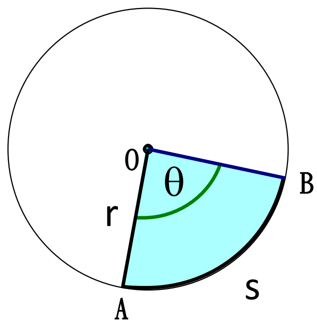
雙語教學主題(國中九年級教材): 圓-2 (圓心角、圓周角)
Topic: Introducing circles-2 (central angle, inscribed angle)
(central angle, inscribed angle, and distance from center to chord)

The teaching materials for introducing circles in our textbooks have been changed quite a bit due to the **108 syllabus.** Therefore, the content of introducing circles here is based on the official textbooks-NANI, KANG HSUAN and HANLIN.
由於 108 新綱教材大改，所以這個單元參考 108 新課綱及南一、康軒及翰林版國中數學課本第五冊

這個單元常用到的一些用語
The vocabulary we will use in this topic
Center of a circle, circumference, radius, diameter, arc, major /minor arc 優/劣弧, chord, sector, segment in a circle, semicircle, tangent, secant, central angle, inscribed angle, area, perimeter, bisector, perpendicular, perpendicular bisector 中垂線, right angle 直角, intersect 相交, external, internal, point of tangency 切點, subtend, congruent, congruence, exterior angle, interior angle, , semicircle, supplementary angle, cyclic quadrilateral,

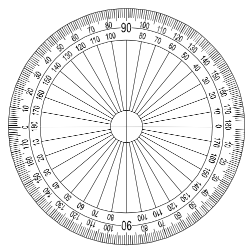
We are going to introduce some angles in a circle and some angle properties. Including central angles, inscribed angles, and cyclic quadrilaterals.

We first look at the central angles and the subtending arcs in a circle.
See the given circle O, ~~the~~ angle θ
(the pronunciation is / th-e-ta /,
我們絕大部分的人都跟著我們的老師們唸 /系他/
至少我是，哈哈)
is formed by two radii $\overline{OA}$ and $\overline{OB}$ and the vertex of the angle is at the center of the circle O. It's called the central angle. $0° \leq \theta \leq 360°$. A round angle is 360°.



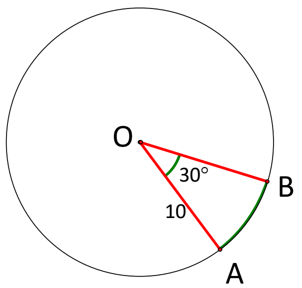
$\widehat{AB}$ is an arc subtended by the central angle θ,
$\widehat{AB}$ represents the degree of arc AB and also the length of arc AB.

We put two protractors together and form a circle.
We can see that the circle is divided into 360 even parts.
Each part of the circumference is an arc. Each arc subtends the angle at the center is $\frac{360°}{360}$=1°.



And the measure of this arc is also defined as 1°.
So the measure of an arc is the degree of its subtending central angle.

Example:

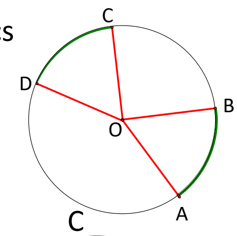


The radius of the given circle O is 10.
The central angle AOB is 30°.
Find the measure of $\widehat{AB}$ and the length of $\widehat{AB}$
Sol:
Since ∠AOB=30°., the measure of $\widehat{AB}$=30°.

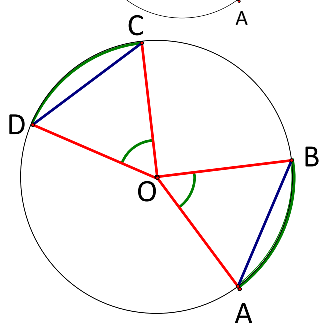
The length of $\widehat{AB}=(\frac{30}{360})2\pi\cdot 10$ (2π·10 is the length of the circumference)

$\widehat{AB}=(\frac{5}{3})\pi$

In the same circle, when $\widehat{AB}=\widehat{CD}$, it means the measure of two arcs AB and CD is the same or the length of these two arcs is equal.



We can see from the given circle O, when $\widehat{AB}=\widehat{CD}$, the corresponding chords with the same end points seem to be equal.
So the question is, when $\widehat{AB}=\widehat{CD}$, if $\overline{AB}=\overline{CD}$?



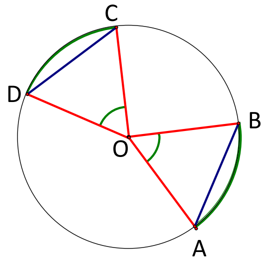
Let students think it over, it's not hard reasoning though. Give students a hint to do the reasoning by using triangle congruence if they have no idea.

To teachers

$\widehat{AB}=\widehat{CD}$ in the given circle O.

Prove that $\overline{AB}=\overline{CD}$

Pf:



In circle O, $\widehat{AB}=\widehat{CD}$

Then ∠AOB=∠COD (subtending central angles respectively)

And $\overline{OA}=\overline{OB}=\overline{OC}=\overline{OD}$ (the same radius of circle O

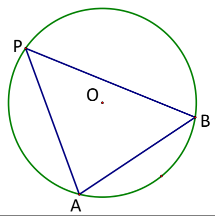
Thus ΔAOB ≅ΔCOD (SAS)

We get $\overline{AB}=\overline{CD}$ (the corresponding line segments are equal)

Therefore, we get the conclusion that in the same circle, ~~the~~ equal arcs subtend equal chords; and ~~the~~ equal chords subtend equal arcs correspondingly.

There's one more important angle in a circle we will introduce before we start to talk about the properties of angles in a circle.

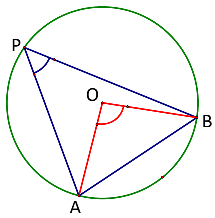
As shown in the figure, an inscribed angle ∠P is formed by two chords $\overline{PA}$ and $\overline{PB}$ in circle O and its vertex is at a point P on the circle.



We will now introduce the very important angle property in a circle.

Angle property:

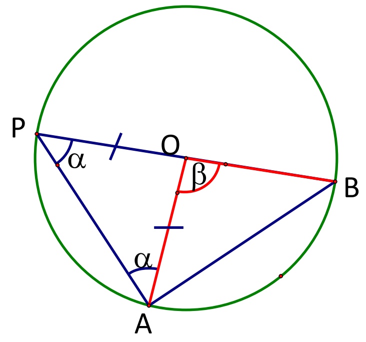
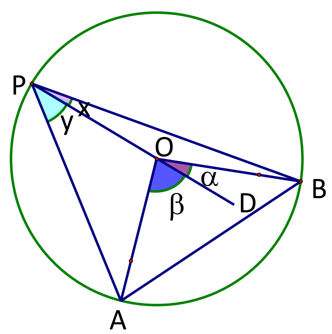
The measure of the central angle ∠AOB subtended by $\widehat{AB}$ is twice the measure of the inscribed angle ∠APB subtended by the same arc $\widehat{AB}$.



That is, $\angle APB=\frac{1}{2}\angle AOB$

We will prove the property based on three different conditions:

1. the center O is on the side $\overline{PB}$ of ΔAPB
2. the center O is inside ΔAPB
3. the center O is outside ΔAPB.

1. the center O is on the side $\overline{PB}$ In circle O shown in the figure, the central angle ∠AOB is subtended by $\widehat{AB}$,and ∠APB, the inscribed angle with its vertex P on the circumference, is also subtended by the same arc $\widehat{AB}$. Prove that $\angle APB=\frac{1}{2}\angle AOB$ Pf: Let ∠APB=α, ∠AOB=β Since $\overline{OA}$=$\overline{OP}$= the radius of circle O ΔAOP is an isosceles triangle. So ∠OAP=∠OPA=α The exterior angle ∠AOB of ΔAOP is the sum of two opposite interior angles of ΔAOP. ∠AOB=∠OAP+∠OPA β=2α Or $\alpha=\frac{1}{2}\beta$ That is $\angle APB=\frac{1}{2}\angle AOB$#	 (angle AOB equals angle OAP plus angle OPA) (angle APB is half of angle AOB)
2. the center O is inside ΔAPB In circle O shown in figure 1, the central angle ∠AOB is subtended by $\widehat{AB}$, and ∠APB, the inscribed angle with its vertex P on the circumference, is also subtended by the same arc $\widehat{AB}$. Prove that $\angle APB=\frac{1}{2}\angle AOB$ Pf: Connect $\overrightarrow{PO}$ and point D is on $\overrightarrow{PO}$ as shown in figure 1. Let ∠OPA=y, ∠OPB=x	 Figure 1

$\angle DOA=\beta$, $\angle DOB=\alpha$

Connect $\overline{OA}$ and $\overline{OB}$

$\overline{OA}=\overline{OB}=\overline{OP}$

(all are radii in circle O)

We have isosceles triangles

ΔPOA and ΔPOB

Thus $\angle OAP=\angle OPA=y$

$\angle OBP=\angle OPB=x$

In ΔPOA

The exterior angle $\angle DOA$ is the sum of two opposite interior angels

$\angle OAP$ and $\angle OPA$

That is $\beta=y+y=2y$

And $\alpha=x+x=2x$

We get

$$\begin{aligned}\angle AOB&=\alpha+\beta\\&=2x+2y\\&=2(x+y)\\&=2(\angle OPA+\angle OPB)\\&=\angle APB\end{aligned}$$

Same as $\angle APB=\frac{1}{2}\angle AOB$#

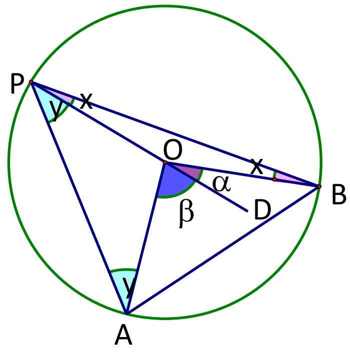


Figure 2

3. the center O is outside ΔAPB

In circle O shown in figure 1, the central angle $\angle AOB$ is subtended by $\overset{\frown}{AB}$,and $\angle APB$, the inscribed angle with its vertex P on the circumference, is also subtended by the same arc $\overset{\frown}{AB}$.

Prove that

$$\angle APB=\frac{1}{2}\angle AOB$$

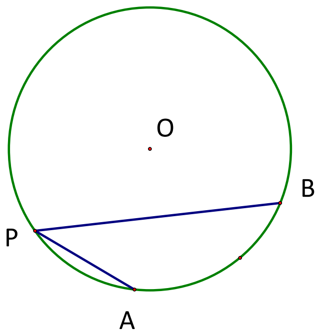


Figure 1

Pf:

In figure 2, connect $\overrightarrow{PO}$, and $\overrightarrow{PO}$ intersects circle O at point C.

Connect radii $\overline{OA}$ and $\overline{OB}$.

From the conclusion of situation 1,

$\angle APC = \frac{1}{2}\widehat{ABC} = \alpha+\beta$ and

$\angle BPC = \frac{1}{2}\widehat{BC} = \beta$ Therefore

$\angle APB = \angle APC - \angle BPC$

$= \alpha+\beta-\beta$

$= \frac{1}{2}\widehat{ABC} - \frac{1}{2}\widehat{BC}$

$= \frac{1}{2}\angle AOC - \frac{1}{2}\angle BOC$

$= \frac{1}{2}\angle AOB$#

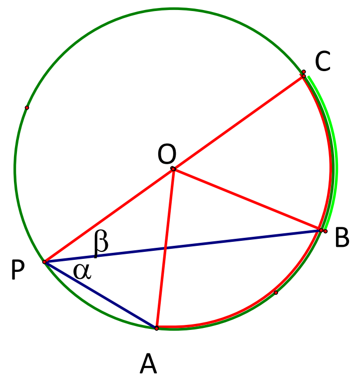


Figure 2

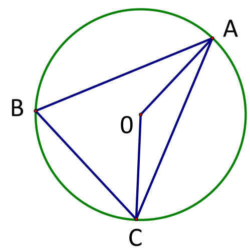
Central angles subtended by $\widehat{ABC}$ and $\widehat{BC}$ respectively.

Let's see an example.

Ex:

As the figure shown, point A, B, and C are on circle O.

$\widehat{BC}=60°$, $\angle ACB=80°$. Then $\angle AOC=?$



Sol:

We can solve this problem in two ways:

Using the measure of arcs or the measure of angles

(1) measure of arcs

Since $\angle ACB=80°$, the measure of $\widehat{AB}=160°$

The measure of the arc of a round circle is 360°,

$\widehat{AC}=360°-60°-160°=140°$, $\angle AOC=140°$#

(2) measure of angles

$\widehat{BC}=60°$, so $\angle BAC=30°$

The sum of interior angles of a triangle is 180°,

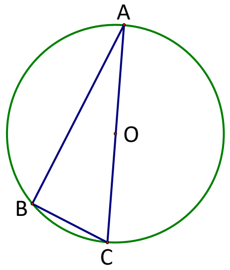
$\angle ABC=180°-30°-80°=70°$

Then

$\angle AOC=70°\text{x}2=140°$# (the central angle subtended by the same arc)

What if an inscribed angle is subtended by a semicircle?

In circle O, $\overline{AC}$ is a diameter. ∠ABC is an inscribed angle subtended by a semicircle. What is the measure of ∠ABC?



(Because the measure of semicircle $\widehat{AC}$ is 180°
From the conclusion we get from the previous discussion

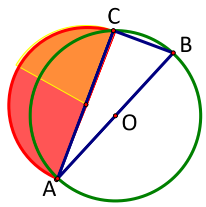
$$\angle ABC=\frac{1}{2}\widehat{AC}=\frac{1}{2}\times 180°=90°)$$

So remember this from now on that
Any inscribed angle subtended by a semicircle is a right angle.

Let's look at an example.

Ex:

As the figure shows, $\overline{AB}$ is a diameter of circle O.
∠ACB is an inscribed angle, $\overline{OB}$=5 and $\overline{BC}$=6.
The red area is a semicircle with the diameter $\overline{AC}$.
Find out the area of the red semicircle.



Sol:

$\overline{AB}=2\overline{OB}=10$

∠ABC is an inscribed angle subtended by the semicircle of circle O,

∠ABC=90°, Δ ABC is a right triangle

By the Pythagorean Theorem, we get

$$\overline{AC}^2=\overline{AB}^2-\overline{BC}^2$$

$$=10^2-6^2$$

$$=64$$

$$\overline{AC}=8$$

The radius of red semicircle is 8/2=4

The area of the red semicircle=$\frac{1}{2}\pi\cdot 4^2=8\pi_{\#}$

Next we will introduce an important topic related to inscribed angles—cyclic quadrilateral.

When a quadrilateral stays inside a circle and its four vertices are on the circle, it's a cyclic quadrilateral.

For instance, in figure 1

Quadrilateral ABCD is inside circle O, and its four vertices A,B,C, and D are on the circumference. Quadrilateral ABCD is a cyclic quadrilateral. We will discuss some important properties here.

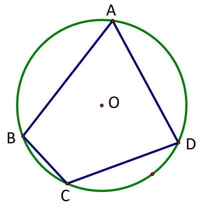


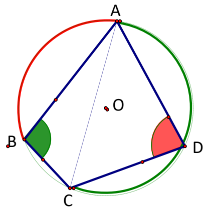
Figure 1

First, there are many inscribed angles : ∠ABC, ∠BCD, ∠CDA, and ∠BAD.

Are they related to each other? Yes!

∠ABC is subtended by $\overset{\frown}{ADC}$ and ∠ADC is subtended by $\overset{\frown}{ABC}$ as shown in figure 2.

Therefore, ∠ABC= $\frac{1}{2}\overset{\frown}{ADC}$ and ∠ADC= $\frac{1}{2}\overset{\frown}{ABC}$



$\overset{\frown}{ADC}$ and $\overset{\frown}{ABC}$ form a complete circumference

Then ∠ABC+∠ADC=$\frac{1}{2}$ ($\overset{\frown}{ADC}$+$\overset{\frown}{ABC}$)

=$\frac{1}{2}$x360°

=180°

Same reason, ∠BCD +∠BAD=180°#

So we get the conclusion

Opposite angles in a cyclic quadrilateral are supplementary to each other.

It's true in any inscribed quadrilaterals.

Let's do another example together.

Ex:

In figure 1, trapezoid ABCD is inscribed in circle O.

$\overline{AD}//\overline{BC}$, ∠ABC=110°, then

(1) ∠BAD=?

(2) ∠BCD=?

(3)Any result do you get from (1) and (2)?

(4) Is it true that $\overset{\frown}{AB}=\overset{\frown}{CD}$?

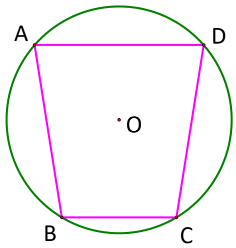


Figure 1

Sol:

(1) ∠BAD+∠ABC=180° ($\overline{AD}//\overline{BC}$, ∠BAD and ∠ABC are interior angles on the same side)

∴∠BAD=180°-∠ABC

=180°-110°

=70°

(2) ∠BAD +∠BCD =180° (trapezoid ABCD is a cyclic quadrilateral)

∠BCD =180°-70°

=110°

(3) From (1) and (2), we get

∠ABC=110°=∠BCD

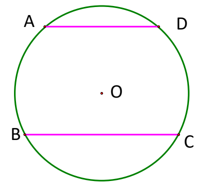
That means trapezoid ABCD is an isosceles quadrilateral, that is

$\overline{AB}=\overline{CD}$

(4)yes, $\overline{AB}=\overline{CD}$ then $\overset{\frown}{AB}=\overset{\frown}{CD}$#

Actually, we have a direct proof to the statement:

In a circle, parallel chords(or secants) intercept equal arcs.



In the circle, point A, B, C, and D are on the circumference.

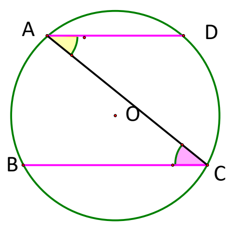
$\overline{AD}$ and $\overline{BC}$ are chords and $\overline{AD}//\overline{BC}$

Prove that $\overset{\frown}{AB}=\overset{\frown}{CD}$

Pf:

Connect $\overline{AC}$

We can apply the property of parallel lines



∠BCA=∠CAD (the alternate interior angles)

Thus, $\overset{\frown}{AB}=\overset{\frown}{CD}$ (subtending arcs)#

Let's do the last example.

Ex:

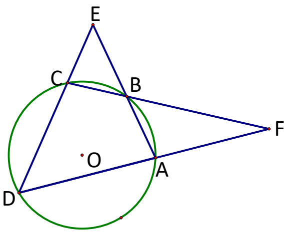
Quadrilateral ABCD is a cyclic quadrilateral in circle O. $\overline{AD}$ is a diameter. $\overrightarrow{DC}$ intersects $\overrightarrow{AB}$ at point E, and $\overrightarrow{DA}$ intersects $\overrightarrow{CB}$ at point F.

∠ADC=50°, ∠AED=60°

Please find the measure of the following angles.

(1) ∠BCE

(2) ∠AFB



Sol:

(1)in ΔADE, ∠ADC+∠AED+∠BAD=180°

∠BAD=180°-50°-60°

=70°

quadrilateral ABCD is a cyclic quadrilateral,

∠BCD+∠BAD=180°

∴∠BCD=180°-∠BAD

=180°-70°

=110°

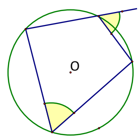
∠BCE+∠BCD=180° (they are straight angles)

∴∠BCE=70°

The measure of ∠BCE is the same as ∠BAD, we use this convenient result a lot.

That is, the measure of an exterior angle of a cyclic quadrilateral is equal to its opposite interior angle in the cyclic quadrilateral.

The proving is pretty simple, please do it yourself.



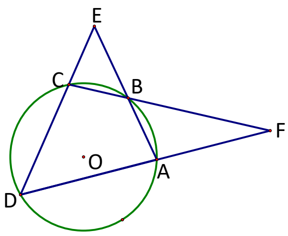
(2)there are many ways to get the measure of ∠AFB,

I share one with you, and ask students to discuss others with their classmates.

in ΔCDF, ∠CFD=180°-∠CDF-∠DCF

(the sum of interior angles in a triangle equals 180°)

∠AFB =∠CFD=180°-50°-110°=20°#



There are so many terms and properties of circles in what we introduced here. Please be patient with yourself and you'll get used to everything. You'll find it's not that hard after all.

Reference:

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教育部審定國民中學數學科南一、康軒以翰林及第五冊課本

製作者 北市 金華國中 郝曉青