微分（一）

Differentiation I

Material	Vocabulary
導數與切線的斜率 (1) 導數的定義： 當極限 $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ 存在時，稱此極限為函數 $f(x)$ 在 $x=a$ 處的**導數**，記作 $f'(a)$，即 $f'(a)=\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$。 (2) 導數與切線的斜率： 若導數 $f'(a)$ 存在，則稱通過點 $P(a, f(a))$ 且斜率為 $f'(a)$ 的直線為函數 $f(x)$ 的圖形在 P 點的**切線**，而 P 點稱為**切點**。	1. tangent (切線), 2. perpendicular (垂直), 3. radial (徑向), 4. curve (曲線) 5. condense (濃縮), 6. approximate (近似的), 7. secant (割線), 8. the point of tangency (切點), 9. slope (斜率), 10. difference (差), 11.quotient (商), 12. denominator (分母), 13. numerator (分子), 14. interval (區間), 15. differentiate (微分), 16. derivative (導數), 17. differentiation (微分), 18. relabel (重新標記), 19. alternate (代替), 20. approach (逼近), 21. associate (關聯), 22. denote (表示), 23. notation (符號), 24. procedure (程序), 25. rationalize (合理化), 26. vertical (垂直), 27 point-slope form (點斜式).

Illustration 1

What does it mean that a line is **tangent**[1] to a curve at a point? For a circle, the tangent line at a point P is the line that **perpendicular**[2] to the **radial**[3] line at point P, as shown Figure 1.

考慮一切線切曲線於一點是什麼意思？見圖 1，以圓為例，在 P 點的切線為垂直在 P 點的半徑的那條線。

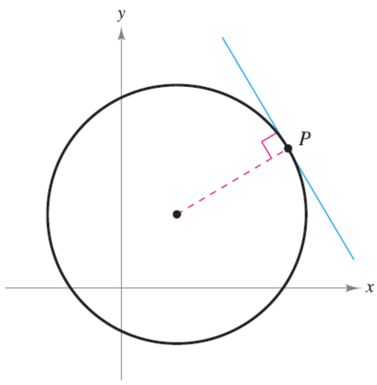


Figure 1

For a general **curve**[4], how would you define a tangent line? See Figure 2 below.

見圖 2，如果是一般的曲線，又應該如何定義呢？

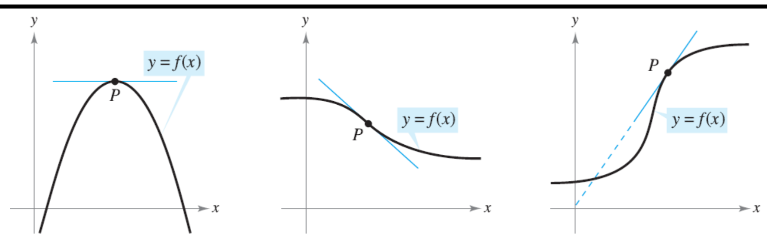


Figure 2

A line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. 若一條線與曲線剛好碰觸到 P 點，沒有穿過曲線，就稱為切線。

The problem of finding the tangent line at a point P **condenses**[5] down to the problem of finding the slope of the tangent line at point P. You can **approximate**[6] this slope using a **secant**[7] line through **the point of tangency**[8] and a second point on the curve, as shown in Figure 3.

如何找到曲線上 P 點的切線，此問題被縮小範圍成如何找到曲線上 P 點的切線斜率。我們可以用通過切點的割線與第二個點來逼近切線，如圖 3。

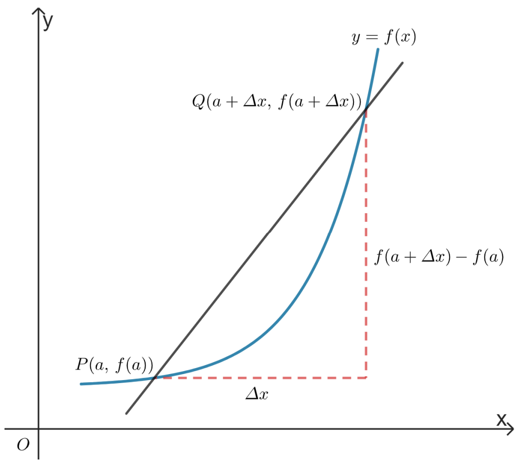


Figure 3

If $P(a, f(a))$ is the point of tangency and $Q(a+\Delta x, f(a+\Delta x))$ is a second point on the graph of f, the slope of the secant line $m_{\sec}$ through the two points using the **slope**[9] formula is:.

函數 f 上，若 $P(a, f(a))$ 是切點，$Q(a+\Delta x, f(a+\Delta x))$ 是第二個點，用計算斜率的公式代入兩點坐標，求出通過這兩個點的割線斜率。

$$m_{\sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

This equation is a **difference**[10] **quotient**[11]. The **denominator**[12] Δx is the change in x, and the **numerator**[13] $\Delta y = f(a+\Delta x) - f(a)$ is the change in y, as shown in Figure 4.

這個算式稱為**差商**。分母 Δx 為 x 的變化量，分子 $\Delta y = f(a+\Delta x) - f(a)$ 為 y 的變化量，

如圖 4。

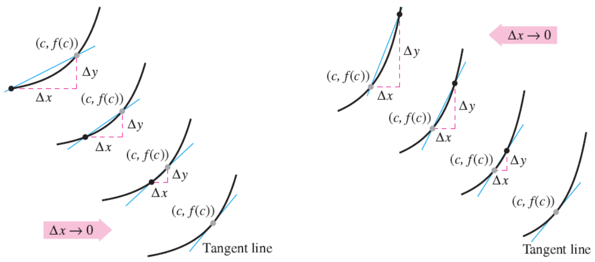


Figure 4

Definition of Tangent Line with Slope *m*

If f is defined on an open interval[14] containing a, and if the limit

$$m=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$$

exists, then the line passing through $P(a,f(a))$ with slope m is the tangent line to the graph of f at the point $P(a,f(a))$.

切線斜率 m 的定義

若函數 f 的定義域為包含 a 值的開區間，且極限值 $m=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$ 存在，則通過點 $P(a,f(a))$ 且斜率為 m 的直線即為函數 f 上通過點 $P(a,f(a))$ 的切線。

The slope of the tangent line to the graph of f at the point $P(a,f(a))$ is also called the slope of the graph of f at $x=a$.

函數 f 上在點 $P(a,f(a))$ 的切線斜率也稱為函數 f 在 $x=a$ 的斜率。

Example 1

Differentiate[15] (that is, find the derivative[16] of) $f(x)=x^3$

Solution

Applying the definition, we have

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left(x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3\right)-x^3}{\Delta x}$$ $(x+\Delta x)^3$ expanded

$$=\lim_{\Delta x\to 0}\frac{\left(3x^2+3x\Delta x+(\Delta x)^2\right)\Delta x}{\Delta x}$$ x^3 s cancelled, h factored out

$$=\lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right)$$

$$=3x^2$$

Illustration 2

The limit used to define the slope of a tangent line is also used to define one of two fundamental operations of calculus—**differentiation**[17].

極值常用於定義切線斜率，也定義微積分的基本運算之一—微分。

Definition of the Derivative of a Function

The derivative of f at x is given by

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x

導函數的定義

函數 f 在 x 的導數極限 $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$ 存在。每一個數 x 的極限均存在，稱 f' 為 x 導函數。

Definition (Alternate) Derivative at a Point

The derivative of $f(x)$ at a point where $x=a$ is found by taking the limit as $\Delta x\to 0$ of slopes of secant lines, as shown in Figure 6.

導函數在一點上的定義(替代公式)

函數 $f(x)$ 在點 $x=a$ 上的導數為當 $\Delta x\to 0$ 時割線斜率的極限值，如圖6。

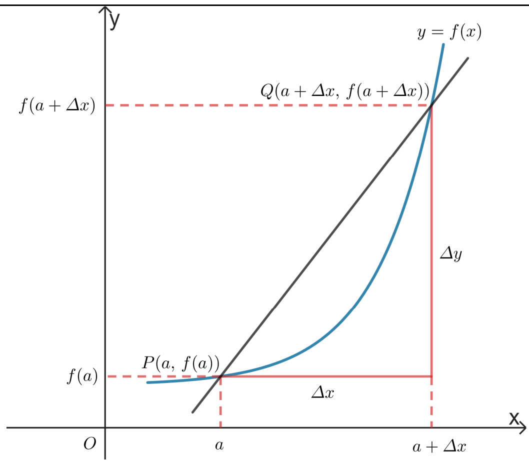


Figure 6

By **relabeling**[18] the picture as in Figure 7, we arrive at a useful **alternate**[19] formula for calculating the derivative. The limit is taken as x **approaches**[20] a .

如圖7，將其代號重新命名，我們會得到一實用的導數的替代公式。此極限為 x 逼近 a 之值。

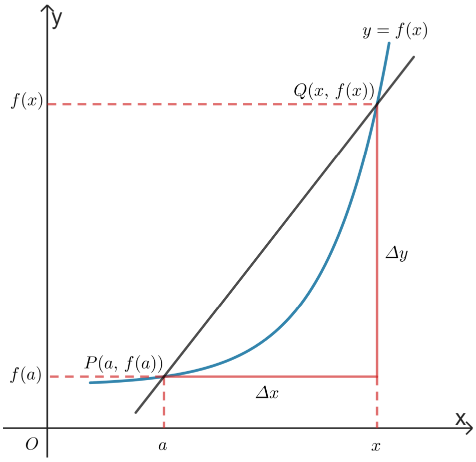


Figure 7

Definition (Alternative definition) of Derivative at A Point.

The derivative of the function f at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exits.

導函數在一點上的定義(替代公式)

若極限存在，且此極限 $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ 為函數 f 在點 $x = a$ 的導數。

Other Derivative Notations

The process of finding a derivative is called differentiation. You can think of differentiation as an operation on functions that **associates**[21] a function f' with a function f. When the independent variable is x, the differentiation operation is also commonly **denoted**[22] by

其它導數記號

尋找導數的過程稱為微分，微分是函數 f' 與函數 f 間的運算。當自變數為 x，微分運算符號也可寫成：

Notations[23]	**Readings**
$f'(x)$	f prime of x
y'	y prime
$\dfrac{dy}{dx}$	"dy dx" or "the derivative of y with respect to x"
$\dfrac{df}{dx}$	"df dx" or "the derivative of f with respect to x"
$\dfrac{d}{dx}f(x)$	"d dx of f of x" or "the derivative of f of x"
$D_x[y]$	"D x y" or "D sub x of y"

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0)=\frac{d}{dx}\left[f(x)\right]\Big|_{x=x_0},\ f'(x_0)=D_x\left[f(x)\right]\Big|_{x=x_0},\ f'(x_0)=y'(x_0),\ f'(x_0)=\frac{dy}{dx}\Big|_{x=x_0}$$

Examples 2

Using the derivative to Find the Slope at a Point

Find $f'(x)$ for $f(x)=\sqrt{x}$. Then find the slopes of the graph of f at the points $(1,1)$ and $(4,2)$. Discuss the behavior of f at $(0,0)$.

Solution

Use the **procedure**[24] by **rationalizing**[25] numerators.

$$
\begin{aligned}
f'(x) &= \lim_{x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\
&= \lim_{x \to 0} \left(\frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \right) \left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \right) \\
&= \lim_{x \to 0} \frac{(x+\Delta x) - x}{\Delta x \left(\sqrt{x+\Delta x} + \sqrt{x} \right)} \\
&= \lim_{x \to 0} \frac{\Delta x}{\Delta x \left(\sqrt{x+\Delta x} + \sqrt{x} \right)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}, x > 0
\end{aligned}
$$

Definition of derivative

At the point $(1,1)$, the slope is $f'(1) = \frac{1}{2}$. At the point $(4,2)$, the slope is $f'(4) = \frac{1}{4}$. See Figure 8. At the point $(0,0)$, the slope is undefined. Moreover, the graph of f has a **vertical**[26] tangent line at $(0,0)$.

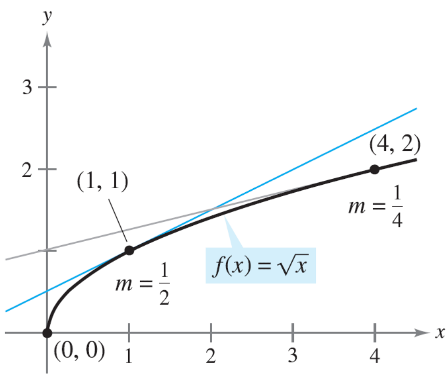


Figure 8

Use the **point-slope form**[27], we can find that the equation of the tangent line to the graph at $(1,1)$ is $y - 1 = \frac{1}{2}(x-1)$, and at $(4,2)$ is $y - 2 = \frac{1}{4}(x-4)$, as shown in Figure 8.

Material	Vocabulary
可微分一定連續 若函數 $f(x)$ 在 $x=a$ 處可微分，則 $f(x)$ 在 $x=a$ 處連續。	28. one-sided (單向), 29. differ (差異), 30. existence (存在), 31. discontinuity (不連續性), 32. observe (觀察), 33. imply (暗示), 34. converse (逆命題), 35. cusp (尖端), 36. infinite (無窮).

Illustrations 3

One-sided[28] Derivatives

The derivative of f at c is

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$ Alternative form of derivative

provided this limit exists, as Figure 9 shown.

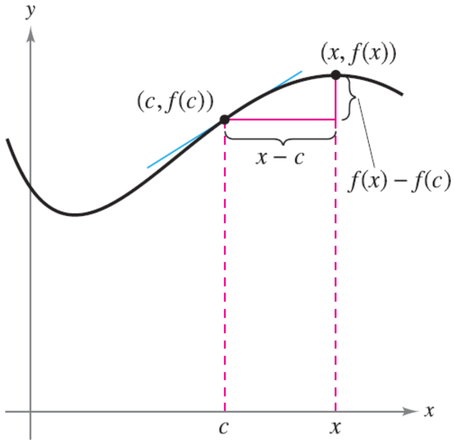
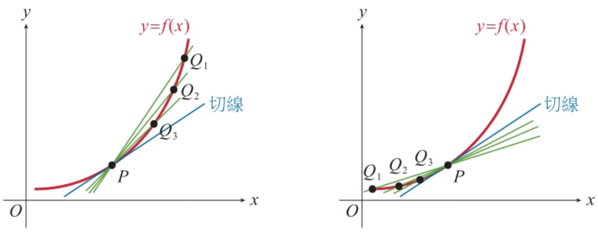


Figure 9

Note that the existence of the limit in this alternative form requires that the one-sided limits

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \text{ and } \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}$$

exist and are equal (see Figure 10). These one-sided limits are called the derivatives from the left and from the right, respectively.



(a) Q 從 P 的右側靠近 (b) Q 從 P 的左側靠近

Figure 10

Examples 3

One-sided Derivatives Can Differ[29] at a Point

Show that the following function has left-hand and right-hand derivatives at $x=0$, but no derivative there (Figure 11).

$$y=\begin{cases} x^2, & x\le 0 \\ 2x, & x>0 \end{cases}$$

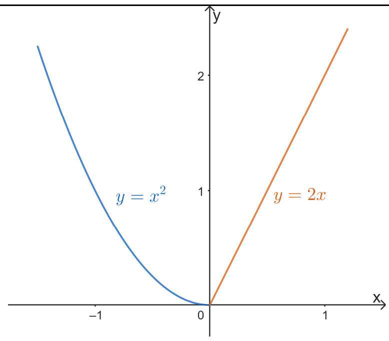


Figure 11

Solution

We verify the **existence**[30] of the left-hand derivative:

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^2-0}{x-0}=\lim_{x\to 0^-}x=0$$

We verify the existence of the right-hand derivative:

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{2x-0}{x-0}=\lim_{x\to 0^+}2=2$$

Since the left-hand derivative equals zero and the right-hand derivative equals 2, the derivatives are not equal at $x=0$. The function does not have a derivative at 0.

Illustrations 4

How $f'(a)$ Might Fail to Exist

A function will not have a derivative at a point $P(a,f(a))$ where the slopes of the secant lines,

$$\frac{f(x)-f(a)}{x-a},$$

fail to approach a limit as x approaches a.

微分 $f'(a)$ 不存在

若當 x 趨近 a ，割線斜率 $\frac{f(x)-f(a)}{x-a}$ 無法逼近極限值，則函數 f 在點 $P(a,f(a))$ 的導數不存在。

A Discontinuity[31]

If a function is not continuous at $x=c$, it is also not differentiable at $x=c$. For instance, the greatest integer function, $f(x)=[x]$,

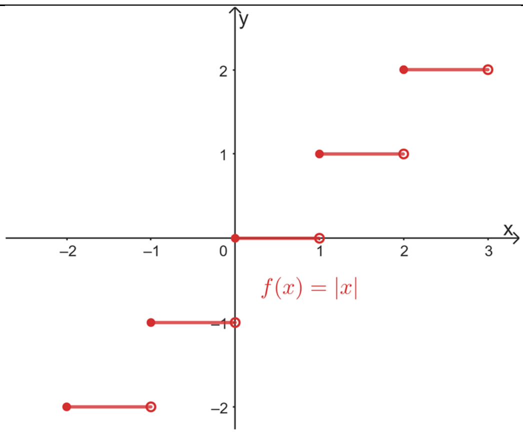


Figure 12

is not continuous at $x=0$, and so it is not differentiable at $x=0$ (See Figure 12). We verify this by **observing**[32] that

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\lfloor x\rfloor-0}{x-0}=\frac{-1}{0^-}=\infty \qquad \text{Derivative from the left}$$

and

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\lfloor x\rfloor-0}{x-0}=\frac{0}{0^+}=0. \qquad \text{Derivative from the right}$$

Although it is true that differentiability **implies**[33] continuity, the **converse**[34] is not true.

A Corner

The one-sided derivatives differ; The function: $f(x)=|x|$

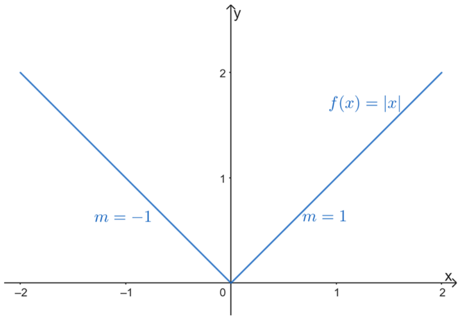


Figure 13

is continuous at $x=0$, as shown in Figure 13. However, the one-sided limits

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{|x|-0}{x-0}=\frac{-x}{x}=-1 \qquad \text{Derivative from the left}$$

and

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{|x|-0}{x-0}=\frac{x}{x}=1 \qquad \text{Derivative from the right}$$

are not equal. So, f is not differentiable at $x=0$ and the graph of f does not have a tangent line at the point $(0,0)$.

A cusp[35]

The slopes of the secant lines approach ∞ from one side and $-\infty$ from the other; The function: $f(x)=x^{\frac{2}{3}}$

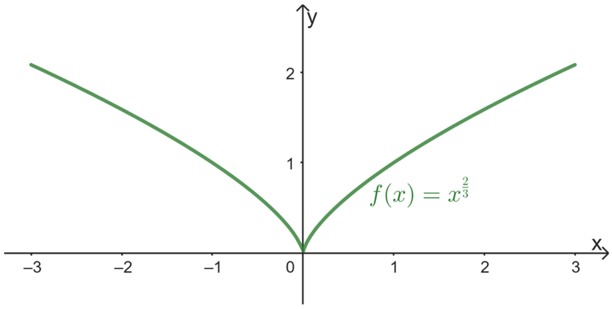


Figure 14

is continuous at $x=0$, as shown in Figure 14. However, the one-sided limits

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x^{\frac{2}{3}}-0}{x-0}=\frac{1}{x^{\frac{1}{3}}}=-\infty$$ Derivative from the left

and

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^{\frac{2}{3}}-0}{x-0}=\frac{1}{x^{\frac{1}{3}}}=\infty$$ Derivative from the right

don't exist and are not equal. So, f is not differentiable at $x=0$.

A Vertical Tangent

The slopes of the secant lines approach either ∞ or $-\infty$ both sides; The function: $f(x)=\sqrt[3]{x}$

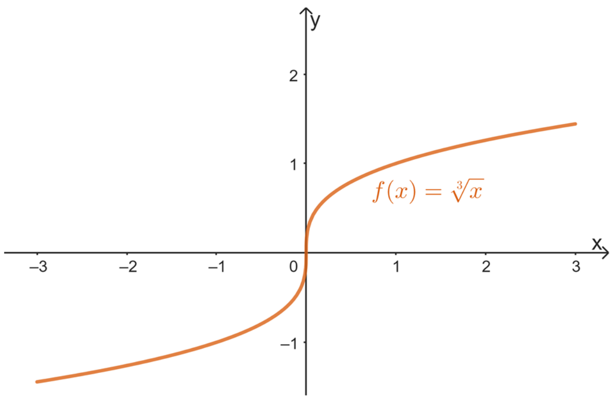


Figure 15

is continuous at $x=0$, as shown in Figure 15. However, because the limit

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{\frac{1}{3}}-0}{x-0}=\frac{1}{x^{\frac{2}{3}}}=\infty$$

is **infinite**[36], we can conclude that the tangent line is vertical at $x=0$. So, f is not differentiable at $x=0$.

From these examples, we see that a function is not differentiable at point at which its graph has a sharp turn or a vertical tangent line.

Differentiability Implies Continuity

If f is differentiable at $x=a$, then f is continuous at $x=a$.

可微分即連續

若函數 $f(x)$ 在 $x=a$ 處可微分，則 $f(x)$ 在 $x=a$ 處連續。

Proof

To prove that f is continuous at $x=a$ by showing that $f(x)$ approaches $f(a)$ as $x\to a$. To do this, use the differentiability of f at $x=a$ and consider the following limit.

$$\begin{aligned}\lim_{x\to c}\left[f(x)-f(a)\right]&=\lim_{x\to a}\left[(x-a)\frac{f(x)-f(a)}{x-a}\right]\\&=\left[\lim_{x\to a}(x-a)\right]\cdot\left[\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right]\\&=(0)\left[f'(a)\right]\\&=0\end{aligned}$$

Because the difference $f(x)-f(a)$ approaches zero as $x\to a$, you can conclude that $\lim_{x\to a}f(x)=f(a)$. So, f is continuous at $x=a$.

The converse of this Theorem is false, as we have already seen. A continuous function might have a discontinuity, a corner, a cusp, or a vertical tangent line, and hence not be differentiable at given point.

逆定理不為真，如前所述，即使函數在某一點連續，圖形有可能為不連續、轉折點、尖點或鉛直切線，不能保證在該點可微分。

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