Integration I

I. Key mathematical terms

Terms	Symbol	Chinese translation
Integration		
Riemann sum		
Definite integration		
Indefinite integration		

II. The area problem

We use the “tangent line” to illustrate the definition of differentiation.
To understand integration, we can start with an area problem:
Find the area of the region *S* that lies under the curve *y=f(x)* from *x=a* to *x=b*.
In **figure1**, the shaded region is a rectangle. We can easily find the area by multiplying the length (*f(x)*) and width (*b-a*). However, if we want to find the area in **figure2** it is not that easy. To find this area, we should chop the function f(x) into small pieces of sections. We can approximate the region S by using the areas of these rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles. The following example illustrates this procedure:

Straight line	Curve line

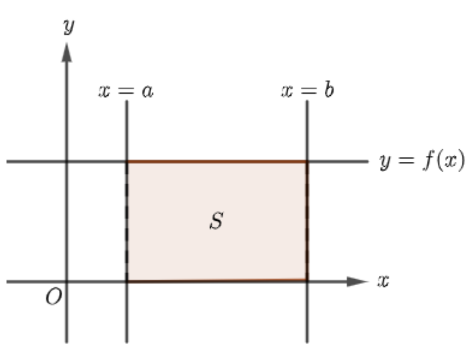
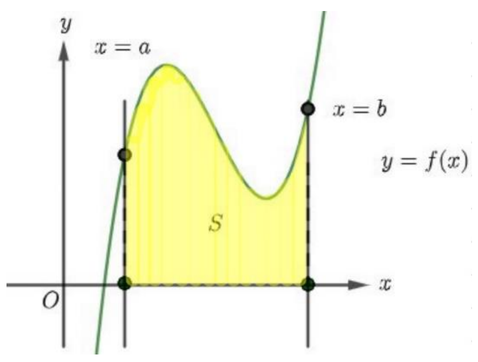
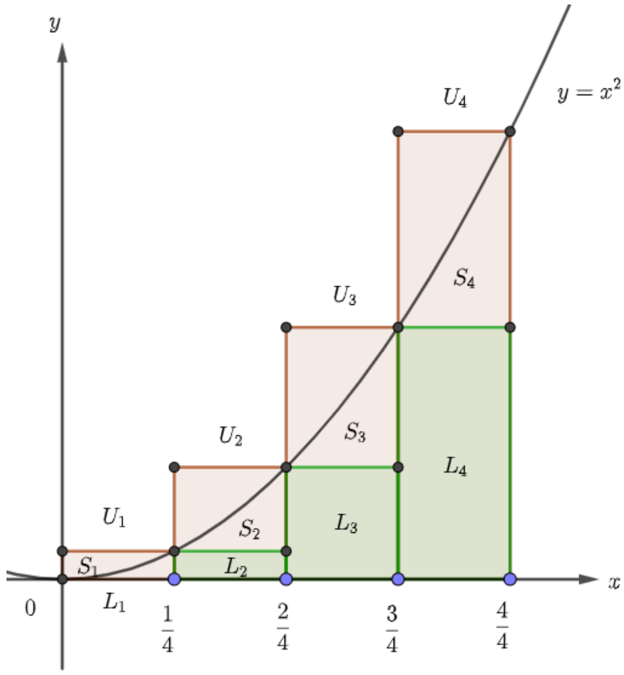

	

Figure1	Figure2

Example1

Use four rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.



<illustration>

To approximate the area of a region, begin by subdividing the interval [0,1] into 4 subintervals. (We can also divide the interval into n subintervals to get a more precise value.) The four subintervals are:

$\left[0,\frac{1}{4}\right]$, $\left[\frac{1}{4},\frac{2}{4}\right]$, $\left[\frac{2}{4},\frac{3}{4}\right]$, $\left[\frac{3}{4},\frac{4}{4}\right]$ (Each subinterval has width $\frac{1-0}{4}$)

The function $y = x^2$ is continuous. We can find the minimum and maximum value of $f(x)$ in each subinterval. That is:

$f(m_i) =$ the minimum value of $f(x)$ in ith subinterval

$f(M_i) =$ the Maximum value of $f(x)$ in ith subinterval

Then we define an **inscribed rectangle** (rectangle L_1, L_2, L_3, L_4) lying inside the ith subregion and a **circumscribed rectangle** (rectangle U_1, U_2, U_3, U_4) extending outside the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of ith circumscribed rectangle is $f(M_i)$. For each i, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle. We have the following relation among the inscribed rectangle, circumscribed rectangle and actual area (S_1, S_2, S_3, S_4) of each section:

Area of inscribed rectangle	$= f(m_i)\times\frac{1}{4}\le$	Actual area	$\le f(M_i)\times\frac{1}{4}=$	Area of circumscribed rectangle

The sum of the areas of the inscribed rectangles is called the **<u>lower sum</u>**, and the sum of the areas of the circumscribed rectangles is called the **<u>upper sum</u>**.

Now we can find the lower sum and upper sum of this question:

Lower sum:

$$\frac{1}{4}f(0)+\frac{1}{4}f(\frac{1}{4})+\frac{1}{4}f(\frac{2}{4})+\frac{1}{4}f(\frac{3}{4})=\frac{1}{4}\left[f(0)+f(\frac{1}{4})+f(\frac{2}{4})+f(\frac{3}{4})\right]=0.21875$$

Upper sum:

$$\frac{1}{4}f(\frac{1}{4})+\frac{1}{4}f(\frac{2}{4})+\frac{1}{4}f(\frac{3}{4})+\frac{1}{4}f(\frac{4}{4})=\frac{1}{4}\left[f(\frac{1}{4})+f(\frac{2}{4})+f(\frac{3}{4})+f(\frac{4}{4})\right]=0.46875$$

<Extension>

Now we divide the interval [0,1] into n subintervals.

Each interval has width $\frac{1}{n}$.

The endpoints of the intervals are $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}$.

We can represent the lower sum and upper sum by n.

Lower sum:

$$L=L_1+L_2+\ldots+L_n=\sum_{i=1}^{n}f(\frac{i-1}{n})\frac{1}{n}=\frac{1}{n}\sum_{i=1}^{n}(\frac{i-1}{n})^2=\frac{1}{n^3}[0^2+1^2+\ldots+(n-1)^2]=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$

Upper sum:

$$U=U_1+U_2+\ldots+U_n=\sum_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}=\frac{1}{n}\sum_{i=1}^{n}(\frac{i}{n})^2=\frac{1}{n^3}(1^2+2^2\ldots+n^2)=\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$$

Hence, we have:

$$L=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\le \text{ Actual area } \le\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}=U$$

If we take the limit of n to infinity, by the squeezing theorem we can have the actual area equals $\frac{1}{3}$.

By the illustration and extension above, we have the following conclusion:

A plane region bounded above by the graph of nonnegative, continuous function $y = f(x)$. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$. To approximate the area of the region, we divide the interval $[a,b]$ into n pieces with width $\frac{b-a}{n}$.

We can find the minimum ($f(m_i)$) and maximum ($f(M_i)$) value of $f(x)$ in each subinterval to get the inscribed and circumscribed rectangles:

Area of inscribed rectangle	$= f(m_i) \times \frac{b-a}{n} \leq$	Actual area	$\leq f(M_i) \times \frac{b-a}{n} =$	Area of circumscribed rectangle

Finally, we take lower sum = $s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$, upper sum = $S(n) = \sum_{i=1}^{n} f(M_i)\Delta x$.

$s(n) \leq$ Area of region $\leq S(n)$

Take the limit n tends to the infinity and pinching theorem we can have the actual area.

Example2

For the general case, chop the interval [0,1] into n pieces in Example1 and answer the following questions:

(1) Use n to represent the lower sum ($s(n)$) for the region.

(2) Use n to represent the Upper sum ($S(n)$) for the region.

(3) Take the limits of $s(n)$ and $S(n)$ to find the real area of the bounded region.

Limit of the lower and upper sum

Let f be continuous and nonnegative on the interval $[a,b]$. The limits as n tends to infinity of both the lower and upper sum exist and are equal to each other. That is

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x = \lim_{n\to\infty} S(n)$$

Where $\Delta x = \dfrac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Area of a region in the plane

Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the x-axis and the vertical lines $x=a$ and $x=b$ is:

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x \quad , x_{i-1} \le c_i \le x_i$$

Where $\Delta x = \dfrac{b-a}{n}$

Example3

Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x=0$ and $x=1$.

Example4

Find the area of the region bounded by the graph $f(x) = 9 - x^2$, the x-axis, and the vertical lines $x=1$ and $x=2$.

III. Riemann sums and definite integrals

With the preceding section, we use the limit of a sum to define the area of a region in the plane. This is one of the many applications of finding area. In this section we'll introduce the relation between the Riemann sums and definite integrals. Let's see the following definition. (In this section, the function f has no restrictions, while the function f in the preceding section was assumed to be continuous and nonnegative.)

Riemann sums

Let f be defined on the closed interval $[a,b]$, and let Δ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

Where Δx_i is the width of the ith subinterval. If c_i is any point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

Is called the Riemann sum of f for the partition Δ.

<key>

Please check the definition of "area of a region in the plane" with "Riemann sums" carefully.

To define the definite integral, consider the following limit:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \quad \text{(The limit of Riemann sums)}$$

If the limit exists, we call this is the definite integral of $f(x)$.

Definite Integral

If f is defined on the closed interval $[a,b]$ and the limit $\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists.

We say f is integrable on $[a,b]$ and the limit is denoted by:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

The limit is called the **definite integral of f from a to b**. The number a is the lower limit of integration, and the number b is the upper limit of integration.

Example5

Evaluate the definite integral $\int_{-1}^{3} 5x dx$ as a limit.

The properties of definite integral

Suppose $f(x),\ g(x)$ are continuous functions defined on the interval $[a,b]$, then the following properties hold true:

(1) $\int_a^b k dx = k(b-a)$, k is a constant.

(2) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$

(3) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, k is a constant.

(4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for $a < c < b$

Definite Integral as the area of a region

If f is continuous and nonnegative on the closed interval $[a,b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines x=a and x=b is given by:

$$\text{Area}=\int_a^b f(x)dx$$

Example6

Use the properties of definite integral to find the following values:

(1) $\int_1^2 (1+x^2)dx$ (2) $\int_2^5 x^4 dx$ (3) $\int_1^3 (x^3+x^2+x+1)dx$

Example7

Suppose $f(x)$ is a continuous function. We know that $\int_1^3 f(x)dx = 5$ and

$\int_1^8 f(x)dx = -2$, find the value of $\int_3^8 f(x)dx$.

Example8

$f(x) = |x-1|$, use the relation between definite integral and area, find the value of

$\int_{-2}^5 f(x)dx$.

IV. Fundamental theorem of calculus

So far, we've learned two major branches of calculus: differential calculus and integral calculus. Actually, these two problems have a very close connection. The relation between these two problems was discovered by Isaac Newton and Gottfried Leibniz and is stated in a theorem, the "Fundamental theorem of calculus".

This theorem states that differentiation and integration are inverse operations, just like the inverse relation between division and multiplication. Let's see the theorem below:

The fundamental theorem of calculus (I)

$f(x)$ is continuous on $[a,b]$. Suppose $g(x) = \int_a^x f(t)dt,\ a \le x \le b,$ then $g(x)$ is differentiable on (a,b) and $g(x)$ has derivative $f(x)$. i.e. $g'(x) = f(x)$

<key> You can check the proof of this theorem from the following video: https://youtu.be/pWtt0AvU0KA

Example9

Suppose $g(x)=\int_{-1}^{x}(t^3-t+4)dt$, find $g'(x)=?$

Definition of antiderivative

(1) If $F'(x)=f(x)$, then we say $F(x)$ is an antiderivative of $f(x)$

(2) Suppose $f(x)$ is a continuous function, then the **indefinite integral** of $f(x)$ is all the possible antiderivative of it.

<key> If function $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then $F(x)=G(x)+C$, C is a constant.

Example10

(1) Find an antiderivative of function x^3-2x+1

(2) Find the integral: $\int(x^3-2x+1)dx$

The fundamental theorem of calculus (II)

$f(x)$ is continuous on $[a,b]$ and $F'(x)=f(x)$, i.e. $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x)dx=F(b)-F(a)$.

<key> You can check the proof of this theorem from the following video:

https://youtu.be/Cz_GWNdf_68

Example11

Find the following definite integral: (Use the fundamental theorem of calculus(II))

(1) $\int_{-2}^{5}3x^2dx$ (2) $\int_{5}^{2}(2x^3-5x+1)dx$ (3) $\int_{2}^{7}(2x+1)^2dx$

<資料來源>

1. **Integration**
   Calculus-9th-Edition-by-Ron-Larson
   Calculus 9/e Metric Version-by-James-Jtewart
   Edexcel as and a level further mathematics core pure mathematics book 1/AS
   https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-integration-optional/v/proof-of-fundamental-theorem-of-calculus

2. 南一書局數學甲下冊

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