Integration II

I. Key mathematical terms

Terms	Symbol	Chinese translation
Displacement		
Net change		
Cylinder		
Cross-section		

II. Application – Net change and average

With the fundamental theorem of calculus (II), if a function $f(x)$ is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

$F(x)$ is any antiderivative of $f(x)$. That is $F'(x)=f(x)$. We can rewrite the equation as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

$F'(x)$ represents the rate of change of $y=F(x)$ with respect to x and $F(b)-F(a)$ is the change in y when x changes from a to b. We can use this principle to rate problems in science and social science. For examples:

1. **<u>Displacement of a particle</u>**
   The velocity of a particle is $V(t) = S'(t)$ ($S(t)$ stands for the distance.), so

   $$\int_a^b V(t)dt = S(b) - S(a)$$

   Is the change in position of the particle from time a to time b.
2. **<u>Net change of a population</u>**
   The rate of growth of a population is $n'(t)$, then

   $$\int_{t_1}^{t_2} n'(t)dt = n(t_2) - n(t_1)$$

   Is the net change of a population from time t_1 to time t_2.

3. Mass of a rod

The mass of a rod measured from the left end to a point x is $m(x)$, for the rod has density $\rho(x) = m'(x)$, so

$$\int_a^b \rho(x)dx = m(b) - m(a)$$

Is the mass of the segment of the rod that lies between $x=a$ and $x=b$.

With the examples above, you may understand how to use the integral to measure the net change of real-world problems. Also, if we divide the net change by the difference of variable x or t. We can get the "average" of net change. Take "**Displacement of a particle**" as an example:

The **velocity** of a particle is $V(t)$.

The **change in position** from time a to time b is $\int_a^b V(t)dt = S(b) - S(a)$.

The **difference of *t*** from time a to time b is $b-a$.

The **"average" of change in position** is $\frac{1}{b-a}\int_a^b V(t)dt = \frac{S(b)-S(a)}{b-a}$, which is the average speed of the particle from time a to time b.

Net change

The integral of a rate of change is the net change.

$$\int_a^b F'(t)dt = F(b) - F(a)$$

The average value of the continuous function

The average value of a continuous function $f(x)$ on the interval $[a,b]$ is,

$$\frac{1}{b-a}\int_a^b f(x)dx$$

($f(x)$ can be any continuous function mentioned above.)

Example 1

The linear density of a rod of length 4 m is given by $\rho(x) = 9 + 3\sqrt{x}$ measured in kilograms per meter, where x is measured in meters from one end of the rode. Find the total mass of the rod.

Example 2

A particle moves along a line so that its velocity at time t is $v(t)=t^2-2t-3$ (measured in meters per second).

(1) Find the displacement of the particle during the time period $2\le t\le 5$.

(2) Find the distance traveled during this time period.
(Hint: You should find out at which period the velocity will be negative.)

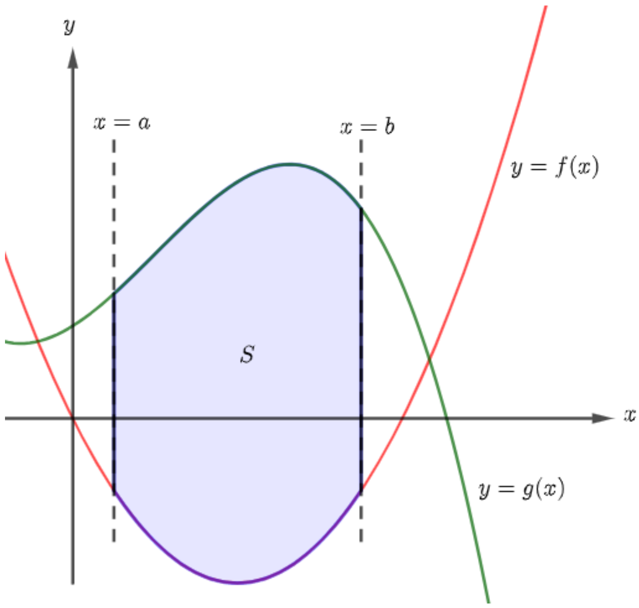
(3) Find the average velocity during this time period.

(4) Find the average speed during this time period.
(Hint: Velocity has both magnitude and direction but speed has magnitude only.)

III. Application – Areas between curves

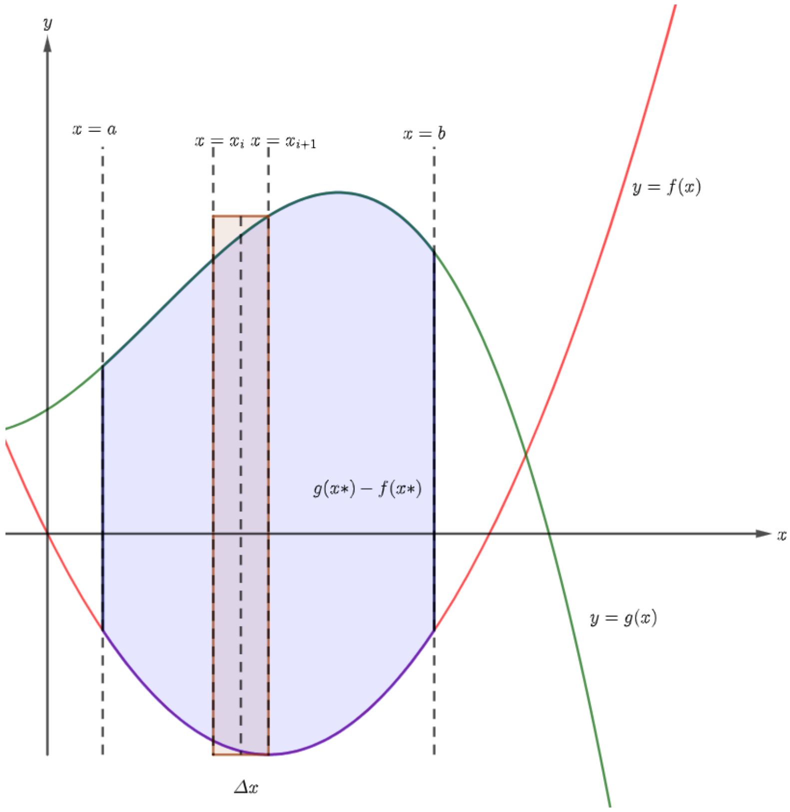
We've talked about the areas of regions that lie under the graphs of functions in Integration I. Here, we use integrals to find the areas of regions that lie between the graphs of two functions.

Consider the region S that lies between two curves $y=f(x)$, $y=g(x)$ and two vertical lines $x=a$, $x=b$ as the figure shown below. The function f and g are both continuous functions and $g(x)\ge f(x)$ for all x in $[a,b]$.



To get the area, we chop the region into n strips of equal width and then we can approximate the ith strip by a rectangle with width $\Delta x = x_{i+1} - x_i = \dfrac{b-a}{n}$ and length $g(x^*) - f(x^*)$. (x^* is the arbitrary point in the interval $[x_i, x_{i+1}]$.)
Then the Riemann sum of the region S equals:

$$\sum_{i=1}^{n}[g(x^*) - f(x^*)]\Delta x$$



To approximate the actual area of region S, take n tends to the infinity. Then we can define the area A of the region S as the limiting value of the sum of the areas of the rectangles.

$$\text{Area of the region} = A = \lim_{n\to\infty}\sum_{i=1}^{n}[g(x^*) - f(x^*)]\Delta x$$

We can find that the limit above is the definite integral of $g(x) - f(x)$ form $x = a$ to $x = b$. Therefore we have the following conclusion:

The area between continuous functions

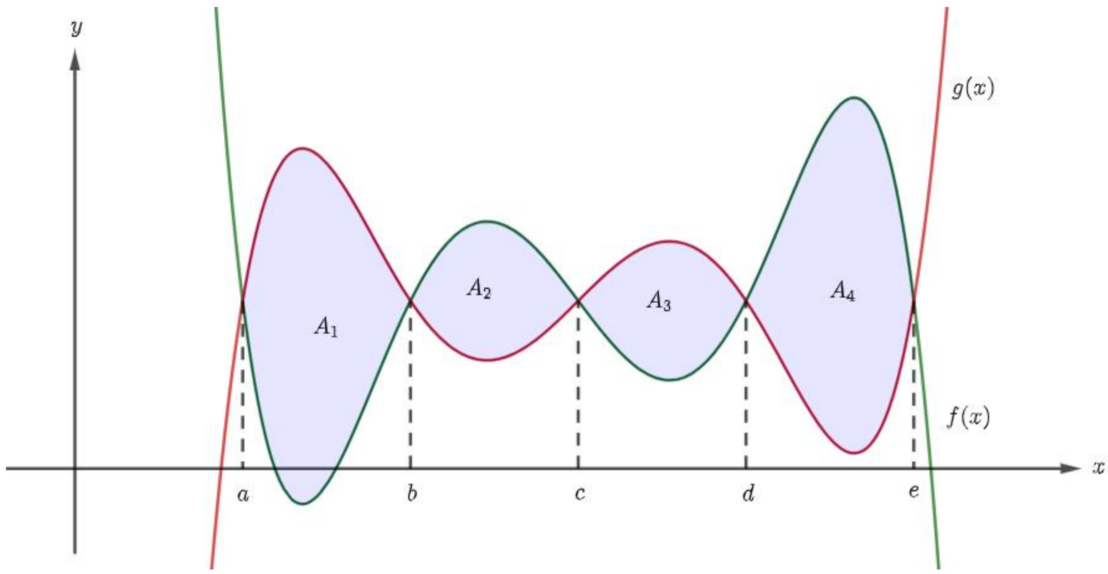
$f(x), g(x)$ are continuous functions and $g(x) \geq f(x)$ for all $a \leq x \leq b$. Then the area of the region bounded by the curves $f(x), g(x)$, and the lines $x = a, x = b$ is:

$$A = \int_a^b [g(x) - f(x)]dx$$

Example 3

(1) Find the area of the region enclosed by the parabola $y = x^2 + 1$ and the straight line $y = 2x + 9$.

(2) Find the area of the region bounded above by the straight line $y = 2x + 9$, bounded below by the parabola $y = x^2 + 1$ and bounded on the sides by $x = 0$ and $x = 3$.



$f(x), g(x)$ are continuous functions, in the interval $[a, e]$. If we are asked to find the area between the curves $f(x), g(x)$, where $f(x) \geq g(x)$ for some values of x and $g(x) \geq f(x)$ for some values of x. We should split the given region S into subregions S_1, S_2, S_3,… with areas A_1, A_2, A_3,… as shown in the figure above. We then defined the area of the region S to the sum of the areas of the smaller regions. Hence we have the area above to be:

$$A_1 + A_2 + A_3 + A_4 = \int_a^b g(x) - f(x)dx + \int_b^c f(x) - g(x)dx + \int_c^d g(x) - f(x)dx + \int_d^e f(x) - g(x)dx$$

$$= \int_a^e \left| f(x) - g(x) \right| dx$$ (To find the area, the integral function should be all positive.)

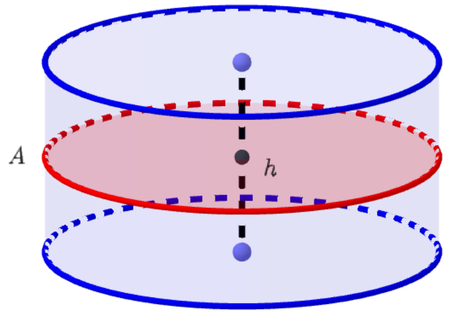
Example 4

Find the area of the region enclosed by the cubic equation $y = x^3$ and the parabola $y = -x^2 + 2x$.

IV. **Application – Volumes**

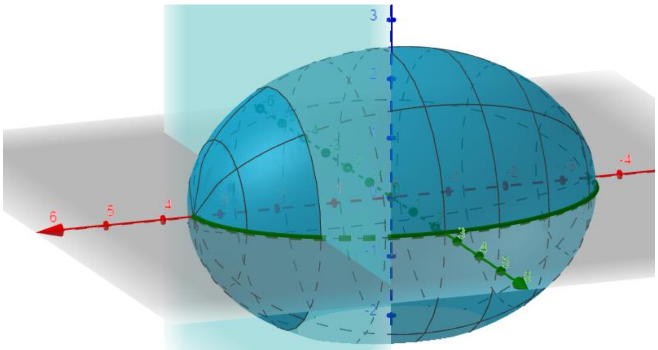
In the previous part, we learned one of the applications of definite integral. Another important application of definite integral is its use in finding volume of a three-dimensional solid. We start with a simple type of solid – a cylinder (as the figure shown below). The cylinder is bounded below by a plane region P_1 (base), and bounded above by another congruent region P_2. If the area of the base is A and the height if the cylinder is h, then the volume V of the cylinder is defined as:

$$V = Ah$$



Volume=Ah

Now we can find the volume of a cylinder with the formula above easily, but how can we get the solid S that isn't a cylinder? First, let's try to "chop" the solid into pieces. Second, we estimate the area of each piece. Finally, we add all the area of each piece to approximate the volume of the solid. (See figure below)



We chop the volume by planes.

We chop the solid region S into n "slabs" of equal width Δx with planes and obtain cross-sections of S. Let the function $A(x)$ be the area of the cross-section of S in a plane P_x perpendicular to the x-axis and passing through the point x, where $a \le x \le b$. The cross-sectional area $A(x)$ will vary as x increases from a to b.

Hence, we have the following:

(1) The height of each slab:

$$\Delta x = \frac{b-a}{n}$$

(2) The cross-section area of each slab:

$$A(x_i^*), x_i^* \in [x_{i-1}, x_i]$$

Then we can find the volume of each slab to be:

$$V(S_i) = A(x_i^*)\Delta x_i$$

Adding the volumes of these slabs, we get the approximate volume to be:

$$\sum_{i=1}^{n} A(x_i^*)\Delta x_i$$

Finally, we take n tends to the infinity to approximate the real volume.

Volumes of solids with cross-sections

Let S be a solid that lies between $x=a$ and $x=b$. If the cross-sectional area of S in plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where $A(x)$ is a continuous function, then the volume of S can be represented as:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i^*)\Delta x_i = \int_a^b A(x)dx$$

We can also take the cross-section that perpendicular to the y-axis between $y=c$ and $y=d$ with area $A(y)$, where $A(y)$ is a continuous function, then the volume of S can be represented as:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(y_i^*)\Delta y_i = \int_c^d A(y)dy$$

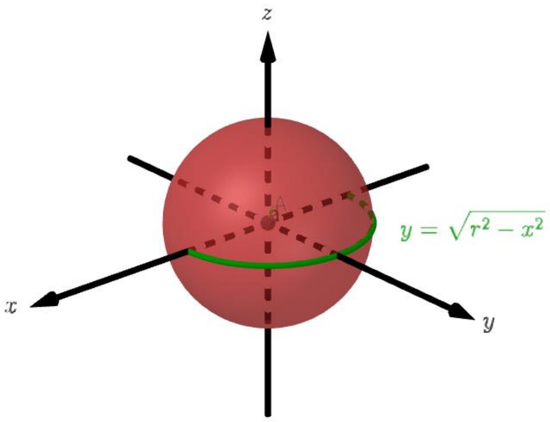
Example 5

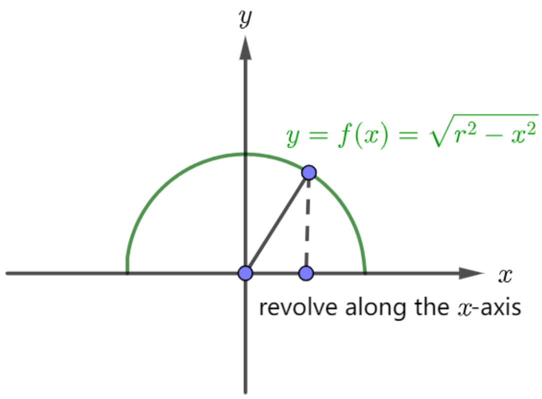
Show that the volume of a pyramid with a square base is $V = \frac{1}{3}a^2 h$, where h is the height of the pyramid and a is the side length of the base.

Example 6

Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$

If we revolve a region about a line, we obtain a solid of revolution. In the example6, a sphere of radius r can be considered as function $y = f(x) = \sqrt{r^2 - x^2}$ revolve along the x-axis, as the figure shown below.





Hence, the volume of a sphere can be considered in another way: $\int_{-r}^{r} \pi [f(x)]^2 \, dx$

We can extend this result to any polynomial function, to get the following result:

Volumes of solids with Revolution

The volume of a solid generated by revolving the region bounded by a function $y = f(x)$ and x-axis on the interval $[a,b]$ about the x-axis is:

$$\int_a^b \pi [f(x)]^2 \, dx \quad , f(x) \geq 0 \text{ for all } x \in [a,b]$$

We can also find the volume of solid generated by revolving region about the y-axis.

<key> This method to find the volume is also as known as “The disk method”.

Example 7

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \frac{r}{h}x$ and the x-axis ($0 \le x \le h$) about the x-axis.

Example 8

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis ($0 \le x \le \pi$) about the x-axis.

Example 9

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

<資料來源>

1. **Integration**
   Calculus-9th-Edition-by-Ron-Larson
   Calculus 9/e Metric Version-by-James-Jtewart
   Edexcel as and a level further mathematics core pure mathematics book 1/AS
   https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-integration-optional/v/proof-of-fundamental-theorem-of-calculus

2. 南一書局數學甲下冊

製作者：國立臺灣師範大學附屬高級中學 蕭煜修