Complex Numbers II

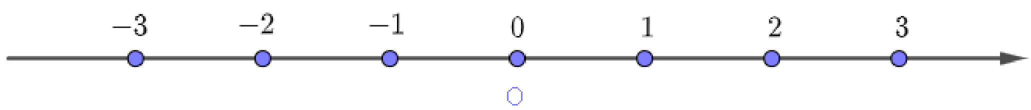
I. Key mathematical terms

Terms	Symbol	Chinese translation
Complex plane		
Real axis		
Complex axis		
Polar form (of complex plane)		

II. Complex plane

We've learned the "unit" imaginary number "i" and the concept of a complex number. The number that can be written as $a+bi$, where i is the imaginary unit and a, b are real numbers. a is called the real part of the number and b is called the imaginary part of the number.

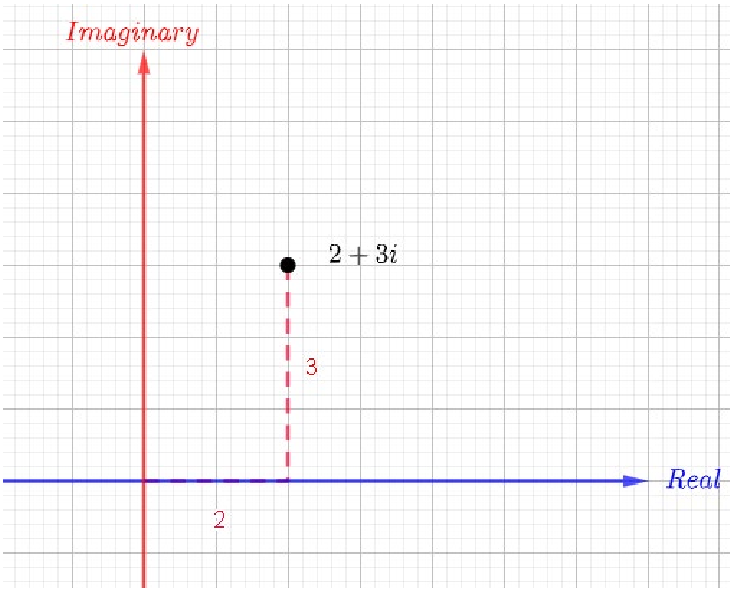
To visualize the set of complex numbers, let's recall the number line we learned in junior high school. We can find all real numbers on the number line.



Number line

How can we put a complex number like $2+3i$?
We augment the horizontal number line with another vertical number line to be the complex plane. The horizontal number line can be compared to the x-axis on the Cartesian plane, and the vertical number line can be compared to the y-axis on the Cartesian plane. Here the horizontal number line stands for the real numbers (**real axis**) and the vertical number line stands for the imaginary numbers (**imaginary axis**).

Now, we can plot the complex number like $2+3i$ on the complex plane:
2 units along the real axis and 3 units up the imaginary axis.
See figure below:

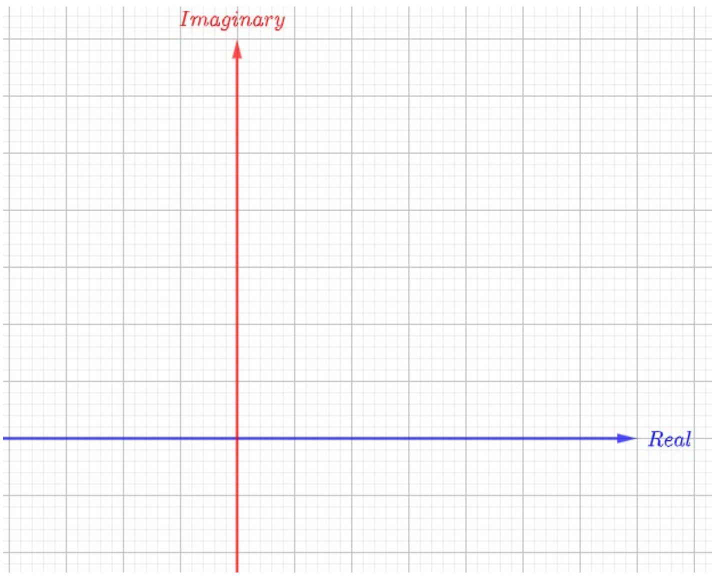


Complex plane

<key> Complex numbers can be represented on the coordinate plane by mapping the real part to the x-axis and the imaginary part to the y-axis. For example, a point P on the complex plane that represents the complex number $z=a+bi$ can be denoted as $P(z)$ or $P(a+bi)$. These are the **complex coordinates**.

Example 1

Plot the complex numbers $z=3+2i$, $\bar{z}$, $-z$, $2z$ on the following complex plane.



<key> $z = a + bi$, $\bar{z} = a - bi$ is the conjugate of the complex number z.

The absolute value of complex numbers

On the number line, the absolute value stands for "how far a number is from the origin." For example, $|x|$ represents the distance of x from the origin. The absolute value must be nonnegative. (It can be positive or zero.)

Similarly, for a complex number $z = a + bi$, the absolute value $|z|$ gives the distance from z to the **origin** in the complex plane, that is:

$$|z| = |a + bi| = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

The absolute value $|x - y|$ represents the distance between x and y on the number line. In the complex plane, $|z_1 - z_2|$ represents the distance between two complex numbers $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$. ($a_1, a_2, b_1, b_2 \in \mathbb{R}$) That is:

$$|z_1 - z_2| = |(a_1 - a_2) + (b_1 - b_2)i| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

The absolute value of complex numbers

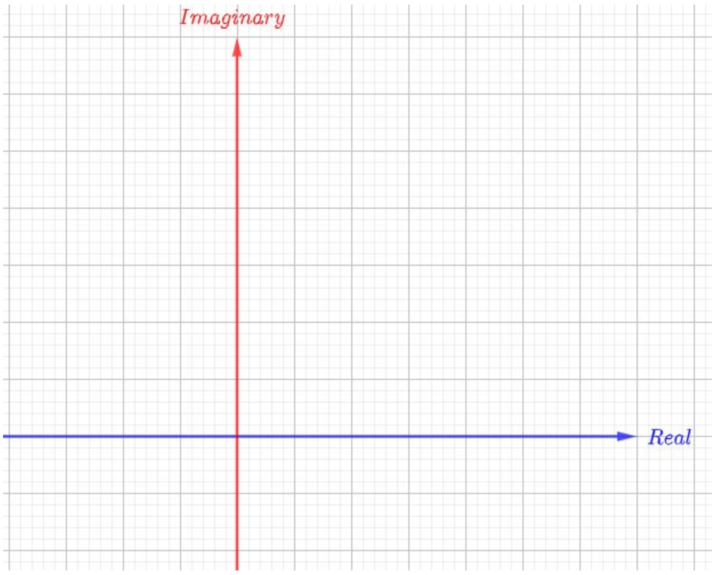
(1) Complex number $z = a + bi$, has absolute value $|z| = |a + bi| = \sqrt{a^2 + b^2}$

(2) The absolute value $|z_1 - z_2|$ represents the distance between z_1, z_2.

Example 2

Plot the complex numbers and find their absolute value:

(1) $3 + 4i$ (2) $-2i$ (3) 7 (4) $-3 - 2i$



Example 3

Interpret the geometric meaning of the following complex equations:

(1) $|z-5|=2$ (Hint: The distance from z to 5 equals 2)

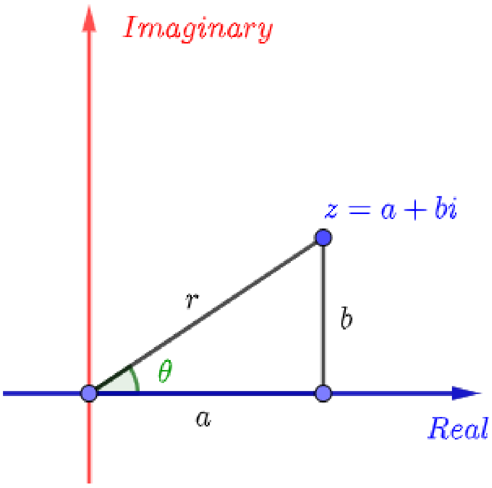
(2) $|z-5|+|z+5|=10$

(3) $|z-5|+|z+5|=12$

(4) $||z-5|-|z+5||=8$

Polar form of a complex number

The form $z=a+bi$, is called the rectangular coordinate form of a complex number. The polar form of a complex number is another way to represent a complex number. As what we mentioned in the first section, we take the horizontal axis to be the real axis and vertical axis to be the imaginary axis. We find the real and complex component in terms of r and θ, where r is the absolute value of the complex number and θ is the angle made with the real axis.



Polar form of a complex number

Nonzero complex number $z = a + bi$ can be represented by $r(\cos\theta + i\sin\theta)$. This is the polar form of the complex number.

(1) $r = |a + bi| = \sqrt{a^2 + b^2}$

(2) $a = r\cos\theta, b = r\sin\theta$

(3) θ is the argument of the complex number z. When $0 \le \theta < 2\pi$, we denote argument θ by $\theta = Arg(a + bi) = Arg(z)$.

Example 4

Find the polar form of the following complex numbers:
(Use Arg(z) to represent the angles.)

(1) $z_1 = 1 + i$

(2) $z_2 = 1 - \sqrt{3}i$

(3) $z_3 = -5i$

(4) $z_4 = 7$

Example 5

The form $z = 2(\cos 40° + i\sin 40°)$, represent $\bar{z},\ -z,\ iz$ in polar form.

The geometric meaning of complex addition and subtraction

Points on the complex plane can be related to points on the coordinate plane.

For example: Suppose $z = a + bi,\ w = c + di\ (a,b,c,d \in \mathbb{R})$

We have $z + w = (a+c) + (b+d)i,\ z - w = (a-c) + (b-d)i$

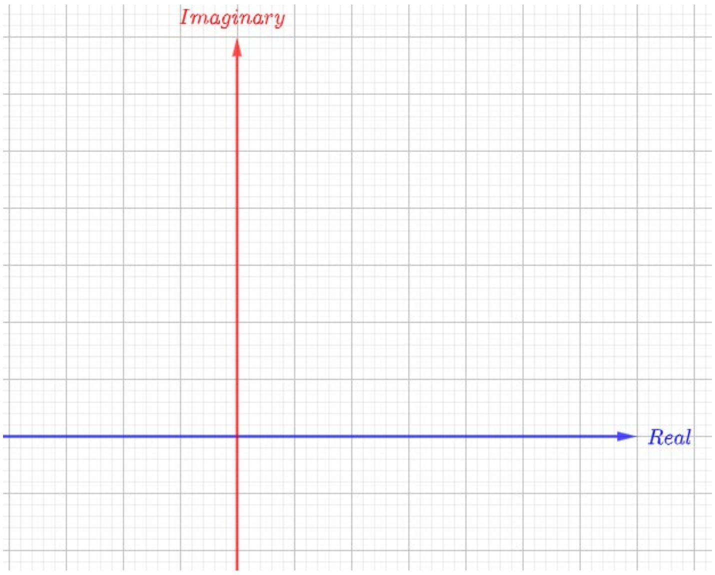
In the complex plane, suppose points P, Q, R, S have complex coordinate $z, w, z+w, z-w$. Then we have the following relationship:

Complex	Coordinate Point	Vector
$z = a + bi$	(a,b)	$\overrightarrow{OP} = (a,b)$
$w = c + di$	(c,d)	$\overrightarrow{OQ} = (c,d)$
$z + w = (a+c) + (b+d)i$	$(a+c, b+d)$	$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$ $= (a+c, b+d)$
$z - w = (a-c) + (b-d)i$	$(a-c, b-d)$	$\overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OS}$ $= (a-c, b-d)$

Example 6

A point $P(z)$ on the complex plane,

A point $P(z)$ on the complex plane and $z = 3 + 2i$. Plot the point (1) $2z$ (2) $-iz$ (3) $(2-i)z$ on the complex plane.



The geometric meaning of complex multiplication and division

As what we've done in the previous part, we can represent the complex numbers in the polar form, we have:

$$z = a + bi,\ |z| = r_1,\ z = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$w = c + di,\ |w| = r_2,\ \mathrm{w} = r_2(\cos\theta_2 + i\sin\theta_2)$$

(1) Complex multiplication:

$$zw = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

(2) Complex reciprocal:

Suppose $w \neq 0$,

$$\frac{1}{w} = \frac{1}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} = \frac{1}{r_2}(\cos\theta_2 - i\sin\theta_2) = \frac{1}{r_2}[\cos(-\theta_2) + i\sin(-\theta_2)]$$

(3) Complex division:

$$\frac{z}{w} = z(\frac{1}{w})$$

$$= [r_1(\cos\theta_1 + i\sin\theta_1)][\frac{1}{r_2(\cos\theta_2 + i\sin\theta_2)}] = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$ (by(2))

The formula of complex multiplication and division(polar form)

Complex numbers z_1, z_2 has polar form:

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

(1) multiplication: $z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

(2) division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$. $(z_2 \neq 0)$

Example 7

Find the following complex numbers:

(1) $[2(\cos 27° + i\sin 27°)][5(\cos 18° + i\sin 18°)]$

(2) $\dfrac{[4(\cos 110° + i\sin 110°)][2(\cos 40° + i\sin 40°)]}{[\cos(-20°) - i\sin(20°)][5(\cos 50° + i\sin 50°)]}$

Example 8

Suppose $z_1 = \sqrt{3} + i,\ z_2 = 6(\cos\dfrac{\pi}{12} - i\sin\dfrac{\pi}{12})$, find the polar form of the following complex numbers:

(1) $z_1 z_2$

(2) $\dfrac{z_1}{z_2}$

<key>

A complex number z is multiplied by another complex number $w = r(\cos\theta + i\sin\theta)$. This multiplication can be considered as z rotates about the origin by angle θ and dilates center at the origin by r.

The nth root of a complex number (De Moivre's Theorem)

De Moivre's theorem is an important theorem when working with complex numbers. We can use this theorem to find the powers and roots of complex numbers in polar form.

Suppose we have a complex number $z = r(\cos\theta + i\sin\theta)$, according to De Moivre's theorem we can easily raise z to the **power** of n. By the complex multiplication in polar form, we have:

(1) $z = r(\cos\theta + i\sin\theta)$

(2) $z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$

(3) $z^3 = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)$ ...

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

This means that to raise $z = r(\cos\theta + i\sin\theta)$ to the power of n, we only need:

* Raise the modulus, r, by the power of n.
* Multiply the value of θ inside the parathesis by n.

Also, we can find the **roots** of the complex numbers using De Moivre't theorem:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right).$$

This means that to find the nth root of z, we only need:

* Taking the nth root of the modulus, r.
* Divide the values of the angle by n.

Example 9

Find the value of $\left(-1 + \sqrt{3}i\right)^5$

Example 10

Find the value of $\left[\dfrac{1}{2(\cos 12° + i\sin 12°)}\right]^5$

Example 11

Find the fourth root of the following complex numbers:

(1) 1

(2) $-8+8\sqrt{3}i$

<資料來源>

1. **Complex Numbers**
   https://www.mathsisfun.com/numbers/complex-numbers.html
   https://www.cuemath.com/numbers/complex-numbers/
   https://byjus.com/maths/complex-numbers/v
   https://brilliant.org/wiki/complex-numbers/
   https://www.storyofmathematics.com/de-moivres-theorem/

2. **Edexcel as and a level further mathematics core pure mathematics book 1/AS**

3. 南一書局數學甲上冊

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