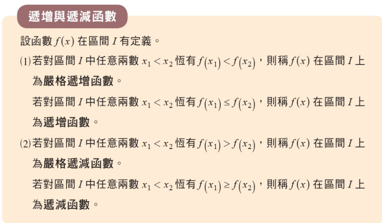
# 微分的應用（一）

# Applications of Differentiation I

| Material | Vocabulary |
|---|---|
| <br>遞增與遞減函數<br>設函數 $f(x)$ 在區間 $I$ 有定義。<br>(1)若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) < f(x_2)$，則稱 $f(x)$ 在區間 $I$ 上為**嚴格遞增函數**。<br>若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) \le f(x_2)$，則稱 $f(x)$ 在區間 $I$ 上為**遞增函數**。<br>(2)若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) > f(x_2)$，則稱 $f(x)$ 在區間 $I$ 上為**嚴格遞減函數**。<br>若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) \ge f(x_2)$，則稱 $f(x)$ 在區間 $I$ 上為**遞減函數**。 | 1. increasing (遞增), 2. decreasing (遞減), 3. strictly (嚴格地), 4. interval (區間), 5. tangent (切線), 6. negative (負的), 7. derivative (導數), 8. constant (常數), 9. positive (正的). |

## Illustrations I

### Increasing[1] and Decreasing[2] Functions

**Definitions of Increasing and Decreasing Functions**

(i) A function $f$ is **strictly[3] increasing** on an interval[4] $I$ if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function $f$ is **increasing** on an interval $I$ if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) \le f(x_2)$.

若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) < f(x_2)$，則稱 $f$ 在區間 $I$ 上為**嚴格遞增函數**。

若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) \le f(x_2)$，則稱 $f$ 在區間 $I$ 上為**遞增函數**。

(ii) A function $f$ is **strictly decreasing** on an interval $I$ if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function $f$ is **decreasing** on an interval $I$ if for any two numbers $x_1$ and $x_2$ in the interval, $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$.

若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) > f(x_2)$，則稱 $f$ 在區間 $I$ 上為**嚴格遞減函數**。

若對區間 $I$ 中任意兩數 $x_1 < x_2$ 恆有 $f(x_1) \ge f(x_2)$，則稱 $f$ 在區間 $I$ 上為**遞減函數**。

A function $f$ is increasing if, as $x$ moves to the right, its graph moves up, and is decreasing if its graph moves down.

函數 $f$ 為遞增，若 $x$ 往右，其圖形由左往右上升；若為遞減，其圖形由左往右下降。

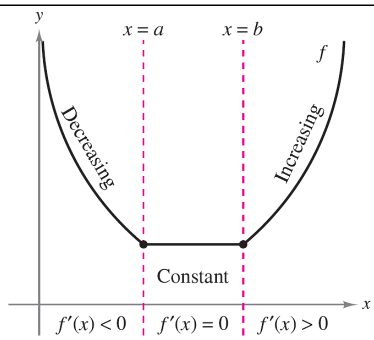



*Figure 1*

For example, in Figure 1 the **tangent**[5] of the function $f$ has a **negative**[6] slope for all $x$ on the interval $(-\infty, a)$, and a negative **derivative**[7] implies that the function is decreasing on the interval $(-\infty, a)$.

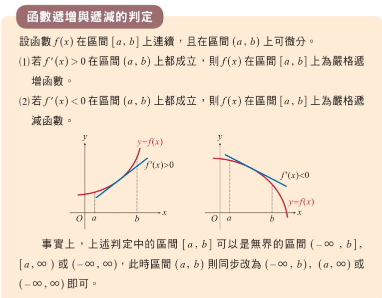
如圖一，函數 $f$ 在區間 $(-\infty, a)$ 上的切線斜率皆為負，而負的斜率代表其函數在區間 $(-\infty, a)$ 上遞減。

The function shown in Figure 1, the tangent of the function $f$ is zero for all $x$ on the interval $(a, b)$, then $f$ is **constant**[8] on $(a, b)$.

函數 $f$ 在區間 $(a, b)$ 上的切線斜率皆為零，代表其函數在區間 $(a, b)$ 上為常數函數。

The function shown in Figure 1 , the tangent of the function $f$ has a **positive**[9] slope for all $x$ on the interval $(a, \infty)$, and a positive derivative implies that the function is increasing on the interval $(a, \infty)$.

函數 $f$ 在區間 $(a, \infty)$ 上的切線斜率皆為正，而正的斜率代表其函數在區間 $(a, \infty)$ 上遞增。

| Material | Vocabulary |
| --- | --- |
| <br>函數遞增與遞減的判定<br>設函數 $f(x)$ 在區間 $[a, b]$ 上連續，且在區間 $(a, b)$ 上可微分。<br>(1)若 $f'(x) > 0$ 在區間 $(a, b)$ 上都成立，則 $f(x)$ 在區間 $[a, b]$ 上為嚴格遞增函數。<br>(2)若 $f'(x) < 0$ 在區間 $(a, b)$ 上都成立，則 $f(x)$ 在區間 $[a, b]$ 上為嚴格遞減函數。<br>$y=f(x)$ $f'(x)>0$ $f'(x)<0$ O a b x y<br>事實上，上述判定中的區間 $[a, b]$ 可以是無界的區間 $(-\infty, b]$, $[a, \infty)$ 或 $(-\infty, \infty)$，此時區間 $(a, b)$ 則同步改為 $(-\infty, b)$, $(a, \infty)$ 或 $(-\infty, \infty)$ 即可。 | 10. sign (符號), 11. continuous (連續的), 12. differentiable (可微分的), 13. entire (全部的), 14. critical number (臨界值), 15. monotonic (單調的). |

## Illustrations II

In fact, the sign[10] of the derivative indicates whether the function's trend is increasing or decreasing.

函數在某區間上「導數為正數或負數」與「函數遞增或遞減」關係密切。

**Test for Increasing and Decreasing Functions**

Let $f$ be a function that is continuous[11] on the closed interval $[a,b]$ and differentiable[12] on the open interval $(a,b)$.

(i) If $f'(x)>0$ for all $x$ in $(a,b)$, then $f$ is **strictly increasing** on $[a,b]$.

(ii) If $f'(x)<0$ for all $x$ in $(a,b)$, then $f$ is **strictly decreasing** on $[a,b]$.

(iii) If $f'(x)=0$ for all $x$ in $(a,b)$, then $f$ is constant on $[a,b]$.

設函數 $f$ 在區間 $[a,b]$ 上連續，且在區間 $(a,b)$ 上可微分。

(i) 若 $f'(x)>0$ 在區間 $(a,b)$ 上都成立，則 $f$ 在區間 $[a,b]$ 上為**嚴格遞增函數**。

(ii) 若 $f'(x)<0$ 在區間 $(a,b)$ 上都成立，則 $f$ 在區間 $[a,b]$ 上為**嚴格遞減函數**。

(iii) 若 $f'(x)=0$ 在區間 $(a,b)$ 上都成立，則 $f$ 在區間 $[a,b]$ 上為常數函數。

It is noteworthy that the sign of the derivative will be negative or equal to 0 when the function is decreasing and positive or equal to 0 when the function is increasing.

值得注意的是，當函數遞減時其導數小於或等於0；函數遞增時其導數大於或等於0。

## Examples I

Find the open intervals on which $f(x)=x^3-3x+2$ is increasing or decreasing.

**Solution**

Note that $f$ is differentiable on the entire[13] real number line. To determine the critical numbers[14] of $f$, set $f'(x)$ equal to zero.

$f(x)=x^3-3x+2$

$f'(x)=3x^2-3=0$ Differentiate and set $f'(x)$ equal to $0$.

$3(x+1)(x-1)=0$ Factorize.

$x=1,\ -1$ Critical numbers.

Because there are no points for which $f'$ does not exist, you can conclude that $x=1$ and $x=-1$ are the only critical numbers. The table summarizes of the three intervals determined by these two critical numbers.

| Interval | $-\infty<x<-1$ | $-1$ | $-1<x<1$ | $1$ | $1<x<\infty$ |
|---|---|---|---|---|---|
| Sign of $f'(x)$ | $+$ | 0 | $-$ | 0 | $+$ |
| conclusion | Increasing | | decreasing | | Increasing |

So, $f$ is increasing on the intervals $(-\infty,-1)$ and $(1,\infty)$ and decreasing on the interval $(-1,1)$, as shown in Figure 2.

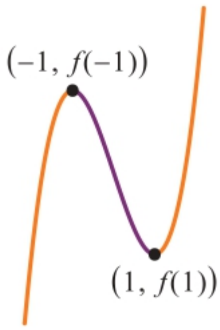



*Figure 2*

A function is strictly **monotonic**[15] on an interval if it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x)=x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 3.

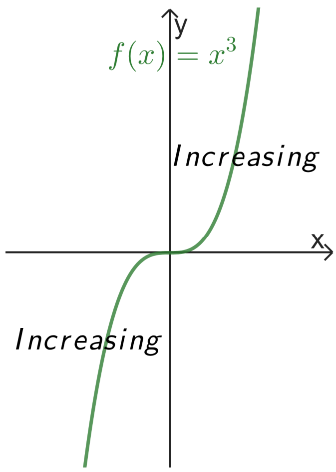



*Figure 3*

The function $f(x)=\begin{cases} -x^2 & ,x<0 \\ 0 & ,0\le x\le 1 \\ (x-1)^2 & ,x>1 \end{cases}$ shown in Figure 4 is not strictly monotonic on the entire real number line because it is constant on the interval $[0,1]$.

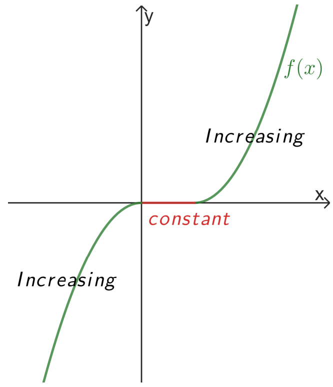
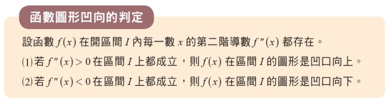
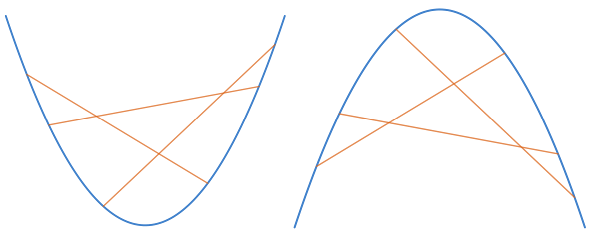



*Figure 4*

| Material | Vocabulary |
| --- | --- |
| <br>函數圖形凹向的判定<br>設函數 $f(x)$ 在開區間 $I$ 內每一數 $x$ 的第二階導數 $f''(x)$ 都存在。<br>(1)若 $f''(x)>0$ 在區間 $I$ 上都成立，則 $f(x)$ 在區間 $I$ 的圖形是凹口向上。<br>(2)若 $f''(x)<0$ 在區間 $I$ 上都成立，則 $f(x)$ 在區間 $I$ 的圖形是凹口向下。 | 16. concavity (凹向性), 17. concave (凹向), 18. convex (凸的), 19. above (在…之上), 20. symmetrically (對稱地), 21. below (在…之下). |

**Illustrations III**

**Definition of Concavity[16]**



(a) (b)

*Figure 5*

A function $f$ is **concave[17]** up (or **convex[18]**, convex down) if every line segment joining two points on its graph lies **above[19]** the graph at any point, see figure 5(a). **Symmetrically[20]**, a function $f$ is concave down (or concave, convex up) if every line segment joining two points on its graph lies **below[21]** the graph at any point, see figure 5(b).

若函數 $f$ 圖形上任相異兩點所連成的線段恆在圖形的上方，則稱圖形凹口向上，如圖 5(a)；反之，若線段恆在圖形的下方，則稱圖形凹口向下，如圖 5(b)。

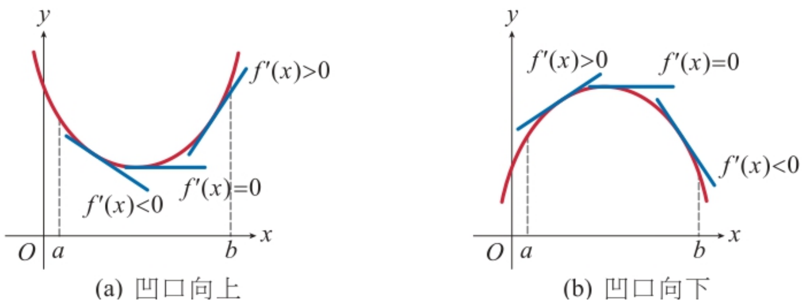



*Figure 6*

(i) Let $f$ be differentiable on an open interval $I$. If the graph of $f$ is concave **up** on $I$, then the graph of $f$ lies **above** all of its tangent lines on $I$. See Figure 6(a).

設函數 $f$ 在開區間 $I$ 上可微分。若函數圖形在的其切線的**上方**，則稱圖形在區間 $I$ 上凹口**向上**。見圖 6(a)。

(ii) Let $f$ be differentiable on an open interval $I$. If the graph of $f$ is concave **down** on $I$, then the graph of $f$ lies **below** all of its tangent lines on $I$. See Figure 6(b).

設函數 $f$ 在開區間 $I$ 上可微分。若函數圖形在的其切線的**下方**，則稱圖形在區間 $I$ 上凹口**向下**。見圖 6(b)。

Let $f$ be differentiable on an open interval $I$. The graph of $f$ is concave up on $I$ if $f'$ is increasing on the interval and concave down on $I$ if $f'$ is decreasing on the interval.

設函數 $f$ 在開區間 $I$ 上可微分。若圖形之導函數 $f'$ 遞增，則稱圖形凹口向上；反之，若圖形之導函數 $f'$ 遞減，則稱圖形凹口向下。

To find the open intervals on which the graph of a function $f$ is concave up or concave down, you need to find the intervals on which $f'$ is increasing or decreasing.

在開區間 $I$ ，函數 $f'$ 遞增或遞減可決定函數 $f$ 圖形的凹口**向上**或**向下**。

By the test for increasing and decreasing functions, the sign of $f''$ can determine whether $f'$ is decreasing or increasing. We describe the relationship of “The Sign of the Second Derivative $f''$ ” and “ The Concavity of Function $f$ ” as follows.

根據函數遞增與遞減的判定，導函數 $f'$ 的遞增與遞減可由二階導函數 $f''$ 函數值的正負來決定。我們將「二階導函數 $f''$ 函數值的正負」與「函數 $f$ 圖形的凹向」的關係敘述如下。

**Test for Concavity**

Let $f$ be a function whose second derivative exists on an open interval $I$.

(i) If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave **up** on $I$.

(ii) If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave **down** on $I$.

**函數圖形凹向的判定**

設函數 $f$ 在開區間 $I$ 內每一數 $x$ 的第二階導數 $f''(x)$ 都存在。

(i)若 $f''(x) > 0$ 在區間 $I$ 上都成立，則 $f$ 在區間 $I$ 的圖形是凹口**向上**。

(ii)若 $f''(x) < 0$ 在區間 $I$ 上都成立，則 $f$ 在區間 $I$ 的圖形是凹口**向下**。

It is noteworthy that if a function is concave up on an interval, then the 2nd derivative is greater than or equal to 0. Similarly, if a function is concave down on an interval, then its 2nd derivative is less than or equal to 0.

特別注意，若函數凹口向上，則其二階導數大於或等於 0；凹口向下，則其二階導數小於或等於 0。

**Examples II**

**Determining Concavity**

Determine the open intervals on which the graph of $f(x) = x^3 - 3x + 2$ is concave up or down.

**Solution**

Begin by observing that $f$ is continuous on the entire real line. Next, find the second derivative of $f$.

$f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$ First derivative.

$f''(x) = 6x$ Second derivative.

Because $f''(x) = 6x = 0$ when $x = 0$ and $f''$ is defined on the entire real line, we test $f''$ in the intervals $(-\infty, 0)$ and $(0, \infty)$. The results are shown in the table and in Figure 7.

| Interval | $(-\infty, 0)$ | 0 | $(0, \infty)$ |
|---|---|---|---|
| Sign of $f''(x)$ | $-$ | 0 | $+$ |
| Conclusion | Concave down | | Concave up |

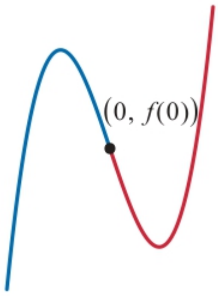



*Figure 7*

| Material | Vocabulary |
|---|---|
| **反曲點的定義**<br>在 $a$ 的附近，當 $f(x)$ 在 $x=a$ 處連續，且 $f(x)$ 的圖形在 $x<a$ 與 $x>a$ 的凹向相反時，稱點 $(a, f(a))$ 為函數 $f(x)$ 圖形的反曲點。<br>**反曲點的性質**<br>若 $(a, f(a))$ 為多項式函數 $f(x)$ 圖形的一個反曲點，則 $f''(a)=0$。 | 22. points of inflection (反曲點), 23. occur (發生). |

| Illustrations IV |
|---|

## Points of Inflection[22]

The graph in Figure 8 has one point at which the concavity changes. If the tangent line to the graph exists at such a point, which is **a point of inflection (or an inflection point)**. There are three types of points of inflection are shown.

圖8中各有一點，其左側和右側凹向相反，我們稱點是函數 $f$ 圖形的反曲點。在反曲點上之三種切線情形如圖8所示。

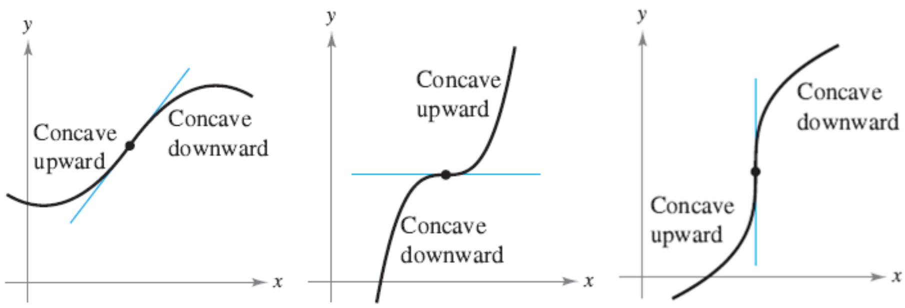



*Figure 8*

**Definition of Inflection Points**

Let $f$ be a function that is continuous on an open interval and let $a$ be a point in the interval. If the graph of $f$ has a tangent line at this point $(a, f(a))$, then this point is a **point of inflection** of the graph of $f$ if the concavity of $f$ changes from upward to downward (or downward to upward) at the point.

**反曲點的定義**

在 $a$ 的附近，當 $f$ 在 $x=a$ 處連續，且 $f$ 的圖形在 $x<a$ 與 $x>a$ 的凹向相反時，稱點 $(a,f(a))$ 為函數 $f$ 圖形的**反曲點**。

**Keys of Inflection Points**

If $(a,f(a))$ is a point of inflection of the graph of $f$, then $f''(a)=0$.

**反曲點的性質**

若 $(a,f(a))$ 為多項式函數 $f$ 圖形的一個反曲點，則 $f''(a)=0$ 。

**Examples III**

Determine the points of inflection and discuss the concavity of the graph of $f(x)=x^4-6x^2+5$

**Solution**

Differentiating twice produces the following

$f(x)=x^4-6x^2+5$

$f'(x)=4x^3-12x$ First derivative.

$f''(x)=12x^2-12$ Second derivative.

Setting $f''(x)=12x^2-12=0$, we can determine that the possible points of inflection occur[23] at $x=-1$ and $x=1$. A summary is shown in the table and the graph of $f$ is shown in Figure 8.

| Interval | $(-\infty,-1)$ | $-1$ | $(-1,1)$ | $1$ | $(1,\infty)$ |
|---|---|---|---|---|---|
| Sign of $f''(x)$ | $+$ | 0 | $-$ | 0 | $+$ |
| Conclusion | Concave up | | Concave down | | Concave up |

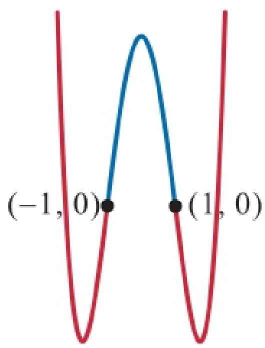



*Figure 8*

**References**

1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。數學甲：單元 4 函數性質的判定。龍騰文化。
2. Ron Larson & Bruce H. Edwards (2009). Calculus 9th. Brooks/Cole
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