雙語教學主題(國中九年級上學期教材)：比例線段

Topic: introducing proportional line segments

Vocabulary

Proportional line segments, Continued ratio, continued proportion,

In this lesson, we are going to talk about proportional line segments in terms of the ratio of the area of triangles, midpoint segment property which we learned last year, and parallel lines as well. Please be patient to learn all these proportional line segments properties because they are very helpful when we learn similarity of triangles and solve proportion questions.

在這一節課，我們會提到關於三角形面積比、三角形中點連線性質以及平行線來討論線段的比例情形。

Let's look at the first situation below to see how proportional line segments are related to the area of triangles.

Ex 1:

In Figure 1, point D is on $\overline{BC}$ in ΔABC. Given $\overline{BD}:\overline{CD}=2:5$

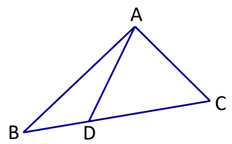


Figure 1

Please find the continued ratio of the area of ΔABD to the area of ΔACD, and to the area of ΔABC

Sol:

Construct $\overline{AH}\perp\overline{BC}$ (as shown in Figure 2)

and intersects $\overline{BC}$ at point H.

Then $\overline{AH}$ is the height of ΔABD, ΔACD, and ΔABC.

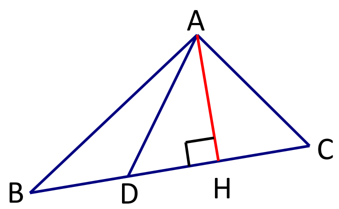


Figure 2

We get

the area of $\Delta ABD=\frac{1}{2}\overline{AH}\cdot\overline{BD}$

the area of $\Delta ACD=\frac{1}{2}\overline{AH}\cdot\overline{CD}$

the area of $\Delta ABC=\frac{1}{2}\overline{AH}\cdot\overline{BC}$

Then, the area of ΔABD: the area of ΔACD: the area of ΔABC

$$=\frac{1}{2}\overline{AH}\cdot\overline{BD}:\frac{1}{2}\overline{AH}\cdot\overline{CD}:\frac{1}{2}\overline{AH}\cdot\overline{BC}\quad \text{(divided by } \frac{1}{2}\overline{AH}\text{)}$$

$$=\overline{BD}:\overline{CD}:\overline{BC}$$

=2:5:(2+5)

=2:5:7#

The result above shows that:

The continued ratio of the area of triangles with the same height is the continued ratio of the length of their bases correspondingly.

Rewrite it in the general form:

In Figure 1, point D is on $\overline{BC}$ in ΔABC. Given $\overline{BD}:\overline{CD}$=m:n

Then the continued ratio of

the area of ΔABD: the area of ΔACD: the area of ΔABC

=m:n:(m+n)

The result above generates most properties coming after. Before we have further discussions, let's review a fact below.

Given L//M, points A and B are on line M and points C and D are on line L. Then

the area of ΔABC=the area of ΔABD

It is obvious!

The distance between two parallel lines is constant.

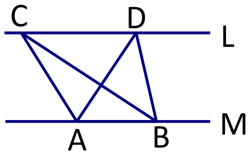


Figure 1

That means the height of ΔABC and the height of ΔABD are congruent.

Let h be the distance between lines L and M. See Figure 2.

And these two triangles share the same base segment AB.

Therefore,

$$\text{the area of } \Delta ABC=\frac{1}{2}\overline{AB}\cdot h=\text{the area of } \Delta ABD$$

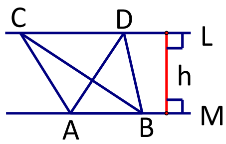


Figure 2

Let's look into the first important proportional line segments in terms of parallel lines in triangles.

As Figure 1 shows, given $\overline{DE}//\overline{BC}$ and intersects $\overline{AB}$ at point D, $\overline{AC}$ at point E. Then

$$\frac{\overline{AD}}{\overline{BD}}=\frac{\overline{AE}}{\overline{CE}}$$

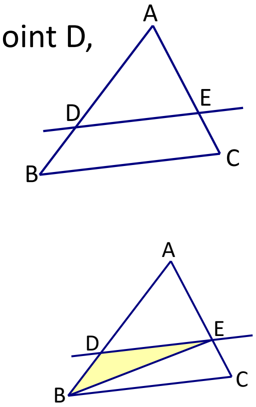


Figure 1

Pf:

Connect $\overline{BE}$ as shown in Figure 1.

From example 1, we get

the area of ΔADE: the area of ΔBDE= $\overline{AD}:\overline{BD}$

And connect $\overline{DC}$ as shown in Figure 2, we get

the area of ΔADE: the area of ΔCDE= $\overline{AE}:\overline{CE}$

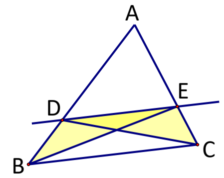


Figure 2

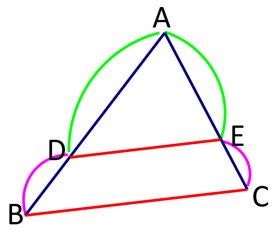
Since the area of ΔADE is always the same, and the area of ΔBDE= the area of ΔCDE (same base with the congruent heights)

We can easily get the result :

$\overline{AD}:\overline{BD}$ = the area of ΔADE: the area of ΔBDE

= the area of ΔADE: the area of ΔCDE

$=\overline{AE}:\overline{CE}$



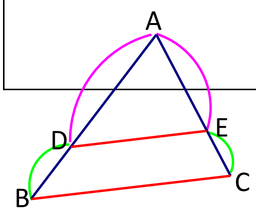
Let's wrap up the conclusion:

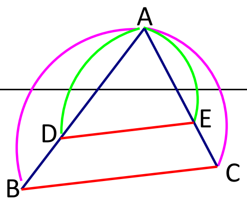
In ΔABC, if $\overline{DE}//\overline{BC}$, then $\frac{\overline{AD}}{\overline{BD}}=\frac{\overline{AE}}{\overline{CE}}$

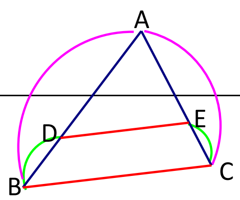
You can do your proving and get the following results.

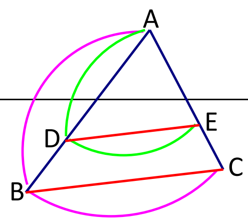
In ΔABC, if $\overline{DE}//\overline{BC}$, then

$\frac{\overline{BD}}{\overline{AD}}=\frac{\overline{CE}}{\overline{AE}}$ $\frac{\overline{AD}}{\overline{AB}}=\frac{\overline{AE}}{\overline{AC}}$ $\frac{\overline{BD}}{\overline{AB}}=\frac{\overline{CE}}{\overline{AC}}$ $\frac{\overline{AD}}{\overline{AB}}=\frac{\overline{DE}}{\overline{BC}}$









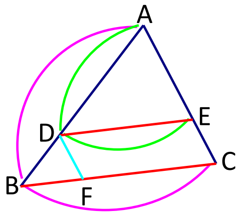
Hint for the proof of the last result.

Construct $\overline{DF} // \overline{CE}$, quadrilateral DFCE is a parallelogram.

(two pairs of opposite sides are parallel)

Then you can do the rest of it.

Go!



We can get so many useful results on proportional line segments in terms of parallel lines in a triangle.

But we want to know if the converse is true.

What I mean is: the statement

In a triangle,

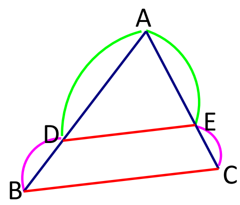
Parallel lines ⇒ proportional line segments (we know it's true)

If the converse

Proportional line segments ⇒ parallel lines is true?

If so, the statement will be very powerful. Let's discuss the following:

In △ABC, given $\frac{\overline{AD}}{\overline{BD}} = \frac{\overline{AE}}{\overline{CE}}$, then $\overline{DE} // \overline{BC}$.



Hint: it's not easy to prove two lines are parallel if we have no Information of angles and we have not learned about the similarity of two triangles. The only information we have here is if the heights of triangles that share the same base are congruent when two areas of triangles are equal, then we can connect to the definition of parallel lines. Because heights only happen when two segments are perpendicular.

提示:如果我們沒有學過相似三角形或者我們沒有角度的資料，我們很難去明兩條直線的平行與否。由上面的討論，我們可以利用三角形面積相等得到它們的高也相等，進而嘗試可不可以推得兩直線的平行。因為平行常常跟垂直有關。

Pf:

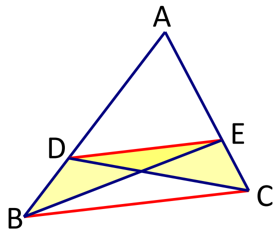


Figure 1

As Figure 1 shows, connect $\overline{BE}$ and $\overline{DC}$ in ΔABC.

the area of ΔADE: the area of $\Delta BDE = \overline{AD} : \overline{BD}$

And the area of ΔADE: the area of $\Delta CDE = \overline{AE} : \overline{CE}$

Then the area of ΔADE: the area of ΔBDE

$= \overline{AD} : \overline{BD} = \overline{AE} : \overline{CE}$ (given)

= the area of ΔADE: the area of ΔCDE

We get

the area of ΔBDE = the area of ΔCDE

Now construct $\overline{BM} \perp \overline{DE}$ and $\overline{CN} \perp \overline{DE}$ (shown in Figure 2) and intersection points are point M and point N on line DE.

$\frac{1}{2} \overline{BM} \cdot \overline{DE}$ = the area of ΔBDE = the area of ΔCDE

$= \frac{1}{2} \overline{CN} \cdot \overline{DE}$

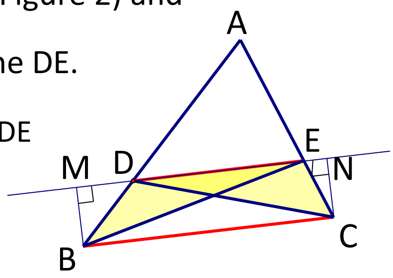


Figure 2

So $\overline{BM} = \overline{CN}$

And $\overline{BM} // \overline{CN}$ (both are perpendicular to the same segment $\overline{DE}$)

Therefore, quadrilateral BCMN is a parallelogram (a pair of opposite sides of a quadrilateral are congruent and parallel to each other, it is a parallelogram)

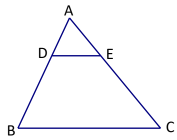
We get

$\overline{DE} // \overline{BC}$ (a pair of opposite sides of a parallelogram are parallel)

So the converse is true.

Let's see an example here.

Given ΔABC, $\overline{DE} // \overline{BC}$, $\frac{\overline{DE}}{\overline{BC}} = \frac{1}{3}$, find the ratios of



1) $\frac{\overline{AD}}{\overline{AB}}$ 2) $\frac{\overline{CE}}{\overline{AE}}$

Sol:

1) $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{DE}}{\overline{BC}} = \frac{1}{3}$ $(\because \overline{DE} // \overline{BC})$

2) since $\frac{\overline{AC}}{\overline{AE}} = \frac{\overline{AB}}{\overline{AD}} = \frac{3}{1}$ then $\overline{AC} = 3\overline{AE}$

That is $\overline{AC} = \overline{AE} + \overline{CE} = 3\overline{AE}$

We have $\overline{CE} = 2\overline{AE}$

So $\frac{\overline{CE}}{\overline{AE}} = \frac{2}{1}$ #

We can work on many questions from the above proportional line segments properties by applying these properties.

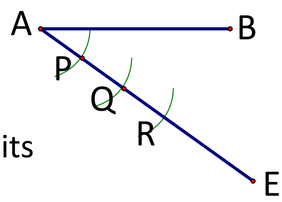
Remember when we learned to construct perpendicular bisectors of line segments, we can easily divide a line segment into 2^n parts. Now we know parallel lines can generate proportional line segments, we can divide a line segment into almost any proportion theoretically.

Dividing a line segment

Given $\overline{AB}$ as shown in Figure 1.

Figure 1

Find a point C on $\overline{AB}$ such that $\overline{AC}:\overline{BC}=1:2$



First draw ray AE, and construct a circle with point A as its center and a certain length as its radius. See Figure 2. Circle A intersects ray AE at point P.

Figure 2

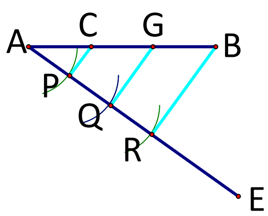
Next, construct a circle with point P as its center and $\overline{AP}$ as its radius.

Circle P intersects ray AE at point Q. Repeat the same process, and we get

$\overline{AP}=\overline{PQ}=\overline{QR}$

Connect $\overline{BR}$, and construct lines parallel to $\overline{BR}$ through point P and point Q and intersects $\overline{AB}$ at point C and point G respectively.

See Figure 3.



Then $\frac{\overline{AC}}{\overline{BC}}=\frac{\overline{AP}}{\overline{PR}}=\frac{1}{2}$ ($\because$ $\overline{AP}=\overline{PQ}=\overline{QR}$, $\overline{AP}:\overline{QR}=1:2$)

Figure 3

So point C is the answer.

Another useful proportional line segments in parallel lines is shown below:

In Figure 1, given $L_1//L_2//L_3$, line M and line N are transversals as shown in Figure 1, line M intersects at points A, B, C and line N intersects at points D, E, F respectively with L_1, L_2, and L_3.

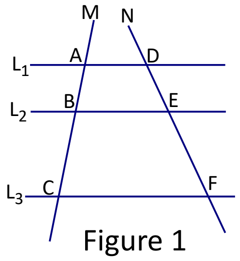


Figure 1

Then $\dfrac{\overline{AB}}{\overline{BC}}=\dfrac{\overline{DE}}{\overline{EF}}$

Pf:

The proof is quite easy. We only need to construct a parallel line.

shown in Figure 2, we construct $\overleftrightarrow{AH}//N$ and intersects L_2 and L_3 at point G and point H respectively.

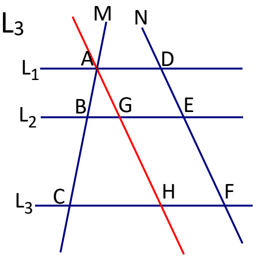


Figure 2

In $\triangle ACH$, $\dfrac{\overline{AB}}{\overline{BC}}=\dfrac{\overline{AG}}{\overline{GH}}$ $\cdots\cdots$ (1)($L_2//L_3$)

Quadrilateral AGED and quadrilateral GHFE are both parallelograms.

So $\dfrac{\overline{AG}}{\overline{GH}}=\dfrac{\overline{DE}}{\overline{EF}}$ $\cdots\cdots$ (2)(opposite sides of parallelograms are parallel)

From (1) and (2), we get

$$\frac{\overline{AB}}{\overline{BC}}=\frac{\overline{DE}}{\overline{EF}}\#$$

One thing we have to be very careful about. Proportional line segments can always get parallel lines in triangles except in the following situation.

In $\triangle ABC$ as Figure 1 shows, given $\dfrac{\overline{AD}}{\overline{AB}}=\dfrac{\overline{DE}}{\overline{BC}}$, then $\overline{DE}//\overline{BC}$.

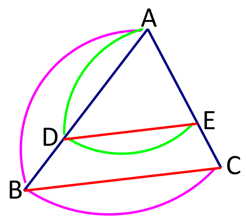


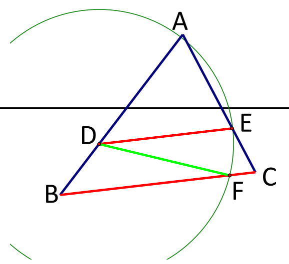
Figure 1

The statement is **wrong**!

Let's see the reason.

Take point D as a center, $\overline{DE}$ as a radius to construct a circle D.

Circle D intersects $\overline{BC}$ at point F, connect $\overline{DF}$



Then

$$\frac{\overline{AD}}{\overline{AB}}=\frac{\overline{DE}}{\overline{BC}}=\frac{\overline{DF}}{\overline{BC}} \quad (\overline{DE}=\overline{DF}=\text{the radius of circle D})$$

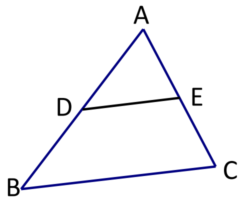
Figure 2

We can easily tell in Figure 2, $\overline{DF}$ is not parallel to $\overline{BC}$.

When a math statement is sometimes wrong, we say the statement is not true.

After we have learned so much about proportional line segments, we are going to introduce in my opinion, the most important proportional line segments property—the midsegment theorem of a triangle.

Triangle midsegment theorem



In Figure 1, point D and point E are midpoints of side AB and side AC in ΔABC respectively.

Then $\overline{DE} // \overline{BC}$ and $\overline{DE}=\frac{1}{2}\overline{BC}$

Pf:

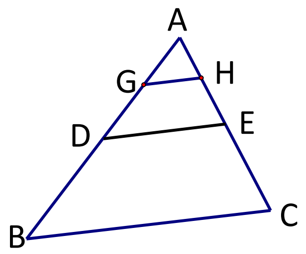
$\frac{\overline{AD}}{\overline{BD}}=\frac{1}{1}=\frac{\overline{AE}}{\overline{CE}}$, then $\overline{DE} // \overline{BC}$ (point D and point E are midpoints)

Once we know that $\overline{DE} // \overline{BC}$, then

$$\frac{\overline{DE}}{\overline{BC}}=\frac{\overline{AD}}{\overline{AB}}=\frac{1}{2}, \text{ so } \overline{DE}=\frac{1}{2}\overline{BC}$$

The statement is true.

Let's see an example here.



Given ΔABC, point D is the midpoint of $\overline{AB}$

and point E is the midpoint of $\overline{AC}$.

$\overline{GH}//\overline{DE}//\overline{BC}$ and $\frac{\overline{GD}}{\overline{DB}}=\frac{3}{5}$.

If $\overline{AC}=10$, and $\overline{BC}=15$, Find the following length:

(1) $\overline{DE}$ (2) $\overline{AH}$

Sol:

(1) $\overline{DE}=\frac{1}{2}\overline{BC}=\frac{15}{2}$ (triangle midsegment theorem)

(2) let $\overline{GD}=3k$, $\overline{DB}=5k$, $k\neq 0$ ($\frac{\overline{GD}}{\overline{DB}}=\frac{3}{5}$.)

$\overline{AG}=\overline{AD}-\overline{GD}=\overline{DB}-\overline{GD}=5k-3k=2k$ (point D is the midpoint of $\overline{AB}$)

$\frac{\overline{AH}}{\overline{AC}}=\frac{\overline{AG}}{\overline{AB}}=\frac{2K}{10K}=\frac{1}{5}$

$\overline{AH}=\frac{1}{5}\overline{AC}=\frac{1}{5}\cdot 10=2$#

Please review all the properties we learn above thoroughly. We are going to use them a lot in the next class when we introduce similar triangles.

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