是集合 B 的元素時，稱 A 是 B 的一個子集，並用符號$A \subseteq B$（讀作 A 包含於 B）表示。例如$\{1,2,3\} \subseteq \{1,2,3,4,5\}$。且規定空集合$\varnothing$為任一集合 A 的子集。

Venn diagram

A Venn diagram is a diagram that helps us visualize the logical relationship between sets and their element. A Venn diagram typically uses intersecting and non-intersecting circles to denote the relationship between sets.

In Figure 1, we use a Venn diagram to show the relationship of $A \subseteq B$:

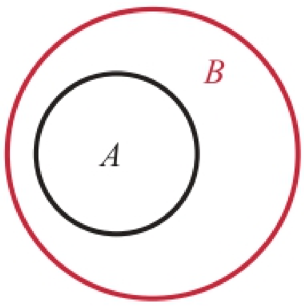


Figure 1

文氏圖使我們能直觀的表示集合關係。利用圓形是否有交集表示集合之間的關係。如圖 1，其關係為 A 包含於 B，$A \subseteq B$。

Some standard sets in math are:

Set of natural numbers, $\mathbb{N} = \{1,2,3,\cdots\}$

Set of integers, $\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$

Set of rational numbers, $\mathbb{Q}=\left\{\frac{q}{p}\middle| q \text{ is an integer and } q\neq 0\right\}$

Set of real numbers, $\mathbb{R}$

By the definition of subset, we have

$$\mathbb{N}\subseteq\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}.$$

In Figure 2, which is represented by a Venn diagram.

Figure 2

Set Intersection, Union and Difference

(a) **Intersection:** The intersection of two sets contains only the elements that are in both sets.

The intersection is notated $A\cap B=\{x|x\in A \text{ and } x\in \text{B}\}$, as shown in figure 3(a).

交集：A 與 B 共同元素組成的集合，稱為 A 與 B 的**交集**，記作 $A\cap B$，如圖 3(a)著色區域，即 $A\cap B=\{x|x\in A \text{ and } x\in \text{B}\}$。

(b) **Union:** The union of two sets contains all the elements contained in either set (or both sets).

The union is notated $A\cup B=\{x|x\in A \text{ or } x\in \text{B}\}$, as shown in figure 3(b).

聯集：A 與 B 共同元素組成的集合，稱為 A 與 B 的**聯集**，記作 $A\cup B$，如圖 3(b)著色區域，即 $A\cup B=\{x|x\in A \text{ or } x\in \text{B}\}$。

(c) **Difference:** The union of two sets contains the elements in set A but not in set B. The union is notated $A-B=\{x|x\in A \text{ and } x\notin \text{B}\}$, as shown in figure 3(c).

差集：A 與 B 共同元素組成的集合，稱為 A 對 B 的**差集**，記作 $A-B$，如圖 3(c)著色區域，即 $A-B=\{x|x\in A \text{ and } x\notin \text{B}\}$。

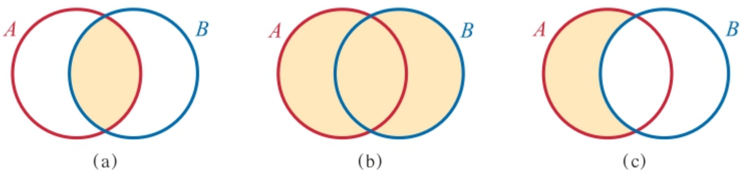


Figure 3

For instance, let $A=\{1,2,3\}$ and $B=\{2,3,4\}$, and we have

$$A\cap B=\{2,3\},\ A\cup B=\{1,2,3,4\},\ A-B=\{1\}.$$

Universal and Complement Set

(a) **Universal:** The universal set is the set of all elements or members of all related sets. It is usually denoted by the symbol U, as shown in figure 4.

宇集：當所探討的集合都是某個集合 U 的子集，稱 U 為**宇集**，如圖 4 中整個矩形區域為宇集 U。

(b) **Complement:** The complement of a set A contains everything that is not in the set A.

The complement is notated $A'=\{x|x\in U, x\notin A\}$, or A^c, as shown in Figure 4.

補集：當 A 是宇集 U 的子集時，不在 A 中的元素組成的集合，稱為 A 在 U 中的**補集**，記作 A’，如圖 4 中的綠色區域為補集 A’，即 $A'=\{x|x\in U, x\notin A\}$。

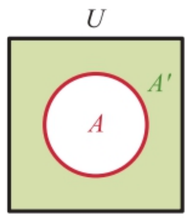


Figure 4

Examples

Which of the following are statements? If it is a statement, **determine**[7] whether it's true or false (if possible).

(A) The diameter of the earth is 1 inch or I ate a pizza. (B) $0=1$

(C) Do you have a pork barbecue sandwich? (D) Give me a cafe mocha! (E) $1+1=2$

Solution

(A) "The diameter of the earth is 1 inch or I ate a pizza." is a statement.

The first part ("The diameter of the earth is 1 inch") is false, but you would need to know something about my recent meals to know whether "I ate a pizza" is true or false.

(B) $0 = 1$ is a statement which is false (assuming that "0" and "1" refer to the real numbers 0 and 1).

(C) “Do you have a pork barbecue sandwich?” is not a statement.

(D) “Give me a cafe mocha!” is not a statement.

(E) “$1 + 1 = 2$” is a statement, and it’s true.

Material	Vocabulary
充分條件與必要條件 當命題「若 P，則 Q」是正確時，稱 P 是 Q 的充分條件，且稱 Q 是 P 的必要條件。	8. hypothesis (假設), 9. conclusion (結論), 10. conditional (有條件的), 11. converse (相反的), 12. inverse (逆), 13. contrapositive (對立的), 14. biconditional (雙條件的), 15. switch (交換), 16. interchange (互換), 17. simultaneously (同時地).

Translations

If-Then statement

A **hypothesis**[8] (if) and a **conclusion**[9] (then) are the two main parts that form a **conditional**[10] statement. Let us consider the example to understand a conditional statement.

Conditional Statement: If today is Monday, then yesterday was Sunday.

Hypothesis: If today is Monday.

Conclusion: Then yesterday was Sunday.

「若…，則…」為「若 P，則 Q」命題，其中 P 為命題的**前提**（或假設），Q 為命題的**結論**。

條件陳述：如果今天是星期一，則昨天是星期日。

前提：如果今天是星期一。

結論：則昨天是星期日。

Illustrations

Let us consider a hypothesis as statement P and a conclusion as statement Q.

Following are the observations (**converse**[11], **inverse**[12], **contrapositive**[13], **biconditional**[14]) made:

Conditional Statement	If P, then Q.	$P \Rightarrow Q$
Converse	If Q, then P.	$Q \Rightarrow P$
Inverse	If not P, then not Q.	$\sim P \Rightarrow \sim Q$
Contrapositive	If not Q, then not P.	$\sim Q \Rightarrow \sim P$
Biconditional	P if and only if Q	$P \Leftrightarrow Q$

Converse of Statement (逆命題)

When a hypothesis and a conclusion are **switched**[15] or **interchanged**[16], it is termed as a converse statement.

Converse: If yesterday was Sunday, then today is Monday.

Inverse (否命題) : If today is not Monday, then yesterday was not Sunday.

Contrapositive Statement (逆否命題)

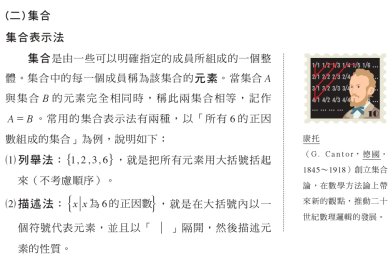
When the hypothesis and conclusion are negated and **simultaneously**[17] interchanged, then the statement is contrapositive.

Contrapositive: If yesterday was not Sunday, then today is not Monday.

Biconditional Statement (雙條件)

The statement is a biconditional statement when a statement satisfies both the conditions as true, being conditional and converse at the same time.

Biconditional: Today is Monday if and only if yesterday was Sunday.

<table>
<tr><th>Material</th><th>Vocabulary</th></tr>
<tr><td>
(二)集合
集合表示法
集合是由一些可以明確指定的成員所組成的一個整體。集合中的每一個成員稱為該集合的**元素**。當集合 A 與集合 B 的元素完全相同時，稱此兩集合相等，記作 $A=B$。常用的集合表示法有兩種，以「所有 6 的正因數組成的集合」為例，說明如下：
(1) **列舉法**：$\{1,2,3,6\}$，就是把所有元素用大括號括起來（不考慮順序）。
(2) **描述法**：$\{x \mid x 為 6 的正因數\}$，就是在大括號內以一個符號代表元素，並且以「 ｜ 」隔開，然後描述元素的性質。
康托
（G. Cantor，德國，1845～1918）創立集合論，在數學方法論上帶來新的觀點，推動二十世紀數理邏輯的發展。</td><td>18. roster form (列舉法), 19. element (元素), 20. comma (逗號), 21. curly bracket (大括號), 22. set builder form (描述法), 23. expression (表達), 24. represent (代表), 25. variable (變數), 26. vertical (垂直的).</td></tr>
</table>

illustrations

Roster Form[18]

A method of listing the **elements**[19] of a set in a row with **comma**[20] separation within **curly brackets**[21] is called roster notation. An example of the roster form of a set is given below:

ex: $\{1,2,3,6\}$.

列舉法

把所有元素用大括號括起來（不考慮順序），而元素間利用逗點隔開。

Set Builder Form[22]

Set builder form uses a statement or an **expression**[23] to **represent**[24] all the elements of a set. In this method, we will write the representative element using a **variable**[25] followed by a **vertical**[26] line and write the general property of the same representative element.

ex: $\{x \mid x \text{ are the positive factors of } 6\}$.

描述法

在大括號內以一個符號代表元素，並且以垂直線「|」隔開，然後描述元素的性質。

Material	Vocabulary
加法原理 若完成某件事的方法，依其性質可分成 k 類，且第 1 類有 m_1 種方法，第 2 類有 m_2 種方法，……，第 k 類有 m_k 種方法，則完成這件事的方法共有 $m_1+m_2+\cdots+m_k$ 種。 **乘法原理** 若完成某件事要經過 k 個步驟，且第 1 步驟中有 m_1 種方法，第 2 步驟中有 m_2 種方法，……，第 k 步驟中有 m_k 種方法，則完成這件事的方法共有 $m_1\times m_2\times\cdots\times m_k$ 種。	27. The Fundamental Counting Principle (基本計數原理), 28. disjoint (互斥), 29. procedure (程序), 30. successive (連續的), 31. stage (階段), 32. outcome (結果), 33. independent (獨立的), 34. previous (以前的), 35. composite (合成的), 36. distinct (相異的), 37. repetition (重複).

Illustrations

The Fundamental Counting Principle[27]

The Addition Principle

If there are m_1 different objects in the first set, m_2 different objects in the second set, . . . , and m_k different objects in the k^{th} set, and if the different sets are **disjoint**[28], then the number of ways to select an object from one of the m sets is $m_1+m_2+\cdots+m_k$.

加法原理

若完成某件事的方法，依其性質可分成 k 類，且第 1 類有 m_1 種方法，第 2 類有 m_2 種方法，……，第 k 類有 m_k 種方法，則完成這件事的方法共有 $m_1+m_2+\cdots+m_k$ 種。

The Multiplication Principle

Suppose a **procedure**[29] can be broken into k **successive**[30] (ordered) **stages**[31], with m_1 different **outcomes**[32] in the first stage, m_2 different outcomes in the second stage, ..., and m_k different outcomes in the k^{th} stage. If the number of outcomes at each stage is **independent**[33] of the choices in **previous**[34] stages and if the **composite**[35] outcomes are all **distinct**[36], then the total procedure has $m_1\times m_2\times\cdots\times m_k$ different composite outcomes.

乘法原理

若完成某件事要經過 k 個步驟，且第 1 步驟中有 m_1 種方法，第 2 步驟中有 m_2 種方法，……，第 k 步驟中有 m_k 種方法，則完成這件事的方法共有 $m_1 \times m_2 \times \cdots \times m_k$ 種。

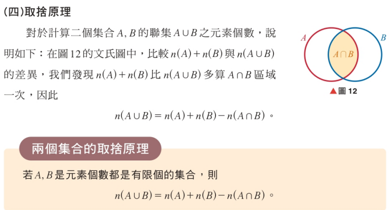
Examples

How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f **(a)** with **repetition**[37] of letters allowed? **(b)** without repetition of any letter?

Solution

(a) With repetition, we have six choices for each successive letter in the sequence. So by the multiplication principle there are $6 \times 6 \times 6 = 216$ three-letter sequences with repetition.

(b) Without repetition, there are six choices for the first letter. For the second letter, there are five choices, corresponding to the five remaining letters (whatever the first choice was). Similarly, for the third letter, there are four choices. Thus, there are $6 \times 5 \times 4 = 120$ three-letter sequences without repetition.

Material	Vocabulary
	38. inclusion–exclusion (取捨).

(四)取捨原理

對於計算二個集合 A, B 的聯集 $A \cup B$ 之元素個數，說明如下：在圖 12 的文氏圖中，比較 $n(A)+n(B)$ 與 $n(A \cup B)$ 的差異，我們發現 $n(A)+n(B)$ 比 $n(A \cup B)$ 多算 $A \cap B$ 區域一次，因此

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) 。$$

▲圖 12

兩個集合的取捨原理

若 A, B 是元素個數都是有限個的集合，則

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) 。$$

Illustrations

The Principle of Inclusion–Exclusion[38]

The subtraction rule is also known as the principle of inclusion–exclusion, especially when it is used to count the number of elements in the union of two sets. Suppose that A and B are sets. Then, there are $n(A)$ ways to select an element from A and $n(B)$ ways to select an element from B. The number of ways to select an element from A or from B, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A and the number of ways to select an element from B, minus the number of ways to select an element that is in both A and B.

Because there are $n(A \cup B)$ ways to select an element in either A or in B, and $n(A \cap B)$ ways to select an element common to both sets, we have

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

對於計算二個集合 A, B 的聯集 $A\cup B$ 之元素個數，說明如下：在文氏圖中，比較 $n(A)+n(B)$ 與 $n(A\cup B)$ 的差異，我們發現 $n(A)+n(B)$ 比 $n(A\cup B)$ 多算了 $A\cap B$ 區域一次。

Next, to count the number of $A\cup B\cup C$, we can find the differences of $n(A)+n(B)+n(C)$ and $n(A\cup B\cup C)$ in Figure 5.

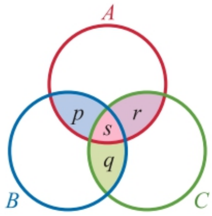


Figure 5

Let $n(A\cap B\cap C)=s$, and *p*, *q* and *r* represent the number of ways to select an element from blue, green and purple area respectively. We notice that counting the number of ways to select an element from $n(A)+n(B)+n(C)$ has more areas than counting $n(A\cup B\cup C)$. The blue, green and purple areas will be counted one more time and $A\cap B\cap C$ will be counted two more times. After subtracting the repeated areas, we have

接下來，計算三個集合 A, B, C 的聯集 $A\cup B\cup C$ 之元素個數，說明如下：在圖 5 的文氏圖中，比較 $n(A)+n(B)+n(C)$ 與 $n(A\cup B\cup C)$ 的差異。令 $n(A\cap B\cap C)=s$，並令 p、q 與 r 分別代表藍、綠、紫三塊區域的元素個數。我們發現 $n(A)+n(B)+n(C)$ 比 $n(A\cup B\cup C)$ 多算藍、綠、紫三塊區域一次及多算 $A\cap B\cap C$ 區域兩次，將重複者扣除，我們得

$$\begin{aligned}
n(A\cup B\cup C)&=n(A)+n(B)+n(C)-1\times(p+q+r)-2\times s\\
&=n(A)+n(B)+n(C)-(p+s)-(q+s)-(r+s)+s\\
&=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)
\end{aligned}$$

Examples

A computer company receives 350 applications for a position to plan a line of new web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business.

How many of these applicants majored neither in computer science nor in business?

Solution

To find the number of these applicants who majored neither in computer science nor in business, we can subtract the number of students who majored either in computer science or in business (or both) from the total number of applicants. Let A be the set of students who majored in computer science and B the set of students who majored in business.

$$n(A)=220,\ n(B)=147.$$

Then $A\cup B$ is the set of students who majored in computer science or business (or both), and $A\cap B$ is the set of students who majored both in computer science and in business.

$$n(A\cap B)=51$$

By the subtraction rule the number of students who majored either in computer science or in business (or both) equals

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)=220+147-51=316.$$

So, the number of applicants majored in neither computer science nor business equals

$$n(U)-n(A\cup B)=350-316=34.$$

References

1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。*數學 2：單元 3 計數原理*。龍騰文化。
2. Ron Larson (2018). *Precalculus with CalcChat and CalcView Tenth Edition.* Cengage Learning.
3. Brilliant. *Principle of Inclusion and Exclusion (PIE).* https://reurl.cc/V8OjXQ
4. ck-12. *16.1 And and Or Statements.* https://reurl.cc/9VekOV

製作者：臺北市立陽明高中 吳柏萱 教師