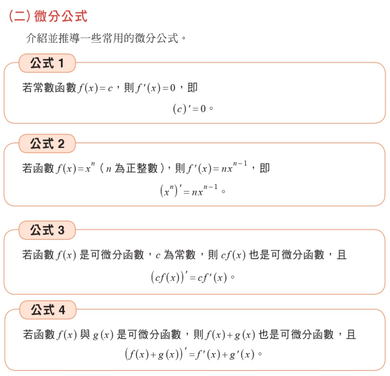
微分（二）

Differentiation II

Material	Vocabulary
(see below)	1. constant (常數), 2. derivative (導數), 3. evaluate (評估), 4. point-slope form (點斜式).



(二) 微分公式

介紹並推導一些常用的微分公式。

公式 1

若常數函數 $f(x)=c$，則 $f'(x)=0$，即

$$(c)'=0\text{。}$$

公式 2

若函數 $f(x)=x^n$（n 為正整數），則 $f'(x)=nx^{n-1}$，即

$$(x^n)'=nx^{n-1}\text{。}$$

公式 3

若函數 $f(x)$ 是可微分函數，c 為常數，則 $cf(x)$ 也是可微分函數，且

$$(cf(x))'=cf'(x)\text{。}$$

公式 4

若函數 $f(x)$ 與 $g(x)$ 是可微分函數，則 $f(x)+g(x)$ 也是可微分函數，且

$$(f(x)+g(x))'=f'(x)+g'(x)\text{。}$$

Illustrations I

The constant[1] rule

The derivative[2] of a constant function is 0. That is, if c is a real number, then

$$(c)'=0.$$

若常數函數 $f(x)=c$，則 $f'(x)=0$，即 $(c)'=0$ 。

The Power Rule

If n is a natural number, then the function $f(x)=x^n$ is differentiable and

$$(x^n)'=nx^{n-1}$$

若函數 $f(x)=x^n$ （ n 為正整數），則 $f'(x)=nx^{n-1}$ ，即 $(x^n)'=nx^{n-1}$ 。

The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$(cf(x))'=cf'(x).$$

若函數 $f(x)$ 是可微分函數， c 為常數，則 $cf(x)$ 也是可微分函數，且 $(cf(x))'=cf'(x)$ 。

The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g,

$$\left(f(x)+g(x)\right)' = f'(x)+g'(x)$$

$$\left(f(x)-g(x)\right)' = f'(x)-g'(x).$$

若函數 $f(x)$ 與 $g(x)$ 是可微分函數，則 $f(x)+g(x)$ （或 $f(x)-g(x)$ ）也是可微分函數，

且 $\left(f(x)+g(x)\right)' = f'(x)+g'(x)$ 、 $\left(f(x)-g(x)\right)' = f'(x)-g'(x)$ 。

Examples I

Using the Rules

Find the derivative of the following functions

(a) $f(x)=\sqrt[3]{x}$ (b) $g(x)=\dfrac{1}{x^2}$ (c) $h(x)=\dfrac{1}{2\sqrt[3]{x^2}}$ (d) $l(x)=2x^3-3x^2-5x+4$

Solution

(a) $f'(x)=\dfrac{d}{dx}\left[x^{\frac{1}{3}}\right]=\dfrac{1}{3}x^{\frac{-2}{3}}=\dfrac{1}{3x^{\frac{2}{3}}}$

(b) $g'(x)=\dfrac{d}{dx}\left[x^{-2}\right]=(-2)x^{-3}=-\dfrac{2}{x^3}$

(c) $h'(x)=\dfrac{d}{dx}\left[\dfrac{1}{2}x^{-\frac{2}{3}}\right]=\dfrac{1}{2}\left(-\dfrac{2}{3}\right)x^{-\frac{5}{3}}=-\dfrac{1}{3x^{\frac{5}{3}}}$

(d) $l'(x)=2\cdot 3x^2-3\cdot 2x-5=6x^2-6x-5$

Examples II

Finding an Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x)=x^3+1$ when $x=1$

Solution

To find the point on the graph of f, **evaluate**[3] the original function at $x=1$.

$(1,f(1))=(1,2)$ Point on graph.

To find the slope of the graph when $x=1$, evaluate the derivative, $f'(x)=3x^2$, at $x=1$.

$m=f'(1)=3$ Slope of graph at $(1,2)$.

Now, using the **point-slope form**[4] of the equation of a line, you can write

$y-2=3(x-1)$ Point-slope form. Substitute for point and slope.

$\Rightarrow y=3x-1$ Simplify.

See Figure 1.

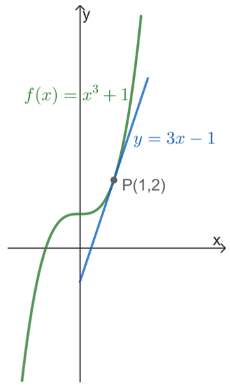
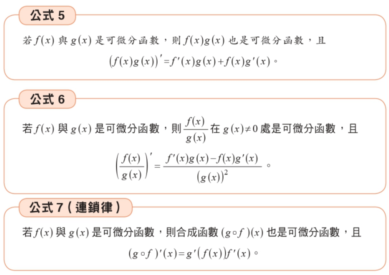


Figure 1

Material	Vocabulary
	6. denominator (分母), 7. numerator (分子), 8. the composition of (構成/合成), 9. equivalently (同等地).

公式 5

若 $f(x)$ 與 $g(x)$ 是可微分函數，則 $f(x)g(x)$ 也是可微分函數，且

$$\left(f(x)g(x)\right)' = f'(x)g(x) + f(x)g'(x)。$$

公式 6

若 $f(x)$ 與 $g(x)$ 是可微分函數，則 $\frac{f(x)}{g(x)}$ 在 $g(x) \neq 0$ 處是可微分函數，且

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}。$$

公式 7（連鎖律）

若 $f(x)$ 與 $g(x)$ 是可微分函數，則合成函數 $(g \circ f)(x)$ 也是可微分函數，且

$$(g \circ f)'(x) = g'\left(f(x)\right)f'(x)。$$

Illustrations II

The Product Rule

The product of two **differentiable**[5] functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

若 $f(x)$ 與 $g(x)$ 是可微分函數，則 $f(x)g(x)$ 也是可微分函數，且

$$\left(f(x)g(x)\right)' = f(x)g'(x) + f'(x)g(x) 。$$

The Quotient Rule

The quotient $\frac{f}{g}$ of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of $\frac{f}{g}$ is given by the **denominator**[6] times the derivative of the numerator minus the **numerator**[7] times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2},\ g(x)\neq 0$$

若 $f(x)$ 與 $g(x)$ 是可微分函數，則 $\dfrac{f(x)}{g(x)}$ 在 $g(x)\neq 0$ 處是可微分函數，且

$$\left(\frac{f(x)}{g(x)}\right)'=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2},\ g(x)\neq 0\text{ 。}$$

The Chain Rule

If $f(x)$ is a differentiable function of $g(x)$ and $g(x)$ is a differentiable function of x, then $g\circ f(x)$ is a differentiable function of x and

$$(g\circ f)'(x)=\frac{d}{dx}\left[g\circ f(x)\right]=f'(g(x))g'(x)$$

若 $f(x)$ 與 $g(x)$ 是可微分函數，則合成函數 $(g\circ f)(x)$ 也是可微分函數，且

$$(g\circ f)'(x)=\frac{d}{dx}\left[g\circ f(x)\right]=f'(g(x))g'(x)\text{ 。}$$

Note:

(1) $(g\circ f)(x)$ is read as g of f of x.

(2) $\dfrac{d}{dx}f(x)$ can be read as d dx of f of x.

Examples III

Using the Product Rule

Find the derivative of $h(x)=(x^2-1)(x^2+x-3)$

Solution

$$h'(x)=\overbrace{(x^2-1)}^{\text{First}}\overbrace{\frac{d}{dx}\left[x^2+x-3\right]}^{\text{Derivative of second}}+\overbrace{\frac{d}{dx}\left[x^2-1\right]}^{\text{Derivative of first}}\overbrace{(x^2+x-3)}^{\text{Second}}$$

Apply Product Rule

$$=(x^2-1)(2x+1)+(2x)(x^2+x-3)$$

$$=4x^3+3x^2-8x-1$$

Applying the General Power Rule

Find the derivative of $h(x)=(x^2+x+1)^3$

Solution

Let $f(x)=x^2+x+1$, $g(x)=x^3$. Then we can see $h(x)$ is **the composition of** [8] $f(x)$ and $g(x)$, or, **equivalently**[9] $h(x)=(g\circ f)(x)$.

By the General Power Rule, the derivative is

$$h'(x)=3\left(x^2+x+1\right)^2\frac{d}{dx}\left[x^2+x+1\right] \qquad \text{Apply General Power Rule}$$

$$=3\left(x^2+x+1\right)^2(2x+1) \qquad \text{Differentiate } x^2+x+1.$$

Material	Vocabulary
丙 高階導函數 對可微分函數 $f(x)$ 而言，求出導函數 $f'(x)$ 後，若可以繼續再求 $f'(x)$ 的導函數，則將「$f'(x)$」的導函數稱為 $f(x)$ 的**二階導函數**，記作 $f''(x)$。仿此，「$f''(x)$」的導函數稱為 $f(x)$ 的**三階導函數**，記作 $f'''(x)$；而四階以上的導函數，記作 $f^{(4)}(x), f^{(5)}(x), \cdots$。	10. relative (相對的), 11. average (平均), 12. Velocity (速度), 13. straight (直的), 14. neglect (忽視), 15. resistance (反抗), 16. influence (影響), 17. gravity (重力), 18. initial (初始), 19. acceleration (加速度), 20. approximately (估計地).

Illustrations III

Rate of Change

The function $s(t)$ that gives the position (**relative**[10] to the origin) of an object as a function of time t is called a position function. If over a period of time Δt, the object changes its position by the amount $\Delta s=s(t+\Delta t)-s(t)$, then

$$\text{Rate} = \frac{\text{distance}}{\text{time}}$$

The average[11] velocity[12] is

$$\frac{\text{Change in distance}}{\text{Change in time}}=\frac{\Delta s}{\Delta t}$$

In general, if $s(t)$ is the position for an object moving along a **straight**[13] line, the velocity of the object at time t is

$$v(t)=\lim_{\Delta t\to 0}\frac{s(t+\Delta t)-s(t)}{\Delta t}=s'(t).$$

The velocity function is the derivative of the position function. The position of a free-falling object (**neglecting**[14] air **resistance**[15]) under the **influence**[16] of **gravity**[17] can be represented by the equation

$$s(t)=\frac{1}{2}gt^2+v_0t+s_0$$

where s_0 is the **initial**[18] height of the object, v_0 is the initial velocity of the object, and g is the **acceleration**[19] due to gravity. On Earth, the value of g is **approximately**[20] -9.8 meters per second squared.

Higher-Order Derivatives

The derivative $f'(x)$ of a function $f(x)$ is a function and may have a derivative of its own. If $f'(x)$ is differentiable, then its derivative is denoted by $f''(x)$ and is called second derivative of $f(x)$. As long as we have differentiability, we can continue the process of differentiating to obtain third, fourth, and even higher derivatives of $f(x)$. These successive derivatives are denoted as follows.

First derivative	y', $f'(x)$
Second derivative	y'', $f''(x)$
Third derivative	y''', $f'''(x)$
Fourth derivative	$y^{(4)}$, $f^{(4)}(x)$
$\vdots$	$\vdots$
nth derivative	$y^{(n)}$, $f^{(n)}(x)$

Examples IV

Acceleration

The velocity of an object in meters per second is $v(t)=36-t^2, 0\le t\le 6$. Find the velocity and acceleration of the object when $t=3$.

Solution

$v(t)=36-t^2$ Velocity function

$v'(t)=-2t$ Acceleration function

substitute $t=3$,

$v(3)=36-3^2=27$

$v'(3)=-2\times 3=-6$

So, when $t=3$ the velocity of the object is 27 meters per second, and the acceleration of the object -6 meters per second squared.

Examples V

Acceleration on Earth vs. Moon

An astronaut standing on the moon throws a rock upward. The height of the rock is

$$s(t) = -\frac{27}{10}t^2 + 27t + 6$$

where s is measured in feet and t is measured in seconds.

(a) Find expressions for the velocity and acceleration of the rock.

(b) Find the time when the rock is at its highest point by finding the time when the velocity is zero. What is the height of the rock at this time?

(c) How does the acceleration of the rock compare with the acceleration due to gravity on Earth?

Solution

(a)

We can obtain a velocity function by differentiating a position function, and an acceleration function by differentiating a velocity function.

$$s'(t) = v(t) = -\frac{27}{5}t + 27 \qquad \text{Velocity function}$$

$$s''(t) = a(t) = -\frac{27}{5} = -5.4 \qquad \text{Acceleration function}$$

(b)

When the velocity is zero, we have $v(t) = 0$. So,

$$v(t) = -\frac{27}{5}t + 27 = 0 \Rightarrow t = 5$$

So, it's at $t = 5$ seconds, the rock is at its highest point. The height of the rock at $t = 5$ is $s(5) = -\frac{27}{10} \times 5^2 + 27 \times 5 + 6 = 73.5$ feet.

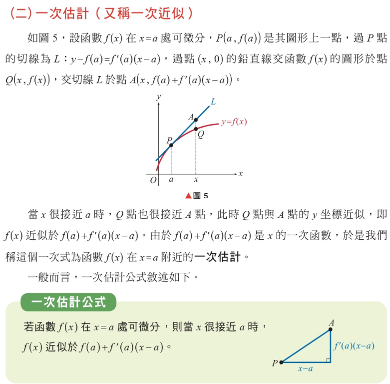
(c)

So, the acceleration due to gravity on the moon is -5.4 feet per second squared. Because the acceleration due to gravity on Earth is -32 feet per second squared, the ration of Earth's gravitational force to the moon's is

$$\frac{\text{Earth's gravitational force}}{\text{Moon's gravitational force}} = \frac{-32}{-5.4} \approx 6.0$$

Note:

On Earth, the value of the acceleration due to gravity g is approximately -32 feet per second squared or -9.8 meters per second squared.

Material	Vocabulary
	21. linear approximation (一次估計), 22. magnify (放大), 23. portion (部份), 24. appearance (外型), 25. vicinity (附近).

(二) 一次估計（又稱一次近似）

如圖 5，設函數 $f(x)$ 在 $x=a$ 處可微分，$P(a,f(a))$ 是其圖形上一點，過 P 點的切線為 L：$y-f(a)=f'(a)(x-a)$，過點 $(x,0)$ 的鉛直線交函數 $f(x)$ 的圖形於點 $Q(x,f(x))$，交切線 L 於點 $A(x,f(a)+f'(a)(x-a))$。

▲圖 5

當 x 很接近 a 時，Q 點也很接近 A 點，此時 Q 點與 A 點的 y 坐標近似，即 $f(x)$ 近似於 $f(a)+f'(a)(x-a)$。由於 $f(a)+f'(a)(x-a)$ 是 x 的一次函數，於是我們稱這個一次式為函數 $f(x)$ 在 $x=a$ 附近的**一次估計**。

一般而言，一次估計公式敘述如下。

一次估計公式

若函數 $f(x)$ 在 $x=a$ 處可微分，則當 x 很接近 a 時，$f(x)$ 近似於 $f(a)+f'(a)(x-a)$。

Illustrations IV

Local Linear Approximation[21]

If a function f is differentiable at a, then a **magnified[22] portion[23]** of the graph of f centered at the point $P(a,f(a))$ takes on the **appearance[24]** of a straight line segment, shown as Figure 2. For this reason, a function that is differentiable at a is sometimes said to be locally linear at a.

設函數 $f(x)$ 在 $x=a$ 處可微分，$P(a,f(a))$ 是其圖形上一點，將圖形部分放大發現 $P(a,f(a))$ 附近圖形為直線，如圖 2。於是我們稱這個函數在 $x=a$ 處可微分，且在 $x=a$ 附近有**一次估計**。

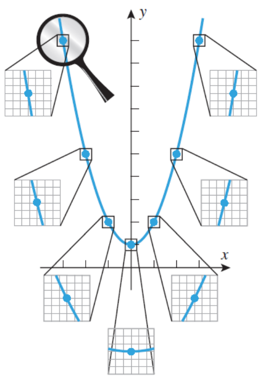


Figure 2

The line that best approximates the graph of f in the **vicinity[25]** of $P(a,f(a))$ is the tangent line to the graph of f at a, given by the equation

$$y=f(a)+f'(a)(x-a).$$

在函數 $f(x)$ 上點 $P(a,f(a))$ 附近最接近的近似圖形為 $f(x)$ 在 $x=a$ 的切線，其方程式為 $y=f(a)+f'(a)(x-a)$。

Thus, for values of x near a we can approximate values of $f(x)$ by

$$f(x)\approx f(a)+f'(a)(x-a)$$

This is called the local linear approximation of f at a.

因此，當 x 很接近 a 時，$f(x)$ 近似於 $f(a)+f'(a)(x-a)$。我們稱這個一次式為函數 $f(x)$ 在 $x=a$ 附近的**一次估計**。

Examples VI

Approximation

A function $f(x)=x^3-10x^2+24x+1$

(a) Find the local linear approximation of $f(x)$ at $x=1$.

(b) Use the local linear approximation obtained in part (a) to approximate $f(1.01)$.

(c) Compare the approximation to the result produced directly by a calculator.

Solution

(a)

The derivative function of $f(x)$ is $f'(x)=3x^2-20x+24$, it follows from $f(x)\approx f(a)+f'(a)(x-a)$, the local linear approximation of $f(x)$ at a point a is

$$f(x)\approx f(1)+f'(1)(x-1)$$

Thus, the local linear approximation at a point $x=1$ is

$$f(x)\approx 16+7(x-1)=9+7x$$

The graphs of $f(x)=x^3-10x^2+24x+1$ and the local linear approximation $y=9+7x$ are shown in Figure 3.

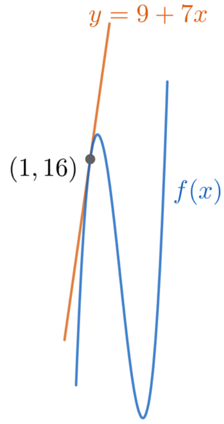


Figure 3

(b)

Applying $f(x)\approx 9+7x$ with $x=1.01$ yields

$$f(1.01)\approx 9+7\times 1.01=16.07.$$

(c)

Using the calculator, we have $f(1.01)=(1.01)^3-10\times(1.01)^2+24\times(1.01)+1=16.069301$.

The approximation error from part (b) is $16.07-16.069301=0.000699$.

References

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